Sobolev embeddings for Riesz potentials of functions in grand Morrey spaces of variable exponents over non-doubling measure spaces

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Abstract

In this paper, we are concerned with Sobolev embeddings for Riesz potentials of functions in grand Morrey spaces of variable exponents over non-doubling measure spaces.

1 Introduction

The space introduced by Morrey [37] in 1938 has become a useful tool of the study for the existence and regularity of partial differential equations (see also [39]). The maximal operator is a classical tool in harmonic analysis and studying Sobolev functions and partial differential equations and plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on (see [4], [29], [44], etc.). Boundedness properties of the maximal operator and Riesz potentials of functions in Morrey spaces were investigated in [1], [5] and [38]. The same problem for the maximal operator and Riesz potentials of functions in Morrey spaces with non-doubling measure was studied in [41] (see also [23] and [40], etc.).

In the mean time, variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth condition. For a survey, see [9]. The boundedness of the maximal operator on variable exponent Lebesgue spaces $L^{p(\cdot)}$ was studied in [6], [7] and [24]. In [8], Sobolev's inequality for variable exponent Lebesgue spaces $L^{p(\cdot)}$ was studied. Then such properties were investigated on variable exponent Morrey spaces in [3], [21], [17], [22] and [35]. For variable exponent Morrey spaces with non-doubling measure in [30].

Grand Lebesgue spaces were introduced in [27] for the sake of study of the Jacobian. The grand Lebesgue spaces play an important role also in the theory of partial differential equations (see [19], [28] and [43], etc.). The generalized grand Lebesgue spaces appeared in [20], where the existence and uniqueness of the non-homogeneous

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N-harmonic equations div $(|\nabla u|^{N-2}\nabla u) = \mu$ were studied. The boundedness of the maximal operator on the grand Lebesgue spaces was studied in [14]. The boundedness of the maximal operator and Sobolev's inequality for grand Morrey spaces with doubling measure were also studied in [32]. See also [15] and [31], etc..

Our first aim in this paper is to establish the boundedness of the maximal operator on grand Morrey spaces of variable exponents over non-doubling measure spaces. As an application of the boundedness of the maximal operator by use of Hedberg's trick [25], we shall give Sobolev type inequalities for Riesz potentials of functions in these spaces.

A famous Trudinger inequality ([45]) insists that Sobolev functions in $W^{1,N}(G)$ satisfy finite exponential integrability, where G is an open bounded set in $\mathbf{R}^N$ (see also [2] and [46]). Great progress on Trudinger type inequalities has been made for Riesz potentials of order α $(0 < \alpha < N)$ in the limiting case $\alpha p = N$ (see e.g. [10], [11], [12], [13], [42]). Trudinger type exponential integrability was investigated on variable exponent Lebesgue spaces $L^{p(\cdot)}$ in [16], [17] and [18] and on variable exponent Morrey spaces in [35]. For related results, see e.g. [33], [34] and [36].

Our second aim in this paper is to establish Trudinger's type exponential integrability for Riesz potentials of functions in grand Morrey spaces of variable exponents over non-doubling measure spaces. Further, in the final section, we are concerned with the continuity for Riesz potentials in our setting.

2 Preliminaries

By a quasi-metric measure space, we mean a triple (X, ρ, μ), where X is a set, ρ is a quasi-metric on X and μ is a complete measure on X. Here, we say that ρ is a quasi-metric on X if ρ satisfies the following conditions:

(ρ1) $\rho(x, y) \ge 0$ and $\rho(x, y) = 0$ if and only if $x = y$;

(ρ2) there exists a constant $a_0 \ge 1$ such that $\rho(x, y) \le a_0\rho(y, x)$ for all $x, y \in X$;

(ρ3) there exists a constant $a_1 > 0$ such that $\rho(x, y) \le a_1(\rho(x, z) + \rho(z, y))$ for all $x, y, z \in X$.

We denote $B(x, r) = \{y \in X : \rho(x, y) < r\}$ and $d_X = \sup\{\rho(x, y) : x, y \in X\}$. In this paper, we assume that $0 < d_X < \infty$ and $0 < \mu(B(x, r)) < \infty$ for all $x \in X$ and $r > 0$. This implies $\mu(X) < \infty$.

We say that a measure μ is lower Ahlfors q-regular if there exists a constant $c_0 > 0$ such that

$$\mu(B(x, r)) \ge c_0 r^q \tag{2.1}$$

for all $x \in X$ and $0 < r < d_X$. Further, μ is said to be a doubling measure if there exists a constant $c_1 > 0$ such that $\mu(B(x, 2r)) \le c_1\mu(B(x, r))$ for every $x \in X$ and $0 < r < d_X$. By the doubling property, if $0 < r \le R < d_X$, then there exist constants $C_Q > 0$ and $Q \ge 0$ such that

$$\frac{\mu(B(x, r))}{\mu(B(x, R))} \ge C_Q \left(\frac{r}{R}\right)^Q \tag{2.2}$$

for all $x \in X$ (see e.g. [26]).

For $\alpha > 0, k \ge 1$ and a locally integrable function f on X, we define the Riesz potential $U_{\alpha,k}f$ of order α by

$$U_{\alpha,k}f(x) = \int_X \frac{\rho(x,y)^\alpha}{\mu(B(x,k\rho(x,y)))} f(y)\, d\mu(y).$$

Let $p(\cdot)$ be a measurable function on X such that

(P1) $1 < p^- := \inf_{x\in X} p(x) \le \sup_{x\in X} p(x) =: p^+ < \infty$

and

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{c_p}{\log(e + 1/\rho(x,y))} \quad \text{for } x, y \in X$$

with a constant $c_p \ge 0$. Here note from (ρ2) that

(P2')

$$|p(x) - p(y)| \le \frac{c'_p}{\log(e + 1/\rho(y,x))} \quad \text{for } x, y \in X$$

with a constant $c'_p \ge 0$.

For a locally integrable function f on X, set

$$\|f\|_{L^{p(\cdot)}(X)} = \inf\left\{\lambda > 0 : \int_X \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} d\mu(y) \le 1\right\}.$$

For $0 < \varepsilon < p^- - 1$, set

$$p_\varepsilon(x) = p(x) - \varepsilon.$$

For $\nu > 0, \theta > 0$ and $k \ge 1$, we denote by $L^{p(\cdot)-0,\nu,\theta;k}(X)$ the class of locally integrable functions f on X satisfying

$$\|f\|_{L^{p(\cdot)-0,\nu,\theta;k}(X)} = \sup_{x\in X, 0<r<d_X, 0<\varepsilon<p^--1} \varepsilon^\theta \left(\frac{r^\nu}{\mu(B(x,kr))}\right)^{1/p_\varepsilon(x)} \|f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} < \infty.$$

Throughout this paper, let C denote various constants independent of the variables in question. $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 0$.

LEMMA 2.1. *Let $k \ge 1$. If μ is lower Ahlfors q-reqular, then*

$$\mu(B(x,kr))^{p_\varepsilon(y)} \sim \mu(B(x,kr))^{p_\varepsilon(x)}$$

whenever $y \in B(x,r)$.

Proof. Since $p_\varepsilon(\cdot)$ satisfies the condition (P2), we see from (2.1) that

$$\begin{aligned}\left(\frac{\mu(B(x,kr))}{\mu(X)}\right)^{-|p_\varepsilon(x)-p_\varepsilon(y)|} &\le \exp\left(\frac{c_p}{\log(e+1/\rho(x,y))}\log\frac{\mu(X)}{\mu(B(x,kr))}\right)\\ &\le \exp\left(\frac{c_p}{\log(e+1/r)}\log\frac{\mu(X)}{c_0(kr)^q}\right) \le C\end{aligned}$$

whenever $y \in B(x,r)$. Hence, we obtain the required result. □

LEMMA 2.2. *Let $k \geq 1$. If μ is lower Ahlfors q-regular and $0 < \varepsilon_0 < p^- - 1$, then*

$$\sup_{x \in X, 0<r<d_X, 0<\varepsilon<\varepsilon_0} \varepsilon^\theta \left(\frac{r^\nu}{\mu(B(x,kr))} \right)^{1/p_\varepsilon(x)} \|f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \sim \|f\|_{L^{p(\cdot)-0,\nu,\theta;k}(X)}$$

for all $f \in L^1_{loc}(X)$.

Proof. We may assume that

$$\sup_{x \in X, 0<r<d_X, 0<\varepsilon<\varepsilon_0} \varepsilon^\theta \left(\frac{r^\nu}{\mu(B(x,kr))} \right)^{1/p_\varepsilon(x)} \|f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \leq 1.$$

Then note from Lemma 2.1 that

$$\frac{1}{\mu(B(x,kr))} \int_{B(x,r)} f(y)^{p_{\varepsilon_0/2}(y)}\, d\mu(y) \leq C r^{-\nu}$$

for all $x \in X$ and $0 < r < d_X$. To end the proof, it is sufficient to show that there exists a constant $C > 0$ such that

$$\frac{1}{\mu(B(x,kr))} \int_{B(x,r)} f(y)^{p_{\varepsilon_1}(y)}\, d\mu(y) \leq C r^{-\nu}$$

for all $\varepsilon_0 \leq \varepsilon_1 < p^- - 1$. For this, we see that

$$\begin{aligned} &\frac{1}{\mu(B(x,kr))} \int_{B(x,r)} f(y)^{p_{\varepsilon_1}(y)}\, d\mu(y) \\ &\leq 1 + \frac{1}{\mu(B(x,kr))} \int_{B(x,r)} f(y)^{p_{\varepsilon_0/2}(y)}\, d\mu(y) \leq C r^{-\nu}. \end{aligned}$$

Thus the required result is proved. □

LEMMA 2.3. *If μ is lower Ahlfors q-regular, then*

$$\|1\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \sim \mu(B(x,r))^{1/p_\varepsilon(x)}$$

for all $x \in X, 0 < r < d_X$ and $0 < \varepsilon < p^- - 1$.

Proof. By Lemma 2.1, we have

$$\int_{B(x,r)} \left(\frac{1}{\mu(B(x,r))^{1/p_\varepsilon(x)}} \right)^{p_\varepsilon(y)} d\mu(y) \sim 1$$

for all $x \in X, 0 < r < d_X$ and $0 < \varepsilon < p^- - 1$, as required. □

3 Boundedness of the maximal operator

From now on, we assume that μ is lower Ahlfors q-regular. For a locally integrable functions f on X, we consider the maximal function $M_2 f$ defined by

$$M_2 f(x) = \sup_{r>0} \frac{1}{\mu(B(x,2r))} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

We first show the boundedness of the maximal operator on grand Morrey spaces of variable exponents over non-doubling measure spaces, as an extension of Meskhi [32, Theorem 3.1].

Let j_0 be the smallest integer satisfying $2^{j_0} > a_1$.

THEOREM 3.1. *The maximal operator : $f \to M_2 f$ is bounded from $L^{p(\cdot)-0,\nu,\theta;2}(X)$ to $L^{p(\cdot)-0,\nu,\theta;2^{j_0+1}}(X)$, that is,*

$$\|M_2 f\|_{L^{p(\cdot)-0,\nu,\theta;2^{j_0+1}}(X)} \le C\|f\|_{L^{p(\cdot)-0,\nu,\theta;2}(X)} \qquad \text{for all } f \in L^{p(\cdot)-0,\nu,\theta;2}(X).$$

To show Theorem 3.1, we need the following results.

LEMMA 3.2. *Let $k \ge 1$. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta;k}(X)} \le 1$. Then there exists a constant $C > 0$ such that*

$$\frac{1}{\mu(B(x,kr))} \int_{B(x,r)} g(y)\,d\mu(y) \le C r^{-\nu/p_\varepsilon(x)}$$

for all $x \in X, 0 < r < d_X$ and $0 < \varepsilon < p^- - 1$, where $g(y) = \varepsilon^\theta f(y)$.

Proof. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta;k}(X)} \le 1$. Then note that

$$\frac{1}{\mu(B(x,kr))} \int_{B(x,r)} g(y)^{p_\varepsilon(y)}\,d\mu(y) \le C r^{-\nu}$$

for all $x \in X, 0 < r < d_X$ and $0 < \varepsilon < p^- - 1$. Hence, we find

$$\begin{aligned}
&\frac{1}{\mu(B(x,kr))} \int_{B(x,r)} g(y)\,d\mu(y) \\
&\le r^{-\nu/p_\varepsilon(x)} + \frac{1}{\mu(B(x,kr))} \int_{B(x,r)} g(y) \left(\frac{g(y)}{r^{-\nu/p_\varepsilon(x)}}\right)^{p_\varepsilon(y)-1} d\mu(y) \\
&\le r^{-\nu/p_\varepsilon(x)} + C r^{\nu(p_\varepsilon(x)-1)/p_\varepsilon(x)} \frac{1}{\mu(B(x,kr))} \int_{B(x,r)} g(y)^{p_\varepsilon(y)}\,d\mu(y) \\
&\le C r^{-\nu/p_\varepsilon(x)},
\end{aligned}$$

as required. □

We denote by χ_E the characteristic function of E.

LEMMA 3.3. *Let $j \ge j_0$. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta;2}(X)} \le 1$. Set $g_j(y) = \varepsilon^\theta f(y)\chi_{B(x,2^{j+1}r)\setminus B(x,2^j r)}(y)$ for $0 < \varepsilon < p^- - 1$. Then there exists a constant $C > 0$ such that*

$$M_2 g_j(z) \le C 2^{-\nu j/p^+} r^{-\nu/p_\varepsilon(x)}$$

for all $z \in B(x,r)$ and $0 < \varepsilon < p^- - 1$.

Proof. Let $z \in B(x,r)$. Noting that $g_j(y) = 0$ for $y \in B(z,(2^j/a_1 - 1)r)$, we have by Lemma 3.2 and (P2)

$$\begin{aligned}
M_2 g_j(z) &= \sup_{t > (2^j/a_1 - 1)r} \frac{1}{\mu(B(z,2t))} \int_{B(z,t)} g_j(y)\,d\mu(y) \\
&\le C \sup_{t > (2^j/a_1 - 1)r} t^{-\nu/p_\varepsilon(z)} \\
&\le C 2^{-\nu j/p^+} r^{-\nu/p_\varepsilon(x)},
\end{aligned}$$

as required. □

Lemma 3.4 (cf. [30, Theorem 3.1]). *Suppose that* $p_0(\cdot)$ *is a function on* X *such that*

$$1 < p_0^- := \inf_{x\in X} p_0(x) \le \sup_{x\in X} p_0(x) =: p_0^+ < \infty$$

and

$$|p_0(x) - p_0(y)| \le \frac{c_{p_0}}{\log(e + 1/\rho(x,y))}$$

for all $x, y \in X$ *and some constant* $c_{p_0} \ge 0$. *Then there exists a constant* $c_0 > 0$ *depending only on* p_0^-, p_0^+, c_{p_0} *and* $\mu(X)$ *such that*

$$\|M_2 f\|_{L^{p_0(\cdot)}(X)} \le c_0 \|f\|_{L^{p_0(\cdot)}(X)}$$

for all $f \in L^{p_0(\cdot)}(X)$.

Proof of Theorem 3.1. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta;2}(X)} \le 1$. Let $x \in X, 0 < r < d_X$ and $0 < \varepsilon < (p^- - 1)/2$ be fixed. Set $g(y) = \varepsilon^\theta f(y)$.

For positive integers $j \ge j_0$, set

$$g_j = g\chi_{B(x,2^{j+1}r)\setminus B(x,2^j r)}(y)$$

and $g_0 = g\chi_{B(x,2^{j_0}r)}(y)$.

Here, we find by Lemmas 3.3 and 2.3

$$\begin{aligned}\|M_2 g_j\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} &\le C2^{-\nu j/p^+} r^{-\nu/p_\varepsilon(x)} \|1\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \\ &\le C2^{-\nu j/p^+} r^{-\nu/p_\varepsilon(x)} \mu(B(x,r))^{1/p_\varepsilon(x)}\end{aligned}$$

for $j \ge j_0$. Since $p_\varepsilon^- > (p^- + 1)/2 > 1$, we see from Lemma 3.4 that

$$\begin{aligned}&\|M_2 g\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \\ &\le \|M_2 g_0\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} + \sum_{j=j_0}^{\infty} \|M_2 g_j\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \\ &\le C\left\{ \|g_0\|_{L^{p_\varepsilon(\cdot)}(B(x,2^{j_0}r))} + \mu(B(x,r))^{1/p_\varepsilon(x)} r^{-\nu/p_\varepsilon(x)} \sum_{j=j_0}^{\infty} 2^{-\nu j/p^+} \right\} \\ &\le C\left\{ \mu(B(x,2^{j_0+1}r))^{1/p_\varepsilon(x)} (2^{j_0}r)^{-\nu/p_\varepsilon(x)} + \mu(B(x,r))^{1/p_\varepsilon(x)} r^{-\nu/p_\varepsilon(x)} \right\} \\ &\le C\mu(B(x,2^{j_0+1}r))^{1/p_\varepsilon(x)} r^{-\nu/p_\varepsilon(x)},\end{aligned}$$

so that

$$\sup_{x\in X, 0<r<d_X, 0<\varepsilon<(p^- -1)/2} \varepsilon^\theta \left(\frac{r^\nu}{\mu(B(x,2^{j_0+1}r))} \right)^{1/p_\varepsilon(x)} \|M_2 f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} \le C.$$

Hence, we obtain the required result by Lemma 2.2. □

4 Sobolev's inequality

Now we show the Sobolev type inequality for Riesz potentials in grand Morrey spaces of variable exponents over non-doubling measure spaces, as an extension of Meskhi [32, Theorems 5.3 and 5.4].

THEOREM 4.1. *Suppose* $1/p^*(x) = 1/p(x) - \alpha/\nu \geq 1/p^+ - \alpha/\nu > 0$. *Then there exists a constant* $C > 0$ *such that*

$$\|U_{\alpha,4} f\|_{L^{p^*(\cdot)-0,\nu,\theta;2^{j_0+1}}(X)} \leq C\|f\|_{L^{p(\cdot)-0,\nu,\theta;2}(X)}.$$

Proof. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\nu,\theta;2}(X)} \leq 1$. Let $x \in X, 0 < r < d_X$ and $0 < \varepsilon < \min\{p^- - 1, ((p^*)^- - 1)/\gamma\}$ be fixed, where

$$\gamma = \sup_{z \in X, 0<\varepsilon<p^- -1} (p_\varepsilon)^*(z)p^*(z)/(p_\varepsilon(z)p(z)).$$

For $z \in B(x, r)$ and $\delta > 0$, we write

$$\begin{aligned} U_{\alpha,4} f(z) &= \int_{B(z,\delta)} \frac{\rho(z, y)^\alpha}{\mu(B(z, 4\rho(z, y)))} f(y)\, d\mu(y) + \int_{X\setminus B(z,\delta)} \frac{\rho(z, y)^\alpha}{\mu(B(z, 4\rho(z, y)))} f(y)\, d\mu(y) \\ &= U_1(z) + U_2(z). \end{aligned}$$

First we have

$$\begin{aligned} U_1(z) &= \sum_{j=1}^{\infty} \int_{B(z,2^{-j+1}\delta)\setminus B(z,2^{-j}\delta)} \frac{\rho(z, y)^\alpha}{\mu(B(z, 4\rho(z, y)))} f(y)\, d\mu(y) \\ &\leq \sum_{j=1}^{\infty} \int_{B(z,2^{-j+1}\delta)} \frac{(2^{-j+1}\delta)^\alpha}{\mu(B(z, 2^{-j+2}\delta))} f(y)\, d\mu(y) \\ &\leq \sum_{j=1}^{\infty} (2^{-j+1}\delta)^\alpha M_2 f(z) \\ &\leq C\delta^\alpha M_2 f(z). \end{aligned}$$

To estimate U_2, set $g(y) = \varepsilon^\theta f(y)$. Then we have by Lemma 3.2

$$\begin{aligned} \varepsilon^\theta U_2(z) &= \sum_{j=1}^{\infty} \int_{X\cap(B(z,2^j\delta)\setminus B(z,2^{j-1}\delta))} \frac{\rho(z, y)^\alpha}{\mu(B(z, 4\rho(z, y)))} g(y)\, d\mu(y) \\ &\leq C \sum_{j=1}^{\infty} (2^j\delta)^\alpha \frac{1}{\mu(B(z, 2^{j+1}\delta))} \int_{B(z,2^j\delta)} g(y)\, d\mu(y) \\ &\leq C \sum_{j=1}^{\infty} (2^j\delta)^{\alpha-\nu/p_\varepsilon(z)} \\ &\leq C\delta^{\alpha-\nu/p_\varepsilon(z)}. \end{aligned}$$

Hence

$$U_{\alpha,4} g(z) \leq C\left\{\delta^\alpha M_2 g(z) + \delta^{\alpha-\nu/p_\varepsilon(z)}\right\}.$$

Here, letting $\delta = M_2 g(z)^{-p_\varepsilon(z)/\nu}$, we establish

$$U_{\alpha,4} g(z) \le C M_2 g(z)^{1-\alpha p_\varepsilon(z)/\nu}.$$

Now Theorem 3.1 gives

$$\begin{aligned}
&\frac{1}{\mu(B(x,2^{j_0+1}r))} \int_{B(x,r)} \{\varepsilon^\theta U_{\alpha,4} f(z)\}^{(p_\varepsilon)^*(z)}\, d\mu(z) \\
&\le \frac{C}{\mu(B(x,2^{j_0+1}r))} \int_{B(x,r)} \{M_2 g(z)\}^{p_\varepsilon(z)}\, d\mu(z) \\
&\le C r^{-\nu}.
\end{aligned}$$

Here one sees that

$$(p_\varepsilon)^*(z) = p^*(z) - \frac{(p_\varepsilon)^*(z) p^*(z)}{p_\varepsilon(z) p(z)} \varepsilon.$$

Setting $\tilde{\varepsilon} = \gamma\varepsilon$, we have

$$\begin{aligned}
&\frac{1}{\mu(B(x,2^{j_0+1}r))} \int_{B(x,r)} \{\tilde{\varepsilon}^\theta U_{\alpha,4} f(z)\}^{(p^*)_{\tilde{\varepsilon}}(z)}\, d\mu(z) \\
&\le C \left[\frac{1}{\mu(B(x,2^{j_0+1}r))} \int_{B(x,r)} \{\varepsilon^\theta U_{\alpha,4} f(z)\}^{(p_\varepsilon)^*(z)}\, d\mu(z) + 1 \right] \\
&\le C r^{-\nu}
\end{aligned}$$

for all $x \in X, 0 < r < d_X$ and $0 < \varepsilon < \min\{p^- - 1, ((p^*)^- - 1)/\gamma\}$, so that we obtain the required result by Lemma 2.2. □

5 Exponential integrability

In this section, we assume that

$$\operatorname*{ess\,sup}_{x \in X} (1/p(x) - \alpha/\nu) \le 0. \tag{5.1}$$

Our aim in this section is to give an exponential integrability of Trudinger type. Recall that j_0 is the smallest integer satisfying $2^{j_0} > a_1$, where $a_1 > 0$ is the constant in (ρ3). Set

$$k_0 = \max\{2a_0 a_1(a_0+1), a_1^2(a_0 + 2^{j_0+1})/(2^{j_0} - a_1), 2\},$$

where $a_0 \ge 1$ is the constant in (ρ2).

Theorem 5.1. *Let $0 < \eta < \alpha$. Suppose that (5.1) holds. Then there exist constants $c_1, c_2 > 0$ such that*

$$\frac{1}{\mu(B(z,2^{j_0}r))} \int_{B(z,r)} \exp\left(c_1 U_{\alpha,k_0} f(x)^{1/(\theta+1)}\right)\, d\mu(x) \le c_2 r^{\eta-\alpha}$$

for all $z \in X$ and $0 < r < d_X$, whenever f is a nonnegative measurable function on X satisfying $\|f\|_{L^{p(\cdot)-0,\nu,\theta;1}(X)} \le 1$.

To prove the theorem, we prepare some lemmas.

LEMMA 5.2. *Let $k \geq 2, \theta > 0$ and $0 < \eta < \alpha$. Let f be a nonnegative function on X such that there exists a constant $C > 0$ such that*

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\, d\mu(y) \leq C r^{-\alpha} (\log(e + 1/r))^{\theta}. \tag{5.2}$$

Then there exists a constant $C > 0$ such that

$$\int_{X \setminus B(x,\delta)} \frac{\rho(x,y)^{\eta}}{\mu(B(x, k\rho(x,y)))} f(y)\, d\mu(y) \leq C \delta^{\eta-\alpha} (\log(e + 1/\delta))^{\theta}$$

for $x \in X$ and $\delta > 0$.

Proof. Let f be a nonnegative function on X satisfying (5.2). We choose the smallest integer j_1 such that $2^{j_1}\delta \geq d_X$. We have by (5.2)

$$\begin{aligned}
&\int_{X \setminus B(x,\delta)} \frac{\rho(x,y)^{\eta}}{\mu(B(x, k\rho(x,y)))} f(y)\, d\mu(y) \\
&= \sum_{j=1}^{j_1} \int_{B(x,2^j\delta) \setminus B(x,2^{j-1}\delta)} \frac{\rho(x,y)^{\eta}}{\mu(B(x, k\rho(x,y)))} f(y)\, d\mu(y) \\
&\leq \sum_{j=1}^{j_1} (2^j\delta)^{\eta} \frac{1}{\mu(B(x, 2^{j-1}k\delta))} \int_{B(x,2^j\delta)} f(y)\, d\mu(y) \\
&\leq C \sum_{j=1}^{j_1} (2^j\delta)^{\eta-\alpha} (\log(e + 1/(2^j\delta)))^{\theta} \\
&\leq C \sum_{j=1}^{j_1} \int_{2^{j-1}\delta}^{2^j\delta} t^{\eta-\alpha} (\log(e + 1/t))^{\theta} \frac{dt}{t} \\
&\leq C \int_{\delta}^{2d_X} t^{\eta-\alpha} (\log(e + 1/t))^{\theta} \frac{dt}{t}.
\end{aligned} \tag{5.3}$$

Hence we find by $\eta < \alpha$

$$\int_{X \setminus B(x,\delta)} \frac{\rho(x,y)^{\eta}}{\mu(B(x, k\rho(x,y)))} f(y)\, d\mu(y) \leq C \delta^{\eta-\alpha} (\log(e + 1/\delta))^{\theta},$$

as required. □

LEMMA 5.3. *Let $0 < \eta < \alpha$. Let f be a nonnegative function on X satisfying (5.2). Define*

$$I_{\eta} f(x) = \int_X \frac{\rho(x,y)^{\eta}}{\mu(B(x, k_0\rho(x,y)))} f(y)\, d\mu(y).$$

Then there exists a constant $C > 0$ such that

$$\frac{1}{\mu(B(z, 2^{j_0} r))} \int_{B(z,r)} I_{\eta} f(x)\, d\mu(x) \leq C r^{\eta-\alpha} (\log(e + 1/r))^{\theta}$$

for all $z \in X$ and $0 < r < d_X$.

Proof. Write

$$\begin{aligned} & I_\eta f(x) \\ = & \int_{B(z,2^{j_0}r)} \frac{\rho(x,y)^\eta}{\mu(B(x,k_0\rho(x,y)))} f(y)\,d\mu(y) + \int_{X\setminus B(z,2^{j_0}r)} \frac{\rho(x,y)^\eta}{\mu(B(x,k_0\rho(x,y)))} f(y)\,d\mu(y) \\ = & I_1(x) + I_2(x). \end{aligned}$$

Let $a = a_1(2^{j_0}a_0 + 1)$. By Fubini's theorem, we have

$$\begin{aligned} & \int_{B(z,r)} I_1(x)\,d\mu(x) \\ = & \int_{B(z,2^{j_0}r)} \left(\int_{B(z,r)} \frac{\rho(x,y)^\eta}{\mu(B(x,k_0\rho(x,y)))}\,d\mu(x) \right) f(y)\,d\mu(y) \\ \le & \int_{B(z,2^{j_0}r)} \left(\int_{B(y,ar)} \frac{\rho(x,y)^\eta}{\mu(B(x,k_0\rho(x,y)))}\,d\mu(x) \right) f(y)\,d\mu(y) \\ = & \int_{B(z,2^{j_0}r)} \left(\sum_{j=0}^\infty \int_{B(y,2^{-j}ar)\setminus B(y,2^{-j-1}ar)} \frac{\rho(x,y)^\eta}{\mu(B(x,k_0\rho(x,y)))}\,d\mu(x) \right) f(y)\,d\mu(y) \\ \le & \int_{B(z,2^{j_0}r)} \left(\sum_{j=0}^\infty \int_{B(y,2^{-j}ar)\setminus B(y,2^{-j-1}ar)} \frac{(2^{-j}a_0ar)^\eta}{\mu(B(x,2^{-j-1}a_0^{-1}k_0ar))}\,d\mu(x) \right) f(y)\,d\mu(y) \\ \le & \int_{B(z,2^{j_0}r)} \left(\sum_{j=0}^\infty \int_{B(y,2^{-j}ar)\setminus B(y,2^{-j-1}ar)} \frac{(2^{-j}a_0ar)^\eta}{\mu(B(y,2^{-j}ar))}\,d\mu(x) \right) f(y)\,d\mu(y) \\ \le & \int_{B(z,2^{j_0}r)} \left(\sum_{j=0}^\infty (2^{-j}a_0ar)^\eta \right) f(y)\,d\mu(y), \end{aligned}$$

since $B(y,2^{-j}ar) \subset B(x,2^{-j-1}a_0^{-1}k_0ar)$ by the fact that $k_0 \ge 2a_0a_1(a_0+1)$. Using $\eta > 0$ and (5.2), we have

$$\begin{aligned} \int_{B(z,r)} I_1(x)\,d\mu(x) & \le C\int_{B(z,2^{j_0}r)} \left(\sum_{j=1}^\infty (2^{-j}r)^\eta \right) f(y)\,d\mu(y) \\ & \le Cr^\eta \int_{B(z,2^{j_0}r)} f(y)\,d\mu(y) \\ & \le Cr^\eta \mu(B(z,2^{j_0}r))(2^{j_0}r)^{-\alpha}(\log(e+1/(2^{j_0}r)))^\theta \\ & \le Cr^{\eta-\alpha}(\log(e+1/r))^\theta \mu(B(z,2^{j_0}r)). \end{aligned}$$

For $x \in B(z,r)$ and $y \in X \setminus B(z,2^{j_0}r)$, we obtain

$$\rho(z,y) \le \frac{a_1 2^{j_0}}{2^{j_0}-a_1}\rho(x,y). \tag{5.4}$$

and

$$\rho(x,z) \le \frac{a_0a_1}{2^{j_0}-a_1}\rho(x,y) \tag{5.5}$$

Indeed, we have

$$\rho(z,y) \le a_1(\rho(z,x)+\rho(x,y)) \le a_1(r+\rho(x,y)) \le a_1(2^{-j_0}\rho(z,y)+\rho(x,y)),$$

which yields (5.4). Also we have

$$\rho(x,z) \le a_0\rho(z,x) \le a_0 r \le a_0 2^{-j_0}\rho(z,y),$$

which implies (5.5) by (5.4). In view of (5.4), (5.5) and the fact that $k_0 \ge a_1^2(a_0 + 2^{j_0+1})/(2^{j_0} - a_1)$, we have $B(z, 2\rho(z,y)) \subset B(x, k_0\rho(x,y))$. Further, we note

$$\rho(x,y) \le a_1\left(a_0 2^{-j_0} + 1\right)\rho(z,y)$$

for $x \in B(z,r)$ and $y \in X \setminus B(z, 2^{j_0}r)$. Therefore, we obtain

$$\begin{aligned} I_2(x) &\le C\int_{X\setminus B(z,2^{j_0}r)} \frac{\rho(z,y)^\eta}{\mu(B(x,k_0\rho(x,y)))} f(y)\,d\mu(y) \\ &\le C\int_{X\setminus B(z,2^{j_0}r)} \frac{\rho(z,y)^\eta}{\mu(B(z,2\rho(z,y)))} f(y)\,d\mu(y) \end{aligned}$$

for $x \in B(z,r)$. Hence we have by Lemma 5.2

$$I_2(x) \le Cr^{\eta-\alpha}(\log(e+1/r))^\theta.$$

Thus this lemma is proved. □

Proof of Theorem 5.1. Let f be a nonnegative measurable function on X satisfying $\|f\|_{L^{p(\cdot)-0,\nu,\theta;1}(X)} \le 1$. Set $g(y) = \varepsilon^\theta f(y)$. Then we have by Lemma 3.2 and (5.1)

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} g(y)\,d\mu(y) \le Cr^{-\nu/p_\varepsilon(x)} \le Cr^{-\alpha p(x)/p_\varepsilon(x)}.$$

Here we take $\varepsilon = (p^- - 1)(\log(e+1/r))^{-1}$ and obtain

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)\,d\mu(y) \le Cr^{-\alpha}(\log(e+1/r))^\theta \tag{5.6}$$

for all $x \in X$ and $0 < r < d_X$, which is nothing but (5.2). For $x \in B(z,r)$, $\delta > 0$ and $0 < \eta < \alpha$, we find

$$\begin{aligned} & U_{\alpha,k_0} f(x) \\ = & \int_{B(x,\delta)} \frac{\rho(x,y)^\alpha}{\mu(B(x,k_0\rho(x,y)))} f(y)\,d\mu(y) + \int_{X\setminus B(x,\delta)} \frac{\rho(x,y)^\alpha}{\mu(B(x,k_0\rho(x,y)))} f(y)\,d\mu(y) \\ \le & \ \delta^{\alpha-\eta} I_\eta f(x) + U_2(x). \end{aligned}$$

As in the proof of (5.3), it follows that

$$U_2(x) \le C\int_\delta^{2d_X} (\log(e+1/t))^\theta \frac{dt}{t} \le C(\log(e+1/\delta))^{\theta+1},$$

which gives

$$U_{\alpha,k_0} f(x) \le C\left\{\delta^{\alpha-\eta} I_\eta f(x) + (\log(e+1/\delta))^{\theta+1}\right\}.$$

Here, letting ,

$$\delta = \{I_\eta f(x)\}^{-1/(\alpha-\eta)}(\log(e+I_\eta f(x)))^{(\theta+1)/(\alpha-\eta)},$$

we have the inequality

$$U_{\alpha,k_0} f(x) \le C(\log(e + I_\eta f(x)))^{\theta+1}.$$

Then, in view of Lemma 5.3, there exist constants $c_1, c_3 > 0$ such that

$$\begin{aligned}
&\frac{1}{\mu(B(z,2^{j_0}r))} \int_{B(z,r)} \exp\left(c_1 U_{\alpha,k_0} f(x)^{1/(\theta+1)}\right) d\mu(x) \\
&\le C \left\{ \frac{1}{\mu(B(z,2^{j_0}r))} \int_{B(z,r)} I_\eta f(x)\, d\mu(x) + 1 \right\} \\
&\le c_3 r^{\eta-\alpha} (\log(e + 1/r))^\theta
\end{aligned}$$

for all $z \in X$ and $0 < r < d_X$. Since $c_3 r^{\eta-\alpha}(\log(e+1/r))^\theta \le c_2 r^{\eta'-\alpha}$ for all $0 < r < d_X$ and some constant $c_2 > 0$ when $0 < \eta' < \eta$, the proof of the present theorem is completed. □

6 Continuity

In this section, we assume that there exist constants $C_1 > 0$ and $0 < \sigma \le 1$ such that

$$\left| \frac{\rho(x,y)^\alpha}{\mu(B(x,2\rho(x,y)))} - \frac{\rho(z,y)^\alpha}{\mu(B(z,2\rho(z,y)))} \right| \le C_1 \left(\frac{\rho(x,z)}{\rho(x,y)} \right)^\sigma \frac{\rho(x,y)^\alpha}{\mu(B(x,2\rho(x,y)))} \quad (6.1)$$

whenever $\rho(x,z) \le \rho(x,y)/2$.

Let $\omega(\cdot)$ be a positive function on $(0,\infty)$ satisfying the doubling condition

$$\omega(2r) \le C_2 \omega(r) \qquad \text{for all } r > 0$$

and

$$\omega(s) \le C_3 \omega(t) \qquad \text{whenever } 0 < s \le t,$$

where C_2 and C_3 are positive constants. Then, in view of (2.2), one can find constants $Q > 0$ and $C_Q > 0$ such that

$$\omega(r) \ge C_Q r^Q \quad (6.2)$$

for all $0 < r < d_X$.

In this section, for $\theta > 0$, we consider the space $L^{p(\cdot)-0,\omega,\theta}(X)$ of locally integrable functions f on X satisfying

$$\|f\|_{L^{p(\cdot)-0,\omega,\theta}(X)} = \sup_{x\in X, 0<r<d_X, 0<\varepsilon<p^- -1} \varepsilon^\theta \left(\frac{\omega(r)}{\mu(B(x,r))} \right)^{1/p_\varepsilon(x)} \|f\|_{L^{p_\varepsilon(\cdot)}(B(x,r))} < \infty.$$

Set

$$\Omega_*(x,r) = \int_0^r t^\alpha \omega(t)^{-1/p(x)} (\log(e+1/t))^\theta \, \frac{dt}{t}$$

and

$$\Omega^*(x,r) = \int_r^{2d_X} t^{\alpha-\sigma} \omega(t)^{-1/p(x)} (\log(e+1/t))^\theta \, \frac{dt}{t}$$

for $x \in X$ and $0 < r < d_X$.

EXAMPLE 6.1. Let $\omega(r) = r^{\nu}(\log(e+1/r))^{\beta}$. If $p^- \ge \nu/\alpha$ and $\operatorname{ess\,sup}_{x\in X}(-\beta/p(x)+\theta+1) < 0$, then

$$\Omega_*(x,r) + r^{\sigma}\Omega^*(x,r) \le C(\log(e+1/r))^{-\beta/p(x)+\theta+1}$$

for $x \in X$ and $0 < r < d_X$.

Our final goal is to establish the following result, which deals with the continuity for Riesz potentials of functions in grand Morrey spaces of variable exponents over non-doubling measure spaces.

THEOREM 6.2. *Suppose that* (6.1) *holds. Then there exists a constant $C > 0$ such that*

$$|U_{\alpha,2}f(x) - U_{\alpha,2}f(z)| \le C\left\{\Omega_*(x,\rho(x,z)) + \Omega_*(z,\rho(x,z)) + \rho(x,z)^{\sigma}\Omega^*(x,\rho(x,z))\right\}$$

for all $x, z \in X$, whenever f is a nonnegative measurable function on X satisfying $\|f\|_{L^{p(\cdot)-0,\omega,\theta}(X)} \le 1$.

Before the proof of Theorem 6.2, we prepare some lemmas.

Since

$$\omega(r)^{-|p(x)-p(y)|} \le Cr^{-Q|p(x)-p(y)|} \le C$$

for all $y \in B(x,r)$ by (6.2) and (P2), we can show the following result in the same manner as Lemma 3.2 and (5.6).

LEMMA 6.3. *Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\omega,\theta}(X)} \le 1$. Then there exists a constant $C > 0$ such that*

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f(y)\,d\mu(y) \le C\omega(r)^{-1/p(x)}(\log(e+1/r))^{\theta}$$

for all $x \in X$ and $0 < r < d_X$.

LEMMA 6.4. *Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\omega,\theta}(X)} \le 1$. Then there exists a constant $C > 0$ such that*

$$\int_{B(x,\delta)} \frac{\rho(x,y)^{\alpha}}{\mu(B(x,2\rho(x,y)))} f(y)\,d\mu(y) \le C\Omega_*(x,\delta)$$

and

$$\int_{G\setminus B(x,\delta)} \frac{\rho(x,y)^{\alpha-\sigma}}{\mu(B(x,2\rho(x,y)))} f(y)\,d\mu(y) \le C\Omega^*(x,\delta)$$

for all $x \in X$ and $0 < \delta < d_X$.

Proof. Let f be a nonnegative function on X such that $\|f\|_{L^{p(\cdot)-0,\omega,\theta}(X)} \le 1$. We show only the first case. As in the proof of (5.3), we have by Lemma 6.3

$$
\begin{aligned}
&\int_{B(x,\delta)} \frac{\rho(x,y)^\alpha}{\mu(B(x,2\rho(x,y)))} f(y)\, d\mu(y) \\
&= \sum_{j=1}^{\infty} \int_{B(x,2^{-j+1}\delta)\setminus B(x,2^{-j}\delta)} \frac{\rho(x,y)^\alpha}{\mu(B(x,2\rho(x,y)))} f(y)\, d\mu(y) \\
&\le \sum_{j=1}^{\infty} (2^{-j+1}\delta)^\alpha \frac{1}{\mu(B(x,2^{-j+1}\delta))} \int_{B(x,2^{-j+1}\delta)} f(y)\, d\mu(y) \\
&\le C \sum_{j=1}^{\infty} (2^{-j+1}\delta)^\alpha \omega(2^{-j+1}\delta)^{-1/p(x)} (\log(e+1/(2^{-j+1}\delta)))^\theta \\
&\le C \int_0^\delta t^\alpha \omega(t)^{-1/p(x)} (\log(e+1/t))^\theta \frac{dt}{t} = C\Omega_*(x,\delta),
\end{aligned}
$$

as required. □

Proof of Theorem 6.2. Let f be a nonnegative measurable function on X satisfying $\|f\|_{L^{p(\cdot)-0,\omega,\theta}(X)} \le 1$. Write

$$
\begin{aligned}
&U_{\alpha,2}f(x) - U_{\alpha,2}f(z) \\
&= \int_{B(x,2\rho(x,z))} \frac{\rho(x,y)^\alpha}{\mu(B(x,2\rho(x,y)))} f(y)\, d\mu(y) - \int_{B(x,2\rho(x,z))} \frac{\rho(z,y)^\alpha}{\mu(B(z,2\rho(z,y)))} f(y)\, d\mu(y) \\
&\qquad + \int_{X\setminus B(x,2\rho(x,z))} \left(\frac{\rho(x,y)^\alpha}{\mu(B(x,2\rho(x,y)))} - \frac{\rho(z,y)^\alpha}{\mu(B(z,2\rho(z,y)))} \right) f(y)\, d\mu(y)
\end{aligned}
$$

for $x, z \in X$. Using Lemma 6.4, we have

$$
\int_{B(x,2\rho(x,z))} \frac{\rho(x,y)^\alpha}{\mu(B(x,2\rho(x,y)))} f(y)\, d\mu(y) \le C\Omega_*(x,2\rho(x,z)) \le C\Omega_*(x,\rho(x,z))
$$

and

$$
\begin{aligned}
\int_{B(x,2\rho(x,z))} \frac{\rho(z,y)^\alpha}{\mu(B(z,2\rho(z,y)))} f(y)\, d\mu(y) &\le \int_{B(z,a_1(a_0+2)\rho(x,z))} \frac{\rho(z,y)^\alpha}{\mu(B(z,2\rho(z,y)))} f(y)\, d\mu(y) \\
&\le C\Omega_*(z,a_1(a_0+2)\rho(x,z)) \le C\Omega_*(z,\rho(x,z)).
\end{aligned}
$$

On the other hand, by (6.1) and Lemma 6.4, we have

$$
\begin{aligned}
&\int_{X\setminus B(x,2\rho(x,z))} \left| \frac{\rho(x,y)^\alpha}{\mu(B(x,2\rho(x,y)))} - \frac{\rho(z,y)^\alpha}{\mu(B(z,2\rho(z,y)))} \right| f(y)\, d\mu(y) \\
&\le C_1 \rho(x,z)^\sigma \int_{X\setminus B(x,2\rho(x,z))} \frac{\rho(x,y)^{\alpha-\sigma}}{\mu(B(x,2\rho(x,y)))} f(y)\, d\mu(y) \\
&\le C\rho(x,z)^\sigma \Omega^*(x,2\rho(x,z)) \\
&\le C\rho(x,z)^\sigma \Omega^*(x,\rho(x,z)).
\end{aligned}
$$

Then we have the conclusion. □

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