Trudinger’s inequality and continuity for Riesz potentials of functions in grand Musielak-Orlicz-Morrey spaces over non-doubling metric measure spaces

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Abstract
In this paper we are concerned with Trudinger’s inequality and continuity for Riesz potentials of functions in grand Musielak-Orlicz-Morrey spaces over non-doubling metric measure spaces.

1 Introduction

Grand Lebesgue spaces were introduced in [9] for the study of Jacobian. They play important roles also in the theory of partial differential equations (see [5], [10] and [28], etc.). The generalized grand Lebesgue spaces appeared in [7], where the existence and uniqueness of the non-homogeneous $N$-harmonic equations were studied.

For $0 < \alpha < N$, we define the Riesz potential of order $\alpha$ for a locally integrable function $f$ on $\mathbb{R}^N$ by

$$I_\alpha f(x) = \int_{\mathbb{R}^N} |x - y|^\alpha |f(y)| dy.$$ 

The classical Trudinger’s inequality for Riesz potentials of $L^p$-functions (see, e.g. [2, Theorem 3.1.4 (c)]) has been also extended to various function spaces; see [17] and [20] for Morrey spaces of variable exponent, [6] for grand Morrey spaces of variable exponent, [24] for Musielak-Orlicz spaces and [14] for Musielak-Orlicz-Morrey spaces. See also [26] and [27]. Recently, Trudinger’s inequality has been extended to an inequality for Riesz potentials of functions in grand Musielak-Orlicz-Morrey spaces (see [15]).

We denote by $(X, d, \mu)$ a metric measure spaces, where $X$ is a bounded set, $d$ is a metric on $X$ and $\mu$ is a nonnegative complete Borel regular outer measure on $X$ which is finite in every bounded set. For simplicity, we often write $X$ instead of

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For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at $x$ with radius $r$ and $d_X = \sup \{d(x, y) : x, y \in X\}$. We assume that $0 < d_X < \infty$,
\[ \mu(\{x\}) = 0 \]
for $x \in X$ and $\mu(B(x, r)) > 0$ for $x \in X$ and $r > 0$ for simplicity. In the present paper, we do not postulate on $\mu$ the “so called” doubling condition. Recall that a Radon measure $\mu$ is said to be doubling if there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \leq C \mu(B(x, r))$ for all $x \in \text{supp}(\mu) (= X)$ and $r > 0$. Otherwise $\mu$ is said to be non-doubling.

For $\alpha > 0$ and $\tau > 0$, we define the Riesz potential of order $\alpha$ for a locally integrable function $f$ on $X$ by
\[ I_{\alpha, \tau} f(x) = \int_X \frac{d(x, y)^\alpha f(y)}{\mu(B(x, \tau d(x, y)))} d\mu(y) \]
(e.g. see [8] and [22]). Observe that this naturally extends the Riesz potential operator $I_{\alpha, f}(x)$ when $(X, d)$ is the $N$-dimensional Euclidean space and $\mu = dx$.

Our first aim in this paper is to give a general version of Trudinger type exponential integrability for Riesz potentials $I_{\alpha, \tau} f$ of functions in grand Musielak-Orlicz-Morrey spaces $\widetilde{L}_{n, \xi}^\Phi(X)$ over non-doubling metric measure spaces $X$ (see e.g., Corollary 5.5) as an extension of [15, Corollary 6.12] (see Sections 2 and 3 for the definitions of $\phi$, $\kappa$, $\eta$, $\xi$ and $\widetilde{L}_{n, \xi}^\Phi(X)$). Since we discuss the Morrey version, our strategy is to find an estimate of Riesz potentials $I_{\alpha, \tau} f$ by use of another Riesz-type potentials $I_{\gamma, \tau} f$ of order $\gamma (< \alpha)$, which plays a role of the maximal functions (see Section 4). What is new about this paper is that we can pass our results to the non-doubling metric measure setting; the technique developed in [14] still works.

On the other hand, beginning with Sobolev’s embedding theorem (see e.g. [1], [2]), continuity properties of Riesz potentials or Sobolev functions have been studied by many authors. See [18] and [19] for generalized Morrey spaces $L^{1, \phi}$, [21] for Orlicz-Morrey spaces, [21] for variable exponent Morrey spaces and [17] for two variable exponent Morrey spaces.

Our second aim in this paper is to give a general version of continuity for Riesz potentials $I_{\alpha, \tau} f$ of functions in grand Musielak-Orlicz-Morrey spaces over non-doubling metric measure spaces (see e.g., Corollary 6.6), whose counterpart in the Euclidean setting was not considered in [15]. The result is new even for the Euclidean case.

2 Preliminaries

Throughout this paper, let $C$ denote various constants independent of the variables in question.

In this paper, we assume that $X$ is a bounded set, that is $d_X < \infty$. This implies that $\mu(X) < \infty$.

We consider a function
\[ \Phi(x, t) = t \phi(x, t) : X \times [0, \infty) \to [0, \infty) \]
satisfying the following conditions (\Phi1) – (\Phi4):

2
\( \phi(\cdot, t) \) is measurable on \( X \) for each \( t \geq 0 \) and \( \phi(x, \cdot) \) is continuous on \([0, \infty)\) for each \( x \in X \);

(\( \Phi_2 \)) there exists a constant \( A_1 \geq 1 \) such that
\[
A_1^{-1} \leq \phi(x, 1) \leq A_1 \quad \text{for all } x \in X;
\]

(\( \Phi_3 \)) there exists a constant \( \varepsilon_0 > 0 \) such that \( t \mapsto t^{-\varepsilon_0} \phi(x, t) \) is uniformly almost increasing, namely there exists a constant \( A_2 \geq 1 \) such that
\[
t^{-\varepsilon_0} \phi(x, t) \leq A_2 s^{-\varepsilon_0} \phi(x, s)
\]
for all \( x \in X \) whenever \( 0 < t < s \);

(\( \Phi_4 \)) there exists a constant \( A_3 \geq 1 \) such that
\[
\phi(x, 2t) \leq A_3 \phi(x, t) \quad \text{for all } x \in X \text{ and } t > 0.
\]

Note that (\( \Phi_3 \)) implies that
\[
t^{-\varepsilon} \phi(x, t) \leq A_2 s^{-\varepsilon} \phi(x, s)
\]
for all \( x \in X \) and \( 0 < \varepsilon \leq \varepsilon_0 \) whenever \( 0 < t < s \).

Also note that (\( \Phi_2 \)), (\( \Phi_3 \)) and (\( \Phi_4 \)) imply
\[
0 < \inf_{x \in X} \phi(x, t) \leq \sup_{x \in X} \phi(x, t) < \infty
\]
for each \( t > 0 \) and there exists \( \omega > 1 \) such that
\[
(A_1 A_2)^{-1} t^{1+\varepsilon_0} \leq \Phi(x, t) \leq A_1 A_2 A_3 t^\omega
\]
(2.1)
for \( t \geq 1 \); in fact we can take \( \omega \geq 1 + \log A_3 / \log 2 \).

We shall also consider the following condition:

(\( \Phi_5 \)) for every \( \gamma_1, \gamma_2 > 0 \), there exists a constant \( B_{\gamma_1, \gamma_2} \geq 1 \) such that
\[
\phi(x, t) \leq B_{\gamma_1, \gamma_2} \phi(y, t)
\]
whenever \( d(x, y) \leq \gamma_1 t^{-1/\gamma_2} \) and \( t \geq 1 \).

Let \( \phi(x, t) = \sup_{0 \leq s \leq t} \phi(x, s) \) and
\[
\overline{\Phi}(x, t) = \int_0^t \phi(x, r) \, dr
\]
for \( x \in X \) and \( t \geq 0 \). Then \( \overline{\Phi}(x, \cdot) \) is convex and
\[
\frac{1}{2A_3} \Phi(x, t) \leq \overline{\Phi}(x, t) \leq A_2 \Phi(x, t)
\]
for all \( x \in X \) and \( t \geq 0 \).
Example 2.1. Let \( p(\cdot) \) and \( q_j(\cdot), j = 1, \ldots, k \) be measurable functions on \( X \) such that

\[
1 < p^- := \inf_{x \in X} p(x) \leq \sup_{x \in X} p(x) =: p^+ < \infty
\]

and

\[
-\infty < q_j^- := \inf_{x \in X} q_j(x) \leq \sup_{x \in X} q_j(x) =: q_j^+ < \infty \quad j = 1, \ldots, k.
\]

Set \( L(t) := \log(e + t), \ L^{(1)}(t) = L(t) \) and \( L^{(j)}(t) = L(L^{(j-1)}(t)), j = 2, \ldots \). Then,

\[
\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t) = t^{p(x)} \prod_{j=1}^{k} \left( L^{(j)}(t) \right)^{q_j(x)}
\]

satisfies (Φ1), (Φ2), (Φ3) with \( 0 < \varepsilon_0 < p^- - 1 \) and (Φ4). (2.1) holds for any \( \omega > p^+ \). \( \Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t) \) satisfies (Φ5) if \( p(\cdot) \) is log-Hölder continuous, namely

\[
|p(x) - p(y)| \leq \frac{C_p}{L(1/d(x,y))} \quad (x \neq y)
\]

and \( q_j(\cdot) \) is \((j+1)\)-log-Hölder continuous, namely

\[
|q_j(x) - q_j(y)| \leq \frac{C_{q_j}}{L^{(j+1)}(1/d(x,y))} \quad (x \neq y)
\]

for \( j = 1, \ldots, k \) (cf. [13, Example 2.1]).

We also consider a function \( \kappa(x,r) : X \times (0, d_X) \to (0, \infty) \) satisfying the following conditions:

(\( \kappa_1 \)) \( \kappa(x, \cdot) \) is continuous on \((0, d_X)\) for each \( x \in X \) and satisfies the uniform doubling condition: there is a constant \( Q_1 \geq 1 \) such that

\[
Q_1^{-1} \kappa(x, r) \leq \kappa(x, r') \leq Q_1 \kappa(x, r)
\]

for all \( x \in X \) whenever \( 0 < r \leq r' \leq 2r < d_X \);

(\( \kappa_2 \)) \( r \mapsto r^{-\delta} \kappa(x, r) \) is uniformly almost increasing for some \( \delta > 0 \), namely there is a constant \( Q_2 > 0 \) such that

\[
r^{-\delta} \kappa(x, r) \leq Q_2 s^{-\delta} \kappa(x, s)
\]

for all \( x \in X \) whenever \( 0 < r < s < d_X \);

(\( \kappa_3 \)) there are constants \( Q > 0 \) and \( Q_3 \geq 1 \) such that

\[
Q_3^{-1} \min(1, r^Q) \leq \kappa(x, r) \leq Q_3
\]

for all \( x \in X \) and \( 0 < r < d_X \).

Example 2.2. Let \( \nu(\cdot) \) and \( \beta(\cdot) \) be functions on \( X \) such that \( \nu^- := \inf_{x \in X} \nu(x) > 0 \), \( \nu^+ := \sup_{x \in X} \nu(x) \leq Q \) and \(-c(Q - \nu(x)) \leq \beta(x) \leq c\) for all \( x \in X \) and some constant \( c > 0 \). Then \( \kappa(x,r) = r^{\nu(x)}(\log(e + 1/r))^{\beta(x)} \) satisfies (\( \kappa_1 \)), (\( \kappa_2 \)) and (\( \kappa_3 \)); we can take any \( 0 < \delta < \nu^- \) for (\( \kappa_2 \)).
We say that \( f \) is a locally integrable function on \( X \) if \( f \) is an integrable function on all balls \( B \) in \( X \). Given \( \Phi(x,t) \) satisfying (\( \Phi 1 \)), (\( \Phi 2 \)), (\( \Phi 3 \)) and (\( \Phi 4 \)) and \( \kappa(x,r) \) satisfying (\( \kappa 1 \)), (\( \kappa 2 \)) and (\( \kappa 3 \)), we define the Musielak-Orlicz-Morrey space \( L^{\Phi,\kappa}(X) \) by

\[
L^{\Phi,\kappa}(X) = \left\{ f \in L^1_{\text{loc}}(X); \sup_{x \in X, 0 < r < d_X} \frac{\kappa(x,r)}{\mu(B(x,r))^\omega} \int_{B(x,r) \cap X} \Phi(y, |f(y)|) \, d\mu(y) < \infty \right\}.
\]

It is a Banach space with respect to the norm

\[
\| f \|_{\Phi,\kappa;X} = \inf \left\{ \lambda > 0; \sup_{x \in X, 0 < r < d_X} \frac{\kappa(x,r)}{\mu(B(x,r))^\omega} \int_{B(x,r) \cap X} \Phi(y, |f(y)|/\lambda) \, d\mu(y) \leq 1 \right\}
\]

(cf. [23]).

3 Grand Musielak-Orlicz-Morrey space

For \( \varepsilon \geq 0 \), set \( \Phi_{\varepsilon}(x,t) := t^{-\varepsilon} \Phi(x,t) = t^{1-\varepsilon} \phi(x,t) \). Then, \( \Phi_{\varepsilon}(x,t) \) satisfies (\( \Phi 1 \)), (\( \Phi 2 \)) with the same \( A_1 \) and (\( \Phi 4 \)) with the same \( A_3 \). If \( \Phi(x,t) \) satisfies (\( \Phi 5 \)), then so does \( \Phi_{\varepsilon}(x,t) \) with the same \( \{B_{\gamma_1,\gamma_2}\}_{\gamma_1,\gamma_2 > 0} \).

If \( 0 < \varepsilon < \varepsilon_0 \), then \( \Phi_{\varepsilon}(x,t) \) satisfies (\( \Phi 3 \)) with \( \varepsilon_0 \) replaced by \( \varepsilon_0 - \varepsilon \) and the same \( A_2 \). It follows that

\[
\frac{1}{2A_3} \Phi_{\varepsilon}(x,t) \leq \Phi_{\varepsilon}(x,t) \leq A_2 \Phi_{\varepsilon}(x,t)
\]

(3.1)

for all \( x \in X \), \( t \geq 0 \) and \( 0 \leq \varepsilon \leq \varepsilon_0 \).

Let

\[
\tilde{\sigma} = \sup \{ \sigma \geq 0 : r^{Q-\sigma} \kappa(x,r)^{-1} \text{ is bounded on } X \times (0, \min(1,d_X)) \}.
\]

By (\( \kappa 2 \)) and (\( \kappa 3 \)), \( 0 \leq \tilde{\sigma} \leq Q \). If \( \tilde{\sigma} = 0 \), then let \( \sigma_0 = 0 \); otherwise fix any \( \sigma_0 \in (0,\tilde{\sigma}) \). We also take \( \delta_0 \) such that \( 0 < \delta_0 < \delta \) for \( \delta \) in (\( \kappa 2 \)).

For \( -\delta_0 \leq \sigma \leq \sigma_0 \), set

\[
\kappa_\sigma(x,r) = r^\sigma \kappa(x,r)
\]

for \( x \in X \) and \( 0 < r < d_X \). Then \( \kappa_\sigma(x,r) \) satisfies (\( \kappa 1 \)), (\( \kappa 2 \)) and (\( \kappa 3 \)) with constants independent of \( \sigma \).

**Lemma 3.1** ([15, Proposition 3.2]). Assume that \( \Phi(x,t) \) satisfies (\( \Phi 5 \)). If \( 0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \varepsilon_0 \), \( -\delta_0 \leq \sigma_j \leq \sigma_0 \), \( j = 1,2 \) and

\[
\sigma_1 + \frac{\delta - \delta_0}{\omega} \varepsilon_1 \leq \sigma_2 + \frac{\delta - \delta_0}{\omega} \varepsilon_2,
\]

then \( L^{\Phi_{\varepsilon_1,\kappa_1}}(X) \subset L^{\Phi_{\varepsilon_2,\kappa_2}}(X) \) and

\[
\| f \|_{\Phi_{\varepsilon_2,\kappa_2;X}} \leq C \| f \|_{\Phi_{\varepsilon_1,\kappa_1;X}}
\]

for all \( f \in L^{\Phi_{\varepsilon_1,\kappa_1}}(X) \) with \( C > 0 \) independent of \( \varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2 \).

In particular,

\[
L^{\Phi_{\varepsilon,\kappa}}(X) \subset L^{\Phi_{\varepsilon,\kappa}}(X)
\]

if \( 0 \leq \varepsilon \leq \varepsilon_0 \), \( -\delta_0 \leq \sigma \leq \sigma_0 \) and \( \sigma + ((\delta - \delta_0)/\omega) \varepsilon \geq 0 \).
Let $\eta(\varepsilon)$ be an increasing positive function on $(0, \infty)$ such that $\eta(0+) = 0$. Let $\xi(\varepsilon)$ be a function on $(0, \varepsilon_1]$ with some $\varepsilon_1 \in (0, \varepsilon_0/2]$ such that $-\delta_0 \leq \xi(\varepsilon) \leq \sigma_0$ for $0 < \varepsilon \leq \varepsilon_1$, $\xi(0+) = 0$ and $\varepsilon \mapsto \xi(\varepsilon) + ((\delta - \delta_0)/\omega)\varepsilon$ is non-decreasing; in particular, $\xi(\varepsilon) + ((\delta - \delta_0)/\omega)\varepsilon \geq 0$ for $0 < \varepsilon \leq \varepsilon_1$.

Given $\Phi(x, t)$, $\kappa(x, r)$, $\eta(\varepsilon)$ and $\xi(\varepsilon)$, the associated (generalized) grand Musielak-Orlicz-Morrey space is defined by (cf. [11] for generalized grand Morrey space)

$$
\widetilde{L}^{\Phi, \eta, \xi}(X) = \left\{ f \in \bigcap_{0<\varepsilon \leq \varepsilon_1} L^{\Phi, \kappa, \xi(\varepsilon)}(X) : \|f\|_{\Phi, \kappa, \eta, \xi, X} < \infty \right\},
$$

where

$$
\|f\|_{\Phi, \kappa, \eta, \xi, X} = \sup_{0<\varepsilon \leq \varepsilon_1} \eta(\varepsilon) \|f\|_{\Phi, \kappa, \xi(\varepsilon), X}.
$$

$\widetilde{L}^{\Phi, \eta, \xi}(X)$ is a Banach space with the norm $\|f\|_{\Phi, \kappa, \eta, \xi, X}$. Note that, in view of Lemma 3.1, this space is determined independent of the choice of $\varepsilon_1$.

**Remark 3.2.** If $\mu(B(x, r))$ satisfies $(\kappa 1)$, $(\kappa 2)$ and $(\kappa 3)$, then the associated (generalized) grand Musielak-Orlicz-Morrey space $\widetilde{L}^{\Phi, \eta, \xi}(X)$ include the following spaces:

- generalized grand Lebesgue spaces introduced in [3] if $\kappa(x, r) = \mu(B(x, r))$ and $\xi(\varepsilon) \equiv 0$;
- grand Orlicz spaces introduced in [12] if $\kappa(x, r) = \mu(B(x, r)), \xi(\varepsilon) \equiv 0, \Phi(x, t) = \Phi(t)$ and

\[ \sup_{0<\varepsilon \leq \varepsilon_0} \eta(\varepsilon) \int_1^\infty t^{-\varepsilon-\varepsilon} \Phi(t) \frac{dt}{t} < \infty \]

(see also [4]);
- grand Morrey spaces introduced in [16] if $\xi(\varepsilon) \equiv 0$;
- grand grand Morrey spaces introduced in [25] and generalized grand Morrey spaces introduced in [11] if $\xi(\varepsilon)$ is an increasing positive function on $(0, \infty)$.

### 4 Lemmas

**Lemma 4.1 ([13, Lemma 5.1]).** Let $F(x, t)$ be a positive function on $X \times (0, \infty)$ satisfying the following conditions:

- (F1) $F(x, \cdot)$ is continuous on $(0, \infty)$ for each $x \in X$;
- (F2) there exists a constant $K_1 \geq 1$ such that

\[ K_1^{-1} \leq F(x, 1) \leq K_1 \quad \text{for all} \quad x \in X; \]
- (F3) $t \mapsto t^{-\varepsilon} F(x, t)$ is uniformly almost increasing for some $\varepsilon > 0$; namely there exists a constant $K_2 \geq 1$ such that

\[ t^{-\varepsilon} F(x, t) \leq K_2 s^{-\varepsilon} F(x, s) \quad \text{for all} \quad x \in X \quad \text{whenever} \quad 0 < t < s. \]
Set
\[ F^{-1}(x, s) = \sup \{ t > 0 \mid F(x, t) < s \} \]
for \( x \in X \) and \( s > 0 \). Then:

(1) \( F^{-1}(x, \cdot) \) is non-decreasing.

(2)
\[ F^{-1}(x, \lambda t) \leq (K_2 \lambda)^{1/\varepsilon} F^{-1}(x, t) \quad (4.1) \]
for all \( x \in X, t > 0 \) and \( \lambda \geq 1 \).

(3)
\[ F(x, F^{-1}(x, t)) = t \quad (4.2) \]
for all \( x \in X \) and \( t > 0 \).

(4)
\[ K_2^{-1/\varepsilon} t \leq F^{-1}(x, F(x, t)) \leq K_2^{2/\varepsilon} t \]
for all \( x \in X \) and \( t > 0 \).

(5)
\[ \min \left\{ 1, \left( \frac{s}{K_1 K_2} \right)^{1/\varepsilon} \right\} \leq F^{-1}(x, s) \leq \max \{ 1, (K_1 K_2 s)^{1/\varepsilon} \} \quad (4.3) \]
for all \( x \in X \) and \( s > 0 \).

**Remark 4.2.** \( F(x, t) = \Phi(x, t) \) satisfies (F1), (F2) and (F3) with \( K_1 = A_1, K_2 = A_2 \) and \( \varepsilon = 1 \).

By (κ3) and (4.3), we have the following result.

**Lemma 4.3.** There exists a constant \( C > 0 \) such that
\[ C^{-1} \leq \Phi^{-1}(x, \kappa(x, r)^{-1}) \leq Cr^{-Q} \quad (4.4) \]
for all \( x \in X \) and \( 0 < r \leq d_X \), where \( Q \) is a constant appearing in (κ3).

**Lemma 4.4 (cf. [15, Lemma 3.1]).** There exist constants \( C \geq 1 \) and \( r_0 \in (0, \min(1, d_X)) \) such that \( \kappa_\sigma(x, r) \leq Cr^{\delta - \delta_0} \) and
\[ C^{-1} r^{-(\delta - \delta_0)/\omega} \leq \Phi^{-1}_\varepsilon(x, \kappa_\sigma(x, r)^{-1}) \leq Cr^{-Q} \]
for all \( x \in X, 0 < r \leq r_0, -\delta_0 \leq \sigma \leq \sigma_0 \) and \( 0 < \varepsilon \leq \varepsilon_0 \), where \( Q \) is a constant appearing in (κ3).

**Proof.** In view of the proof of [15, Lemma 3.1], we have only to prove that there exists a constant \( C \geq 1 \) such that
\[ \Phi^{-1}_\varepsilon(x, \kappa_\sigma(x, r)^{-1}) \leq Cr^{-Q} \]
for all \( x \in X, 0 < r \leq r_0, -\delta_0 \leq \sigma \leq \sigma_0 \) and \( 0 < \varepsilon \leq \varepsilon_0 \). First note from (Φ3) that there exists a constant \( C \geq 1 \) such that
\[ t^{-\varepsilon} \Phi_\varepsilon(x, t) \leq Cs^{-\varepsilon} \Phi_\varepsilon(x, s) \]
for all $x \in X$ and $0 < \varepsilon' \leq \varepsilon_0 - \varepsilon + 1$ whenever $0 < t < s$. By Lemma 4.1(5) with $\varepsilon' = 1$ and $(\kappa 3)$, we have
\[
\Phi^{-1}_\varepsilon(x, \kappa(x, r)^{-1}) \leq C\kappa(x, r)^{-1} \leq C r^{-Q}
\]
for all $x \in X$, $0 < r \leq r_0$, $-\delta_0 \leq \sigma \leq \sigma_0$ and $0 < \varepsilon \leq \varepsilon_0$, as required.

From now on, we assume:

(\Xi) $\xi(\varepsilon) \leq a \varepsilon$ for $0 < \varepsilon \leq \varepsilon_1$ with some $a \geq 0$.

Recall that $\xi(\varepsilon) = -((\delta - \delta_0)/\omega)\varepsilon$ by assumption.

Let $\varepsilon(r) = (\log(\varepsilon + 1/r))^{-1}$

for $r > 0$ and let $r_1 \in (0, \min(1, d_X))$ such that $\varepsilon(r) \leq \varepsilon_1$ for $0 < r \leq r_1$.

**Lemma 4.5 ([15, Lemma 6.2]).** There exists a constant $C \geq 1$ such that
\[
C^{-1}\Phi^{-1}(x, \kappa(x, r)^{-1}) \leq \Phi^{-1}_\varepsilon(x, \kappa(\varepsilon(r))(x, r)^{-1}) \leq C\Phi^{-1}(x, \kappa(x, r)^{-1})
\]
for all $x \in X$ and $0 < r \leq r_1$.

**Lemma 4.6.** Assume that $\Phi(x, t)$ satisfies $(\Phi 5)$. Then there exists a constant $C > 0$ such that
\[
\frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap X} f(y) \, d\mu(y) \leq C\Phi^{-1}(x, \kappa(x, r)^{-1}) \eta \left((\log(\varepsilon + 1/r))^{-1}\right)^{-1}
\]
for all $x \in X$, $0 < r < d_X$ and nonnegative $f \in \tilde{L}_{\eta, \xi}^\kappa(X)$ with $\|f\|_{\Phi, \kappa; \eta, \xi; X} \leq 1$.

**Proof.** Let $f$ be a nonnegative function with $\|f\|_{\Phi, \kappa; \eta, \xi; X} \leq 1$. Then note from (3.1) that
\[
\frac{\kappa_\eta(\varepsilon(x, r))}{\mu(B(x, r))} \int_{B(x, r) \cap X} \Phi(y, \eta) f(y) \, d\mu(y) \leq 2A_3
\]
for $x \in X$, $0 < r < d_X$ and $0 < \varepsilon < \varepsilon_1$, so that
\[
\frac{\kappa_\eta(\varepsilon(r))(x, r)}{\mu(B(x, r))} \int_{B(x, r) \cap X} \Phi(y, \eta) f(y) \, d\mu(y) \leq 2A_3
\]
for $x \in X$ and $0 < r \leq r_1$. Let $g_r(y) = \eta(\varepsilon(r)) f(y)$ and
\[
K(x, r) = \Phi^{-1}_\varepsilon(x, \kappa(\varepsilon(r))(x, r)^{-1}).
\]
Since there exist constants $C \geq 1$ and $r_0 \in (0, \min(1, d_X))$ such that
\[
1 \leq K(x, r) \leq C r^{-Q}
\]
for all $x \in X$ and $0 < r \leq \min\{r_0, r_1\}$ by Lemma 4.4, we see from $(\Phi 5)$ and (4.2) that
\[
\Phi_\varepsilon(y, K(x, r)) \geq C\Phi_\varepsilon(x, K(x, r)) = C\kappa_\eta(\varepsilon(r))(x, r)^{-1}
\]
for all \( y \in B(x, r) \) and \( 0 < r \leq \min\{r_0, r_1\} \). Therefore, we have by (\( \Phi 3 \))

\[
\frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap X} g_r(y) \, d\mu(y)
\leq K(x, r) + \frac{A_2}{\mu(B(x, r))} \int_{B(x, r) \cap X} g_r(y) \frac{g_r(y)^{-1} \Phi_{\varepsilon(r)}(y, g_r(y))}{K(x, r)^{-1} \Phi_{\varepsilon(r)}(y, K(x, r))} \, d\mu(y)
\leq CK(x, r) \left\{ 1 + \frac{\kappa_{\xi(e(r))}(x, r)}{\mu(B(x, r))} \int_{B(x, r) \cap X} \Phi_{\varepsilon(r)}(y, g_r(y)) \, d\mu(y) \right\}
\leq CK(x, r)
\]

for \( x \in X \) and \( 0 < r \leq \min\{r_0, r_1\} \). Hence, we find by Lemma 4.5

\[
\frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap X} g_r(y) \, d\mu(y) \leq C \Phi^{-1}(x, \kappa(x, r)^{-1})
\]

for all \( x \in X \) and \( 0 < r \leq \min\{r_0, r_1\} \).

In case \( \min\{r_0, r_1\} < r < d_X \), we have by (4.4)

\[
\frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap X} f(y) \, d\mu(y) \leq C \leq C \Phi^{-1}(x, \kappa(x, r)^{-1}) \eta \left( (\log(e + 1/r))^{-1} \right)^{-1},
\]

as required. \( \square \)

Set

\[
\Gamma(x, s) = \int_{1/s}^{d_X} \rho^s \Phi^{-1}(x, \kappa(x, \rho)^{-1}) \eta \left( (\log(e + 1/\rho))^{-1} \right)^{-1} \frac{d\rho}{\rho}
\]

for \( s \geq 2/d_X \) and \( x \in X \). For \( 0 \leq s < 2/d_X \) and \( x \in X \), we set \( \Gamma(x, s) = \Gamma(x, 2/d_X)(d_X/2)^{s} \). Then note that \( \Gamma(x, \cdot) \) is strictly increasing and continuous for each \( x \in X \).

**Lemma 4.7** (cf. [14, Lemma 3.5]). There exists a positive constant \( C' \) such that

\[
\Gamma(x, 2/d_X) \geq C' \text{ for all } x \in X.
\]

**Lemma 4.8.** Assume that \( \Phi(x, t) \) satisfies (\( \Phi 5 \)). Let \( \tau > 1 \). Then there exists a constant \( C > 0 \) such that

\[
\int_{X \setminus B(x, \delta)} \frac{d(x, y)^{\alpha} f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y) \leq C \Gamma \left( x, \frac{1}{\delta} \right)
\]

for all \( x \in X \), \( 0 < \delta \leq d_X/2 \) and nonnegative \( f \in \bar{L}_{\eta, \xi}^{\Phi(x)}(X) \) with \( \|f\|_{\phi_{\kappa}, \eta, \xi, X} \leq 1 \).

**Proof.** Let \( j_0 \) be the smallest positive integer such that \( \tau^{j_0} \delta \geq d_X \). By Lemma 4.6,
we have

\[
\int_{X \setminus B(x, \delta)} \frac{d(x, y)^\alpha f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y)
\]

\[
= \sum_{j=1}^{j_0} \int_{X \cap (B(x, \tau^j \delta) \setminus B(x, \tau^j-1 \delta))} \frac{d(x, y)^\alpha f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y)
\]

\[
\leq \sum_{j=1}^{j_0} (\tau^j \delta)^\alpha \frac{1}{\mu(B(x, \tau^j \delta))} \int_{X \cap B(x, \tau^j \delta)} f(y) \, d\mu(y)
\]

\[
\leq C \left( \sum_{j=1}^{j_0-1} (\tau^j \delta)^\alpha \Phi^{-1}(x, \kappa(x, \tau^j \delta)^{-1}) \eta \left( (\log(e + 1/(\tau^j \delta)))^{-1} \right)^{-1}
\]

\[
+ d_X^\alpha \Phi^{-1}(x, \kappa(x, d_X^{-1}) \eta \left( (\log(e + 1/d_X))^{-1} \right)^{-1}
\]

where we assume that \( \sum_{j=1}^{j_0} a_j = 0 \) for \( a_j \in \mathbb{R} \). By (2.2) and (4.1), we have

\[
\int_{\tau^j \delta}^{\tau^j \delta-1} \frac{t^\alpha \Phi^{-1}(x, \kappa(x, t)^{-1}) \eta \left( (\log(e + 1/t))^{-1} \right)^{-1} dt}{t}
\]

\[
\geq (\tau^j \delta)^\alpha \Phi^{-1}(x, Q_2^{-1} \kappa(x, \tau^j \delta)^{-1}) \eta \left( (\log(e + 1/(\tau^j \delta)))^{-1} \right)^{-1} \log \tau
\]

\[
\geq \frac{(\tau^j \delta)^\alpha \log \tau}{T^2 \alpha T_2} \Phi^{-1}(x, \kappa(x, \tau^j \delta)^{-1}) \eta \left( (\log(e + 1/(\tau^j \delta)))^{-1} \right)^{-1}
\]

\[
= C (\tau^j \delta)^\alpha \log \tau \Phi^{-1}(x, \kappa(x, \tau^j \delta)^{-1}) \eta \left( (\log(e + 1/(\tau^j \delta)))^{-1} \right)^{-1}
\]

and

\[
\int_{d_X/2}^{d_X} \frac{t^\alpha \Phi^{-1}(x, \kappa(x, t)^{-1}) \eta \left( (\log(e + 1/t))^{-1} \right)^{-1} dt}{t}
\]

\[
\geq \frac{d_X^\alpha \log 2}{2^\alpha A_2 Q_2} \Phi^{-1}(x, \kappa(x, d_X)^{-1}) \eta \left( (\log(e + 1/d_X))^{-1} \right)^{-1}
\]

\[
= C d_X^\alpha \Phi^{-1}(x, \kappa(x, d_X)^{-1}) \eta \left( (\log(e + 1/d_X))^{-1} \right)^{-1}.
\]

Hence, we obtain

\[
\int_{X \setminus B(x, \delta)} \frac{d(x, y)^\alpha f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y)
\]

\[
\leq \frac{C}{\log \tau} \left( \sum_{j=1}^{j_0-1} \int_{\tau^j \delta}^{\tau^j \delta-1} t^\alpha \Phi^{-1}(x, \kappa(x, t)^{-1}) \eta \left( (\log(e + 1/t))^{-1} \right)^{-1} \frac{dt}{t}
\]

\[
+ \int_{d_X/2}^{d_X} t^\alpha \Phi^{-1}(x, \kappa(x, t)^{-1}) \eta \left( (\log(e + 1/t))^{-1} \right)^{-1} \frac{dt}{t}
\]

\[
\leq \frac{C}{\log \tau} \Gamma \left( x, \frac{1}{\delta} \right),
\]

as required.
Lemma 4.9. Assume that $\Phi(x,t)$ satisfies (Φ5). Let $\tau > 2$ and $\vartheta > 1$ such that $\tau > (\vartheta + 1)/(\vartheta - 1)$. Let $\gamma > 0$ and define

$$\lambda_\gamma(z,r) = \frac{1}{1 + \int_r^{d_x} \rho^\gamma \Phi^{-1}(z,\kappa(z,\rho)^{-1}) \eta \left( (\log(e + 1/\rho))^{-1} \right)^{-1} \frac{d\rho}{\rho}}$$

for $z \in X$ and $0 < r < d_x$. Then there exists a constant $C_{I,\gamma} > 0$ such that

$$\frac{\lambda_\gamma(z,r)}{\mu(B(z,\vartheta r))} \int_{X \cap B(z,r)} I_{\gamma,\tau} f(x) \, d\mu(x) \leq C_{I,\gamma}$$

for all $z \in X$, $0 < r < d_x$ and nonnegative $f \in \tilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \leq 1$.

Proof. Let $z \in X$ and $0 < r < d_x$. Write

$$I_{\gamma,\tau} f(x) = \int_{X \cap B(z,r)} \frac{d(x,y)^\gamma f(y)}{\mu(B(x,\tau d(x,y)))} \, d\mu(y) + \int_{X \setminus B(z,r)} \frac{d(x,y)^\gamma f(y)}{\mu(B(x,\tau d(x,y)))} \, d\mu(y)$$

for $x \in B(z,r)$. By Fubini’s theorem,

$$\int_{X \cap B(z,r)} I_1(x) \, d\mu(x)$$

$$= \int_{X \cap B(z,r)} \left( \int_{X \cap B(z,r)} \frac{d(x,y)^\gamma}{\mu(B(x,\tau d(x,y)))} \, d\mu(x) \right) f(y) \, d\mu(y)$$

$$\leq \int_{X \cap B(z,r)} \left( \int_{X \cap B(y,\tau d(x,y))} \frac{R_j^\gamma}{\mu(B(x,\tau d(x,y)))} \, d\mu(x) \right) f(y) \, d\mu(y)$$

Hence

$$\int_{X \cap B(z,r)} I_1(x) \, d\mu(x)$$

$$\leq \left( \int_{X \cap B(z,r)} \sum_{j=1}^{\infty} \left( \int_{X \cap B(y,\tau d(x,y))} \frac{R_j^\gamma}{\mu(B(x,\tau d(x,y)))} \, d\mu(x) \right) f(y) \, d\mu(y) \right)$$

$$\leq \left( \int_{X \cap B(z,r)} \sum_{j=1}^{\infty} \left( \int_{X \cap B(y,\tau R_{j+1})} \frac{R_j^\gamma}{\mu(B(y, R_j))} \, d\mu(x) \right) f(y) \, d\mu(y) \right)$$

$$\leq \left( \int_{X \cap B(z,r)} \sum_{j=1}^{\infty} R_j^\gamma \right) f(y) \, d\mu(y)$$

$$= \frac{\vartheta + 1}{\vartheta} (\tau/2)^\gamma - (\tau/2)^\gamma - 1 r^\gamma \int_{X \cap B(z,r)} f(y) \, d\mu(y),$$

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where $R_j = (\vartheta + 1)(\tau/2)^{-j+1}r$. Now, by Lemma 4.6, $(\kappa_2)$ and (4.1), we have

\[
\begin{align*}
    r^\gamma \int_{X \triangleq B(z, \varrho r)} f(y) \, d\mu(y) & \leq C r^\gamma \mu(B(z, \varrho r)) \Phi^{-1}(z, \kappa(z, \varrho r)^{-1}) \eta \left( \log(e + 1/(\varrho r))^{-1} \right)^{-1} \\
    & \leq \frac{C}{\log \vartheta} \mu(B(z, \varrho r)) \int_{r}^{\varrho r} \rho^\gamma \Phi^{-1}(z, \kappa(z, \rho)^{-1}) \eta \left( \log(e + 1/\rho) \right)^{-1} \frac{d\rho}{\rho}
\end{align*}
\]

if $0 < r \leq d_x/\vartheta$ and, by Lemma 4.6 and (4.4), we have

\[
\begin{align*}
    r^\gamma \int_{X \triangleq B(z, \varrho r)} f(y) \, d\mu(y) & = r^\gamma \int_{B(z, d_x)} f(y) \, d\mu(y) \\
    & \leq C d_x^\gamma \mu(B(z, d_x)) \Phi^{-1}(z, \kappa(z, d_x)^{-1}) \eta \left( \log(e + 1/d_x) \right)^{-1} \\
    & \leq C \mu(B(z, \varrho r))
\end{align*}
\]

if $d_x/\vartheta < r < d_x$. Therefore

\[
\int_{X \triangleq B(z, r)} I_1(x) \, d\mu(x) \leq \frac{C}{((\tau/2)^\gamma - 1) \log \vartheta} \frac{\mu(B(z, \varrho r))}{\lambda_x(z, r)}
\]

for all $0 < r < d_x$.

Set $c = (\tau(\vartheta - 1) - 1)/\vartheta > 1$. For $I_2$, first note that $I_2(x) = 0$ if $x \in X$ and $r \geq d_x/\vartheta$. Let $0 < r < d_x/\vartheta$. Let $j_0$ be the smallest positive integer such that $\vartheta e^{j_0} r \geq d_x$. Here we claim that $x \in B(z, r)$ and $y \in X \triangleq B(z, \varrho r)$ imply that

\[
d(y, z) \leq \frac{\vartheta}{\vartheta - 1} d(x, y)
\]

(4.5)

and

\[
B(z, cd(z, y)) \subset B(x, \tau d(x, y)).
\]

(4.6)

Indeed, we have $d(x, z) < r$ and $d(y, z) \geq \vartheta r$. Hence it follows that

\[
d(y, z) \leq d(x, y) + d(x, z) \leq d(x, y) + \frac{1}{\vartheta} d(y, z),
\]

which yields (4.5). Also observe that when $w \in B(z, cd(z, y))$, we have by (4.5)

\[
d(w, x) \leq d(z, x) + d(w, z) \leq \frac{1}{\vartheta} d(z, y) + cd(z, y) \leq \left( c + \frac{1}{\vartheta} \right) \frac{\vartheta}{\vartheta - 1} d(x, y) = \tau d(x, y),
\]

which yields (4.6).

Consequently it follows from (4.6) that

\[
I_2(x) \leq C \int_{X \triangleq B(z, \varrho r)} \frac{d(z, y)^\gamma f(y)}{\mu(B(z, cd(z, y)))} \, d\mu(y) \quad \text{for} \quad x \in X \triangleq B(z, r).
\]
By Lemma 4.6, we have

$$I_2(x) \leq C \sum_{j=1}^{j_0} \int_{B(z, \theta^j r) \setminus B(z, \theta^{j-1} r)} \frac{d(z, y)^\gamma}{\mu(B(z, cd(z, y)))} f(y) \, d\mu(y)$$

$$\leq C \sum_{j=1}^{j_0} (\vartheta^j r)^\gamma \frac{1}{\mu(B(z, \vartheta^j r))} \int_{X \cap B(z, \vartheta^j r)} f(y) \, d\mu(y)$$

$$\leq C \left( \sum_{j=1}^{j_0} (\vartheta^j r)^\gamma \Phi^{-1}(x, \kappa(x, \vartheta^j r)^{-1})(\log(e + 1/(\vartheta^j r)))^{-1} \right)$$

$$+ d_X \Phi^{-1}(x, \kappa(x, d_X)^{-1})(\log(e + 1/d_X))^{-1},$$

where we assume that \( \sum_{j=1}^{j_0} a_j = 0 \) for \( a_j \in R \). As in the proof of Lemma 4.8, we obtain

$$I_2(x) \leq \frac{C}{\log c} \left( \sum_{j=1}^{j_0} \int_{\theta^j r}^{\vartheta^j r} \rho^\gamma \Phi^{-1}(x, \kappa(x, \rho)^{-1})(\log(e + 1/\rho))^{-1} \, \frac{d\rho}{\rho} \right)$$

$$+ \int_{d_X/2}^{d_X} \rho^\gamma \Phi^{-1}(x, \kappa(x, \rho)^{-1})(\log(e + 1/\rho))^{-1} \, \frac{d\rho}{\rho}$$

$$\leq C \int_{r}^{d_X} \rho^\gamma \Phi^{-1}(z, \kappa(z, \rho)^{-1})(\log(e + 1/\rho))^{-1} \, \frac{d\rho}{\rho}$$

$$\leq \frac{C}{\log c} \log \lambda_{\gamma}(z, r)$$

for all \( x \in X \cap B(z, r) \). Hence

$$\int_{X \cap B(z, r)} I_2(x) \, d\mu(x) \leq \frac{C \mu(B(z, r))}{\log c} \leq \frac{C \mu(B(z, \theta r))}{\log c} \frac{\mu(B(z, \theta r))}{\lambda_{\gamma}(z, r)}.$$

Thus this lemma is proved. \( \square \)

5 Trudinger’s inequality for grand Musielak-Orlicz-Morrey spaces

In this section, we deal with the case \( \Gamma(x, t) \) satisfies the uniform log-type condition: \((\Gamma_{log})\) there exists a constant \( c_\Gamma > 0 \) such that

$$\Gamma(x, t^2) \leq c_\Gamma \Gamma(x, t)$$

for all \( x \in X \) and \( t \geq 1 \).

By \((\Gamma_{log})\), together with Lemma 4.7, we see that \( \Gamma(x, t) \) satisfies the uniform doubling condition in \( t \):

**Lemma 5.1** (cf. [14, Lemma 4.2]). Suppose \( \Gamma(x, t) \) satisfies \((\Gamma_{log})\). For every \( a > 1 \), there exists \( b > 0 \) such that \( \Gamma(x, at) \leq b \Gamma(x, t) \) for all \( x \in X \) and \( t > 0 \).
Theorem 5.2. Assume that $\Phi(x,t)$ satisfies ($\Phi 5$) and $\Gamma(x,t)$ satisfies ($\Gamma_{\log}$). For each $x \in X$, let $\gamma(x) = \sup_{s>0} \Gamma(x,s)$. Suppose $\Psi(x,t) : X \times [0, \infty) \to [0, \infty]$ satisfies the following conditions:

(P$\Psi$1) $\Psi(\cdot,t)$ is measurable on $X$ for each $t \in [0, \infty)$ and $\Psi(x, \cdot)$ is continuous on $N(0, \infty)$ for each $x \in X$;

(P$\Psi$2) there is a constant $A'_1 \geq 1$ such that $\Psi(x,t) \leq \Psi(x,A'_1s)$ for all $x \in X$ whenever $0 < t < s$;

(P$\Psi$3) $\Psi(x, \Gamma(x,t)/A'_2) \leq A'_3t$ for all $x \in X$ and $t > 0$ with constants $A'_2$, $A'_3 \geq 1$ independent of $x$.

Let $\tau > 2$ and $\vartheta > 1$ such that $\tau > (\vartheta + 1)/(\vartheta - 1)$. Then, for $0 < \gamma < \alpha$, there exists a constant $C^* > 0$ such that $I_{\alpha,\tau} f(x)/C^* < \gamma(x)$ for a.e. $x \in X$ and

$$\frac{\lambda_\gamma(z,r)}{\mu(B(z,\vartheta r))} \int_{X \cap B(z,r)} \Psi \left( x, \frac{I_{\alpha,\tau} f(x)}{C^*} \right) d\mu(x) \leq 1$$

for all $z \in X, 0 < r < d_X$ and nonnegative $f \in \tilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \leq 1$.

Proof. Let $f \geq 0$ and $\|f\|_{\Phi,\kappa;\eta,\xi;X} \leq 1$. Fix $x \in X$. For $0 < \delta \leq d_X/2$, Lemma 4.8 implies

$$I_{\alpha,\tau} f(x) \leq \int_{X \cap B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))} d\mu(y) + C \Gamma \left( x, \frac{\delta}{\delta} \right)$$

$$\leq \int_{X \cap B(x,\delta)} d(x,y)^{\alpha-\gamma} \frac{d(x,y)^\gamma f(y)}{\mu(B(x,\tau d(x,y)))} d\mu(y) + C \Gamma \left( x, \frac{\delta}{\delta} \right)$$

$$\leq C \left\{ \delta^{\alpha-\gamma} I_{\gamma,\tau} f(x) + \Gamma \left( x, \frac{\delta}{\delta} \right) \right\}$$

with constants $C > 0$ independent of $x$.

If $I_{\gamma,\tau} f(x) \leq 2/d_X$, then we take $\delta = d_X/2$. Then, by Lemma 4.7

$$I_{\alpha,\tau} f(x) \leq C \Gamma \left( x, \frac{2}{d_X} \right).$$

By Lemma 5.1, there exists $C'_1 > 0$ independent of $x$ such that

$$I_{\alpha,\tau} f(x) \leq C'_1 \Gamma \left( x, \frac{1}{2A'_3} \right)$$

if $I_{\gamma,\tau} f(x) \leq 2/d_X$. (5.1)

Next, suppose $2/d_X < I_{\gamma,\tau} f(x) < \infty$. Let $m = \sup_{s \geq 2/d_X} \Gamma(x,s)/s$. By ($\Gamma_{\log}$), $m < \infty$. Define $\delta$ by

$$\delta^{\alpha-\gamma} = \frac{(d_X/2)^{\alpha-\gamma}}{m} \Gamma(x, I_{\gamma,\tau} f(x))(I_{\gamma,\tau} f(x))^{-1}.$$ 

Since $\Gamma(x, I_{\gamma,\tau} f(x))(I_{\gamma,\tau} f(x))^{-1} \leq m$, $0 < \delta \leq d_X/2$. Then by Lemma 4.7

$$\frac{1}{\delta} \leq C \Gamma \left( x, I_{\gamma,\tau} f(x) \right)^{-1/(\alpha-\gamma)} \left( I_{\gamma,\tau} f(x) \right)^{1/(\alpha-\gamma)}$$

$$\leq C \left( x, 2/d_X \right)^{-1/(\alpha-\gamma)} \left( I_{\gamma,\tau} f(x) \right)^{1/(\alpha-\gamma)} \leq C \left( I_{\gamma,\tau} f(x) \right)^{1/(\alpha-\gamma)}.$$
Hence, using \((\Gamma)_{\log}\) and Lemma 5.1, we obtain
\[
\Gamma \left( x, \frac{1}{\delta} \right) \leq \Gamma \left( x, C(I_{\gamma,\tau}f(x) \langle a^{-\gamma} \rangle) \right) \leq C \Gamma \left( x, I_{\gamma,\tau}f(x) \right).
\]
By Lemma 5.1 again, we see that there exists a constant \(C_2 > 0\) independent of \(x\) such that
\[
I_{\alpha,\tau}f(x) \leq C_2 \Gamma \left( x, \frac{1}{2C_{I,\gamma}A_3} I_{\gamma,\tau}f(x) \right) \quad \text{if} \quad 2/d_X < I_{\gamma,\tau}f(x) < \infty,
\]
where \(C_{I,\gamma}\) is the constant given in Lemma 4.9.

Now, let \(C^* = A_1^* A_2^* \max(C_1^*, C_2^*)\). Then, by (5.1) and (5.2),
\[
\frac{I_{\alpha,\tau}f(x)}{C^*} \leq \frac{1}{A_1^* A_2^*} \max \left\{ \Gamma \left( x, \frac{1}{2A_3^*} I_{\gamma,\tau}f(x) \right) \right\}
\]
whenever \(I_{\gamma,\tau}f(x) < \infty\). Since \(I_{\gamma,\tau}f(x) < \infty\) for a.e. \(x \in X\) by Lemma 4.9, \(I_{\alpha,\tau}f(x)/C^* < \gamma(x)\) a.e. \(x \in X\), and by (\(\Psi2\)) and (\(\Psi3\)), we have
\[
\Psi \left( x, \frac{I_{\alpha,\tau}f(x)}{C^*} \right) \\
\leq \max \left\{ \Psi \left( x, \frac{1}{2A_3^*} I_{\gamma,\tau}f(x) \right), \Psi \left( x, \frac{1}{2C_{I,\gamma} A_3^*} I_{\gamma,\tau}f(x) \right) \right\} \\
\leq \frac{1}{2} + \frac{1}{2C_{I,\gamma}} I_{\gamma,\tau}f(x)
\]
for a.e. \(x \in X\). Thus, noting that \(\lambda_\gamma(z, r) \leq 1\) and using Lemma 4.9, we have
\[
\frac{\lambda_\gamma(z, r)}{\mu(B(z, \delta r))} \int_{X \cap B(z, r)} \Psi \left( x, \frac{I_{\alpha,\tau}f(x)}{C^*} \right) d\mu(x) \\
\leq \frac{1}{2} \lambda_\gamma(z, r) + \frac{1}{2C_{I,\gamma} \mu(B(z, \delta r))} \int_{X \cap B(z, r)} I_{\gamma,\tau}f(x) d\mu(x) \\
\leq \frac{1}{2} + \frac{1}{2} = 1
\]
for all \(z \in X\) and \(0 < r < d_X\). \(\square\)

**Remark 5.3.** If \((\Gamma(x, s)\) is bounded, that is,
\[
\sup_{x \in X} \int_0^{d_X} \rho^s \Phi^{-1}(x, \kappa(x, \rho)^{-1}) \eta \left( (\log(e + 1/\rho))^{-1} \right)^{-1} d\rho < \infty,
\]
then by Lemma 4.8 we see that \(I_{\alpha,\tau}|f|\) is bounded for every \(f \in \tilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)\).

**Remark 5.4.** We can not take \(\gamma = \alpha\) in Theorem 5.2. For details, see [18, Remark 2.8].

As in the proof of [14, Corollary 4.6], we obtain the following corollary applying Theorem 5.2 to special \(\Phi\) and \(\kappa\) given in Examples 2.1 and 2.2.
Corollary 5.5. Let $\kappa$ be as in Example 2.2 and let $p(x)$ and $q(x) = q_1(x)$ be as in Examples 2.1. Let $\tau > 2$ and $\vartheta > 1$ such that $\tau > (\vartheta + 1)/(\vartheta - 1)$. Set $\eta(t) = t^\theta$ for $\theta > 0$ and $\Phi(x, t) = t^{p(x)}(\log(e + t))^{q(x)}$.

Assume that
\[ \alpha - \nu(x)/p(x) = 0 \quad \text{for all } x \in X. \]

(1) Suppose
\[ \inf_{x \in X} (-q(x)/p(x) - \beta(x)/p(x) + \theta + 1) > 0. \]
Then for $0 < \gamma < \alpha$ there exist constants $C^* > 0$ and $C^{**} > 0$ such that
\[
\frac{\nu(z)/p(x) - \gamma}{\mu(B(z, \vartheta r))} \int_{B(z, r) \cap X} \exp \left( \left( \frac{I_{\alpha, \tau f(x)}}{C^*} \right)^{p(x)/(p(x) + \theta p(x) - \beta(x) - q(x))} \right) \, d\mu(x) \leq C^{**}
\]
for all $z \in X$, $0 < r \leq d_X$ and nonnegative $f \in \tilde{L}_{\eta, \xi}(X)$ with $\|f\|_{\Phi, \kappa; \eta, \xi; X} \leq 1$.

(2) If
\[ \sup_{x \in X} (-q(x)/p(x) - \beta(x)/p(x) + \theta + 1) \leq 0, \]
then for $0 < \gamma < \alpha$ there exist constants $C^* > 0$ and $C^{**} > 0$ such that
\[
\frac{\nu(z)/p(x) - \gamma}{\mu(B(z, \vartheta r))} \int_{B(z, r) \cap X} \exp \left( \exp \left( \frac{I_{\alpha, \tau f(x)}}{C^*} \right) \right) \, d\mu(x) \leq C^{**}
\]
for all $z \in X$, $0 < r < d_X$ and nonnegative $f \in \tilde{L}_{\eta, \xi}(X)$ with $\|f\|_{\Phi, \kappa; \eta, \xi; X} \leq 1$.

6 Continuity for grand Musielak-Orlicz-Morrey spaces

In this section, we discuss the continuity of Riesz potentials $I_{\alpha, \tau} f$ of functions in grand Musielak-Orlicz-Morrey spaces under the condition: there are constants $\theta > 0$, $\iota > 1$ and $C_0 > 0$ such that
\[
\left| \frac{d(x, y)^{\alpha}}{\mu(B(x, \tau d(x, y)))} - \frac{d(z, y)^{\alpha}}{\mu(B(z, \tau d(z, y)))} \right| \leq C_0 \left( \frac{d(x, z)}{d(x, y)} \right)^{\theta} \frac{d(x, y)^{\alpha}}{\mu(B(x, d(x, y)))} \tag{6.1}
\]
whenever $d(x, z) \leq d(x, y)/2$.

We consider the functions
\[
\omega(x, r) = \int_0^r \rho^{\alpha} \Phi^{-1}(x, \kappa(x, \rho)^{-1}) \eta \left( (\log(e + 1/\rho))^{-1} \right)^{-1} \frac{d\rho}{\rho}
\]
and
\[
\omega_{\theta}(x, r) = r^{\theta} \int_r^{d \cdot x} \rho^{\alpha - \theta} \Phi^{-1}(x, \kappa(x, \rho)^{-1}) \eta \left( (\log(e + 1/\rho))^{-1} \right)^{-1} \frac{d\rho}{\rho}
\]
for $\theta > 0$ and $0 < r \leq d_X$. 

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Lemma 6.1 (cf. [14, Lemma 5.1]). Let \( E \subset X \). If \( \omega(x, r) \to 0 \) as \( r \to 0^+ \) uniformly in \( x \in E \), then \( \omega_\theta(x, r) \to 0 \) as \( r \to 0^+ \) uniformly in \( x \in E \).

Lemma 6.2 (cf. [14, Lemma 5.2]). There exists a constant \( C > 0 \) such that
\[
\omega(x, 2r) \leq C \omega(x, r)
\]
for all \( x \in X \) and \( 0 < r \leq d_X/2 \).

Theorem 6.3. Assume that \( \Phi(x, t) \) satisfies (\( \Phi_5 \)). Let \( \tau > 1 \). Then there exists a constant \( C > 0 \) such that
\[
|I_{\alpha, \tau} f(x) - I_{\alpha, \tau} f(z)| \leq C \{ \omega(x, d(x, z)) + \omega(z, d(x, z)) + \omega_\theta(x, d(x, z)) \}
\]
for all \( x, z \in X \) with \( d(x, z) \leq d_X/4 \) and nonnegative \( f \in \tilde{L}^{\Phi, r}_\eta(X) \) with \( \| f \|_{\Phi, \kappa, \eta, \xi, X} \leq 1 \).

Before giving a proof of Theorem 6.3, we prepare two more lemmas.

Lemma 6.4. Assume that \( \Phi(x, t) \) satisfies (\( \Phi_5 \)). Let \( \tau > 1 \) and let \( f \) be a nonnegative function on \( X \) such that \( \| f \|_{\Phi, \kappa, \eta, \xi, X} \leq 1 \). Then there exists a constant \( C > 0 \) such that
\[
\int_{X \cap B(x, \delta)} \frac{d(x, y)^\alpha f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y) \leq C \omega(x, \delta)
\]
for all \( x \in X \) and \( 0 < \delta \leq d_X \).

Proof. Let \( f \) be a nonnegative function on \( X \) with \( \| f \|_{\Phi, \kappa, \eta, \xi, X} \leq 1 \). As usual we start by decomposing \( B(x, \delta) \) dyadically:
\[
\int_{X \cap B(x, \delta)} \frac{d(x, y)^\alpha f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y) = \sum_{j=1}^{\infty} \int_{X \cap (B(x, \tau^{j+1}\delta) \setminus B(x, \tau^{-j}\delta))} \frac{d(x, y)^\alpha f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y) \leq \sum_{j=1}^{\infty} (\tau^{-j+1}\delta)^\alpha \frac{1}{\mu(B(x, \tau^{-j+1}\delta))} \int_{B(x, \tau^{-j+1}\delta)} f(y) \, d\mu(y).
\]
By Lemma 4.6, we have
\[
\int_{X \cap B(x, \delta)} \frac{d(x, y)^\alpha f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y) \leq \left( \frac{\sum_{j=1}^{\infty} (\tau^{-j+1}\delta)^\alpha \Phi^{-1}(x, \kappa(x, \tau^{-j+1}\delta)^{-1}) \eta \left( (\log(e + 1/(\tau^{-j+1}\delta)))^{-1} \right)^{-1} \right) \frac{C}{\log \tau} \int_{0}^{\delta} \rho^\alpha \Phi^{-1}(x, \kappa(x, \rho)^{-1}) \eta \left( (\log(e + 1/\rho))^{-1} \right)^{-1} \frac{d\rho}{\rho} = C \omega(x, \delta).
\]

\( \square \)
The following lemma can be proved on the same manner as Lemma 4.8.

**Lemma 6.5.** Assume that \( \Phi(x, t) \) satisfies (\( \Phi_5 \)). Let \( \theta \in \mathbb{R} \) and let \( \tau > 1 \). Let \( f \) be a nonnegative function on \( X \) such that \( \| f \|_{\Phi, \kappa; \eta; \xi; X} \leq 1 \). Then there exists a constant \( C > 0 \) such that

\[
\int_{X \setminus B(x, \delta)} \frac{d(x, y)^{\alpha - \theta} f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y) \leq C \delta^{-\theta} \omega_0(x, \delta)
\]

for all \( x \in X \) and \( 0 < \delta \leq d_X/2 \).

**Proof of Theorem 6.3.** Let \( f \) be a nonnegative function on \( X \) with \( \| f \|_{\Phi, \kappa; \eta; \xi; X} \leq 1 \) and let \( x, z \in X \) with \( d(x, z) \leq d_X/4 \). Write

\[
I_{\alpha, \tau} f(x) = \int_{X \cap B(x, 2d(x, z))} \frac{d(x, y)^{\alpha} f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y) - \int_{X \cap B(x, 2d(x, z))} \frac{d(z, y)^{\alpha} f(y)}{\mu(B(z, \tau d(z, y)))} \, d\mu(y)
\]

Using Lemmas 6.2 and 6.4, we have

\[
\int_{X \cap B(x, 2d(x, z))} \frac{d(x, y)^{\alpha} f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y) \leq C \omega(x, 2d(x, z)) \leq C \omega(x, d(x, z))
\]

and

\[
\int_{X \cap B(x, 2d(x, z))} \frac{d(z, y)^{\alpha} f(y)}{\mu(B(z, \tau d(z, y)))} \, d\mu(y) \leq \int_{X \cap B(x, 3d(x, z))} \frac{d(z, y)^{\alpha} f(y)}{\mu(B(z, \tau d(z, y)))} \, d\mu(y) \leq C \omega(z, 3d(x, z)) \leq C \omega(z, d(x, z)).
\]

On the other hand, by (6.1) and Lemma 6.5, we have

\[
\int_{X \setminus B(x, 2d(x, z))} \left| \frac{d(x, y)^{\alpha}}{\mu(B(x, \tau d(x, y)))} - \frac{d(z, y)^{\alpha}}{\mu(B(z, \tau d(z, y)))} \right| f(y) \, d\mu(y) \leq C d(x, z)^{\theta} \int_{X \setminus B(x, 2d(x, z))} \frac{d(x, y)^{\alpha - \theta} f(y)}{\mu(B(x, \tau d(x, y)))} \, d\mu(y) \leq C \omega_0(x, 2d(x, z)) \leq C \omega_0(x, d(x, z)).
\]

Then we have the conclusion. \( \square \)

In view of Lemma 6.1, we obtain the following corollary.

**Corollary 6.6.** Assume that \( \Phi(x, t) \) satisfies (\( \Phi_5 \)). Let \( \tau > 1 \).

(a) Let \( x_0 \in X \) and suppose \( \omega(x, r) \rightarrow 0 \) as \( r \rightarrow 0^+ \) uniformly in \( x \in X \cap B(x_0, \delta) \) for some \( \delta > 0 \). Then \( I_{\alpha, \tau} f \) is continuous at \( x_0 \) for every \( f \in \tilde{L}_{\Phi, \kappa}^{\Phi, \kappa}(X) \).

(b) Suppose \( \omega(x, r) \rightarrow 0 \) as \( r \rightarrow 0^+ \) uniformly in \( x \in X \). Then \( I_{\alpha, \tau} f \) is uniformly continuous on \( X \) for every \( f \in \tilde{L}_{\Phi, \kappa}^{\Phi, \kappa}(X) \).
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