Trudinger's inequality and continuity for Riesz potentials of functions in grand Musielak-Orlicz-Morrey spaces over non-doubling metric measure spaces

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Abstract

In this paper we are concerned with Trudinger's inequality and continuity for Riesz potentials of functions in grand Musielak-Orlicz-Morrey spaces over non-doubling metric measure spaces.

1 Introduction

Grand Lebesgue spaces were introduced in [9] for the study of Jacobian. They play important roles also in the theory of partial differential equations (see [5], [10] and [28], etc.). The generalized grand Lebesgue spaces appeared in [7], where the existence and uniqueness of the non-homogeneous N-harmonic equations were studied.

For $0 < \alpha < N$, we define the Riesz potential of order α for a locally integrable function f on $\mathbf{R}^N$ by

$$I_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\, dy.$$

The classical Trudinger's inequality for Riesz potentials of L^p-functions (see, e.g. [2, Theorem 3.1.4 (c)]) has been also extended to various function spaces; see [17] and [20] for Morrey spaces of variable exponent, [6] for grand Morrey spaces of variable exponent, [24] for Musielak-Orlicz spaces and [14] for Musielak-Orlicz-Morrey spaces. See also [26] and [27]. Recently, Trudinger's inequality has been extended to an inequality for Riesz potentials of functions in grand Musielak-Orlicz-Morrey spaces (see [15]).

We denote by (X, d, μ) a metric measure spaces, where X is a bounded set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of

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(X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x with radius r and $d_X = \sup\{d(x, y) : x, y \in X\}$. We assume that $0 < d_X < \infty$,

$$\mu(\{x\}) = 0$$

for $x \in X$ and $\mu(B(x, r)) > 0$ for $x \in X$ and $r > 0$ for simplicity. In the present paper, we do not postulate on μ the "so called" doubling condition. Recall that a Radon measure μ is said to be doubling if there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \le C\mu(B(x, r))$ for all $x \in \mathrm{supp}(\mu)(= X)$ and $r > 0$. Otherwise μ is said to be non-doubling.

For $\alpha > 0$ and $\tau > 0$, we define the Riesz potential of order α for a locally integrable function f on X by

$$I_{\alpha,\tau} f(x) = \int_X \frac{d(x, y)^\alpha f(y)}{\mu(B(x, \tau d(x, y)))}\, d\mu(y)$$

(e.g. see [8] and [22]). Observe that this naturally extends the Riesz potential operator $I_\alpha f(x)$ when (X, d) is the N-dimensional Euclidean space and $\mu = dx$.

Our first aim in this paper is to give a general version of Trudinger type exponential integrability for Riesz potentials $I_{\alpha,\tau} f$ of functions in grand Musielak-Orlicz-Morrey spaces $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ over non-doubling metric measure spaces X (see e.g., Corollary 5.5) as an extension of [15, Corollary 6.12] (see Sections 2 and 3 for the definitions of Φ, κ, η, ξ and $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$). Since we discuss the Morrey version, our strategy is to find an estimate of Riesz potentials $I_{\alpha,\tau} f$ by use of another Riesz-type potentials $I_{\gamma,\tau} f$ of order γ $(< \alpha)$, which plays a role of the maximal functions (see Section 4). What is new about this paper is that we can pass our results to the non-doubling metric measure setting; the technique developed in [14] still works.

On the other hand, beginning with Sobolev's embedding theorem (see e.g. [1], [2]), continuity properties of Riesz potentials or Sobolev functions have been studied by many authors. See [18] and [19] for generalized Morrey spaces $L^{1,\varphi}$, [21] for Orlicz-Morrey spaces, [21] for variable exponent Morrey spaces and [17] for two variable exponent Morrey spaces.

Our second aim in this paper is to give a general version of continuity for Riesz potentials $I_{\alpha,\tau} f$ of functions in grand Musielak-Orlicz-Morrey spaces over non-doubling metric measure spaces (see e.g., Corollary 6.6), whose counterpart in the Euclidean setting was not considered in [15]. The result is new even for the Euclidean case.

2 Preliminaries

Throughout this paper, let C denote various constants independent of the variables in question.

In this paper, we assume that X is a bounded set, that is $d_X < \infty$. This implies that $\mu(X) < \infty$.

We consider a function

$$\Phi(x, t) = t\phi(x, t) : X \times [0, \infty) \to [0, \infty)$$

satisfying the following conditions $(\Phi1)$ – $(\Phi4)$:

(Φ1) $\phi(\,\cdot\,,t)$ is measurable on X for each $t \ge 0$ and $\phi(x,\,\cdot\,)$ is continuous on $[0,\infty)$ for each $x \in X$;

(Φ2) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \phi(x,1) \le A_1 \quad \text{for all } x \in X;$$

(Φ3) there exists a constant $\varepsilon_0 > 0$ such that $t \mapsto t^{-\varepsilon_0}\phi(x,t)$ is uniformly almost increasing, namely there exists a constant $A_2 \ge 1$ such that

$$t^{-\varepsilon_0}\phi(x,t) \le A_2 s^{-\varepsilon_0}\phi(x,s)$$

for all $x \in X$ whenever $0 < t < s$;

(Φ4) there exists a constant $A_3 \ge 1$ such that

$$\phi(x,2t) \le A_3\phi(x,t) \quad \text{for all } x \in X \text{ and } t > 0.$$

Note that (Φ3) implies that

$$t^{-\varepsilon}\phi(x,t) \le A_2 s^{-\varepsilon}\phi(x,s)$$

for all $x \in X$ and $0 < \varepsilon \le \varepsilon_0$ whenever $0 < t < s$.

Also note that (Φ2), (Φ3) and (Φ4) imply

$$0 < \inf_{x\in X} \phi(x,t) \le \sup_{x\in X} \phi(x,t) < \infty$$

for each $t > 0$ and there exists $\omega > 1$ such that

$$(A_1A_2)^{-1}t^{1+\varepsilon_0} \le \Phi(x,t) \le A_1A_2A_3t^{\omega} \tag{2.1}$$

for $t \ge 1$; in fact we can take $\omega \ge 1 + \log A_3/\log 2$.

We shall also consider the following condition:

(Φ5) for every $\gamma_1, \gamma_2 > 0$, there exists a constant $B_{\gamma_1,\gamma_2} \ge 1$ such that

$$\phi(x,t) \le B_{\gamma_1,\gamma_2}\phi(y,t)$$

whenever $d(x,y) \le \gamma_1 t^{-1/\gamma_2}$ and $t \ge 1$.

Let $\bar{\phi}(x,t) = \sup_{0\le s\le t}\phi(x,s)$ and

$$\overline{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)\,dr$$

for $x \in X$ and $t \ge 0$. Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \le \overline{\Phi}(x,t) \le A_2\Phi(x,t)$$

for all $x \in X$ and $t \ge 0$.

EXAMPLE 2.1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1, \dots, k$ be measurable functions on X such that

$$1 < p^- := \inf_{x \in X} p(x) \le \sup_{x \in X} p(x) =: p^+ < \infty$$

and

$$-\infty < q_j^- := \inf_{x \in X} q_j(x) \le \sup_{x \in X} q_j(x) =: q_j^+ < \infty \quad j = 1, \dots k.$$

Set $L(t) := \log(e + t)$, $L^{(1)}(t) = L(t)$ and $L^{(j)}(t) = L(L^{(j-1)}(t))$, $j = 2, \dots$. Then,

$$\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t) = t^{p(x)} \prod_{j=1}^{k} \big(L^{(j)}(t)\big)^{q_j(x)}$$

satisfies (Φ1), (Φ2), (Φ3) with $0 < \varepsilon_0 < p^- - 1$ and (Φ4). (2.1) holds for any $\omega > p^+$. $\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)$ satisfies (Φ5) if $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{C_p}{L(1/d(x,y))} \quad (x \ne y)$$

and $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely

$$|q_j(x) - q_j(y)| \le \frac{C_{q_j}}{L^{(j+1)}(1/d(x,y))} \quad (x \ne y)$$

for $j = 1, \dots, k$ (cf. [13, Example 2.1]).

We also consider a function $\kappa(x,r) : X \times (0, d_X) \to (0, \infty)$ satisfying the following conditions:

(κ1) $\kappa(x, \cdot)$ is continuous on $(0, d_X)$ for each $x \in X$ and satisfies the uniform doubling condition: there is a constant $Q_1 \ge 1$ such that

$$Q_1^{-1}\kappa(x,r) \le \kappa(x,r') \le Q_1 \kappa(x,r)$$

for all $x \in X$ whenever $0 < r \le r' \le 2r < d_X$;

(κ2) $r \mapsto r^{-\delta}\kappa(x,r)$ is uniformly almost increasing for some $\delta > 0$, namely there is a constant $Q_2 > 0$ such that

$$r^{-\delta}\kappa(x,r) \le Q_2 s^{-\delta}\kappa(x,s)$$

for all $x \in X$ whenever $0 < r < s < d_X$;

(κ3) there are constants $Q > 0$ and $Q_3 \ge 1$ such that

$$Q_3^{-1} \min(1, r^Q) \le \kappa(x,r) \le Q_3$$

for all $x \in X$ and $0 < r < d_X$.

EXAMPLE 2.2. Let $\nu(\cdot)$ and $\beta(\cdot)$ be functions on X such that $\nu^- := \inf_{x \in X} \nu(x) > 0$, $\nu^+ := \sup_{x \in X} \nu(x) \le Q$ and $-c(Q - \nu(x)) \le \beta(x) \le c$ for all $x \in X$ and some constant $c > 0$. Then $\kappa(x,r) = r^{\nu(x)}(\log(e + 1/r))^{\beta(x)}$ satisfies (κ1), (κ2) and (κ3); we can take any $0 < \delta < \nu^-$ for (κ2).

We say that f is a locally integrable function on X if f is an integrable function on all balls B in X. Given $\Phi(x,t)$ satisfying (Φ1), (Φ2), (Φ3) and (Φ4) and $\kappa(x,r)$ satisfying (κ1), (κ2) and (κ3), we define the Musielak-Orlicz-Morrey space $L^{\Phi,\kappa}(X)$ by

$$L^{\Phi,\kappa}(X) = \left\{ f \in L^1_{loc}(X)\,;\ \sup_{x\in X,\,0<r<d_X} \frac{\kappa(x,r)}{\mu(B(x,r))} \int_{B(x,r)\cap X} \Phi\big(y,|f(y)|\big)\,d\mu(y) < \infty \right\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{\Phi,\kappa;X} = \inf\left\{ \lambda > 0\,;\ \sup_{x\in X,\,0<r<d_X} \frac{\kappa(x,r)}{\mu(B(x,r))} \int_{B(x,r)\cap X} \overline{\Phi}\big(y,|f(y)|/\lambda\big)\,d\mu(y) \le 1 \right\}$$

(cf. [23]).

3 Grand Musielak-Orlicz-Morrey space

For $\varepsilon \ge 0$, set $\Phi_\varepsilon(x,t) := t^{-\varepsilon}\Phi(x,t) = t^{1-\varepsilon}\phi(x,t)$. Then, $\Phi_\varepsilon(x,t)$ satisfies (Φ1), (Φ2) with the same A_1 and (Φ4) with the same A_3. If $\Phi(x,t)$ satisfies (Φ5), then so does $\Phi_\varepsilon(x,t)$ with the same $\{B_{\gamma_1,\gamma_2}\}_{\gamma_1,\gamma_2>0}$.

If $0 \le \varepsilon < \varepsilon_0$, then $\Phi_\varepsilon(x,t)$ satisfies (Φ3) with ε_0 replaced by $\varepsilon_0 - \varepsilon$ and the same A_2. It follows that

$$\frac{1}{2A_3}\Phi_\varepsilon(x,t) \le \overline{\Phi_\varepsilon}(x,t) \le A_2\Phi_\varepsilon(x,t) \tag{3.1}$$

for all $x \in X$, $t \ge 0$ and $0 \le \varepsilon \le \varepsilon_0$.

Let

$$\tilde\sigma = \sup\{\sigma \ge 0\ :\ r^{Q-\sigma}\kappa(x,r)^{-1} \text{ is bounded on } X \times (0, \min(1,d_X))\}.$$

By (κ2) and (κ3), $0 \le \tilde\sigma \le Q$. If $\tilde\sigma = 0$, then let $\sigma_0 = 0$; otherwise fix any $\sigma_0 \in (0,\tilde\sigma)$. We also take δ_0 such that $0 < \delta_0 < \delta$ for δ in (κ2).

For $-\delta_0 \le \sigma \le \sigma_0$, set

$$\kappa_\sigma(x,r) = r^\sigma\kappa(x,r)$$

for $x \in X$ and $0 < r < d_X$. Then $\kappa_\sigma(x,r)$ satisfies (κ1), (κ2) and (κ3) with constants independent of σ.

LEMMA 3.1 ([15, Proposition 3.2]). *Assume that $\Phi(x,t)$ satisfies* (Φ5). *If $0 \le \varepsilon_1 \le \varepsilon_2 \le \varepsilon_0$, $-\delta_0 \le \sigma_j \le \sigma_0$, $j = 1,2$ and*

$$\sigma_1 + \frac{\delta-\delta_0}{\omega}\varepsilon_1 \le \sigma_2 + \frac{\delta-\delta_0}{\omega}\varepsilon_2,$$

then $L^{\Phi_{\varepsilon_1},\kappa_{\sigma_1}}(X) \subset L^{\Phi_{\varepsilon_2},\kappa_{\sigma_2}}(X)$ and

$$\|f\|_{\Phi_{\varepsilon_2},\kappa_{\sigma_2};X} \le C\|f\|_{\Phi_{\varepsilon_1},\kappa_{\sigma_1};X}$$

for all $f \in L^{\Phi_{\varepsilon_1},\kappa_{\sigma_1}}(X)$ with $C > 0$ independent of ε_1, ε_2, σ_1, σ_2.

In particular,

$$L^{\Phi,\kappa}(X) \subset L^{\Phi_\varepsilon,\kappa_\sigma}(X)$$

if $0 \le \varepsilon \le \varepsilon_0$, $-\delta_0 \le \sigma \le \sigma_0$ and $\sigma + ((\delta-\delta_0)/\omega)\varepsilon \ge 0$.

Let $\eta(\varepsilon)$ be an increasing positive function on $(0,\infty)$ such that $\eta(0+)=0$. Let $\xi(\varepsilon)$ be a function on $(0,\varepsilon_1]$ with some $\varepsilon_1\in(0,\varepsilon_0/2]$ such that $-\delta_0\le\xi(\varepsilon)\le\sigma_0$ for $0<\varepsilon\le\varepsilon_1$, $\xi(0+)=0$ and $\varepsilon\mapsto\xi(\varepsilon)+((\delta-\delta_0)/\omega)\varepsilon$ is non-decreasing; in particular, $\xi(\varepsilon)+((\delta-\delta_0)/\omega)\varepsilon\ge 0$ for $0<\varepsilon\le\varepsilon_1$.

Given $\Phi(x,t)$, $\kappa(x,r)$, $\eta(\varepsilon)$ and $\xi(\varepsilon)$, the associated (generalized) grand Musielak-Orlicz-Morrey space is defined by (cf. [11] for generalized grand Morrey space)

$$\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)=\left\{f\in\bigcap_{0<\varepsilon\le\varepsilon_1}L^{\Phi_\varepsilon,\kappa_{\xi(\varepsilon)}}(X)\,;\,\|f\|_{\Phi,\kappa;\eta,\xi;X}<\infty\right\},$$

where

$$\|f\|_{\Phi,\kappa;\eta,\xi;X}=\sup_{0<\varepsilon\le\varepsilon_1}\eta(\varepsilon)\|f\|_{\Phi_\varepsilon,\kappa_{\xi(\varepsilon)};X}.$$

$\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ is a Banach space with the norm $\|f\|_{\Phi,\kappa;\eta,\xi;X}$. Note that, in view of Lemma 3.1, this space is determined independent of the choice of ε_1.

Remark 3.2. If $\mu(B(x,r))$ satisfies (κ1), (κ2) and (κ3), then the associated (generalized) grand Musielak-Orlicz-Morrey space $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ include the following spaces:

- generalized grand Lebesgue spaces introduced in [3] if $\kappa(x,r)=\mu(B(x,r))$ and $\xi(\varepsilon)\equiv 0$;
- grand Orlicz spaces introduced in [12] if $\kappa(x,r)=\mu(B(x,r))$, $\xi(\varepsilon)\equiv 0$, $\Phi(x,t)=\Phi(t)$ and
$$\sup_{0<\varepsilon\le\varepsilon_0}\eta(\varepsilon)\int_1^\infty t^{-N-\varepsilon}\Phi(t)\,\frac{dt}{t}<\infty$$
(see also [4]);
- grand Morrey spaces introduced in [16] if $\xi(\varepsilon)\equiv 0$;
- grand grand Morrey spaces introduced in [25] and generalized grand Morrey spaces introduced in [11] if $\xi(\varepsilon)$ is an increasing positive function on $(0,\infty)$.

4 Lemmas

Lemma 4.1 ([13, Lemma 5.1]). *Let $F(x,t)$ be a positive function on $X\times(0,\infty)$ satisfying the following conditions:*

(F1) *$F(x,\cdot)$ is continuous on $(0,\infty)$ for each $x\in X$;*

(F2) *there exists a constant $K_1\ge 1$ such that*

$$K_1^{-1}\le F(x,1)\le K_1\quad\text{for all } x\in X;$$

(F3) *$t\mapsto t^{-\varepsilon}F(x,t)$ is uniformly almost increasing for some $\varepsilon>0$; namely there exists a constant $K_2\ge 1$ such that*

$$t^{-\varepsilon}F(x,t)\le K_2 s^{-\varepsilon}F(x,s)\quad\text{for all } x\in X\quad\text{whenever } 0<t<s.$$

Set

$$F^{-1}(x,s) = \sup\{t > 0\,;\, F(x,t) < s\}$$

for $x \in X$ and $s > 0$. Then:

(1) *$F^{-1}(x,\cdot)$ is non-decreasing.*

(2)

$$F^{-1}(x,\lambda t) \le (K_2\lambda)^{1/\varepsilon} F^{-1}(x,t) \tag{4.1}$$

for all $x \in X$, $t > 0$ and $\lambda \ge 1$.

(3)

$$F(x, F^{-1}(x,t)) = t \tag{4.2}$$

for all $x \in X$ and $t > 0$.

(4)

$$K_2^{-1/\varepsilon} t \le F^{-1}(x, F(x,t)) \le K_2^{2/\varepsilon} t$$

for all $x \in X$ and $t > 0$.

(5)

$$\min\left\{1, \left(\frac{s}{K_1K_2}\right)^{1/\varepsilon}\right\} \le F^{-1}(x,s) \le \max\{1, (K_1K_2 s)^{1/\varepsilon}\} \tag{4.3}$$

for all $x \in X$ and $s > 0$.

Remark 4.2. $F(x,t) = \Phi(x,t)$ satisfies (F1), (F2) and (F3) with $K_1 = A_1, K_2 = A_2$ and $\varepsilon = 1$.

By (κ3) and (4.3), we have the following result.

Lemma 4.3. *There exists a constant $C > 0$ such that*

$$C^{-1} \le \Phi^{-1}(x, \kappa(x,r)^{-1}) \le Cr^{-Q} \tag{4.4}$$

for all $x \in X$ and $0 < r \le d_X$, where Q is a constant appearing in (κ3).

Lemma 4.4 (cf. [15, Lemma 3.1]). *There exist constants $C \ge 1$ and $r_0 \in (0, \min(1, d_X))$ such that $\kappa_\sigma(x,r) \le Cr^{\delta-\delta_0}$ and*

$$C^{-1} r^{-(\delta-\delta_0)/\omega} \le \Phi_\varepsilon^{-1}\big(x, \kappa_\sigma(x,r)^{-1}\big) \le Cr^{-Q}$$

for all $x \in X$, $0 < r \le r_0$, $-\delta_0 \le \sigma \le \sigma_0$ and $0 < \varepsilon \le \varepsilon_0$, where Q is a constant appearing in (κ3).

Proof. In view of the proof of [15, Lemma 3.1], we have only to prove that there exists a constant $C \ge 1$ such that

$$\Phi_\varepsilon^{-1}\big(x, \kappa_\sigma(x,r)^{-1}\big) \le Cr^{-Q}$$

for all $x \in X$, $0 < r \le r_0$, $-\delta_0 \le \sigma \le \sigma_0$ and $0 < \varepsilon \le \varepsilon_0$. First note from ($\Phi$3) that there exists a constant $C \ge 1$ such that

$$t^{-\varepsilon'} \Phi_\varepsilon(x,t) \le C s^{-\varepsilon'} \Phi_\varepsilon(x,s)$$

for all $x \in X$ and $0 < \varepsilon' \le \varepsilon_0 - \varepsilon + 1$ whenever $0 < t < s$. By Lemma 4.1(5) with $\varepsilon' = 1$ and (κ3), we have

$$\Phi_\varepsilon^{-1}\big(x, \kappa_\sigma(x,r)^{-1}\big) \le C\kappa_\sigma(x,r)^{-1} \le Cr^{-Q}$$

for all $x \in X$, $0 < r \le r_0$, $-\delta_0 \le \sigma \le \sigma_0$ and $0 < \varepsilon \le \varepsilon_0$, as required.

□

From now on, we assume:

(Ξ) $\xi(\varepsilon) \le a\varepsilon$ for $0 < \varepsilon \le \varepsilon_1$ with some $a \ge 0$.

Recall that $\xi(\varepsilon) \ge -((\delta - \delta_0)/\omega)\varepsilon$ by assumption.
Let

$$\varepsilon(r) = (\log(e + 1/r))^{-1}$$

for $r > 0$ and let $r_1 \in (0, \min(1, d_X))$ be such that $\varepsilon(r) \le \varepsilon_1$ for $0 < r \le r_1$.

Lemma 4.5 ([15, Lemma 6.2]). *There exists a constant $C \ge 1$ such that*

$$C^{-1}\Phi^{-1}(x, \kappa(x,r)^{-1}) \le \Phi^{-1}_{\varepsilon(r)}(x, \kappa_{\xi(\varepsilon(r))}(x,r)^{-1}) \le C\Phi^{-1}(x, \kappa(x,r)^{-1})$$

for all $x \in X$ and $0 < r \le r_1$.

Lemma 4.6. *Assume that $\Phi(x,t)$ satisfies (Φ5). Then there exists a constant $C > 0$ such that*

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)\cap X} f(y)\, d\mu(y) \le C\Phi^{-1}(x, \kappa(x,r)^{-1})\eta\left((\log(e+1/r))^{-1}\right)^{-1}$$

for all $x \in X$, $0 < r < d_X$ and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$.

Proof. Let f be a nonnegative function with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$. Then note from (3.1) that

$$\frac{\kappa_{\xi(\varepsilon)}(x,r)}{\mu(B(x,r))}\int_{B(x,r)\cap X} \Phi_\varepsilon\left(y, \eta(\varepsilon)f(y)\right)\, d\mu(y) \le 2A_3$$

for $x \in X, 0 < r < d_X$ and $0 < \varepsilon < \varepsilon_1$, so that

$$\frac{\kappa_{\xi(\varepsilon(r))}(x,r)}{\mu(B(x,r))}\int_{B(x,r)\cap X} \Phi_{\varepsilon(r)}\left(y, \eta(\varepsilon(r))f(y)\right)\, d\mu(y) \le 2A_3$$

for $x \in X$ and $0 < r \le r_1$. Let $g_r(y) = \eta(\varepsilon(r))f(y)$ and

$$K(x,r) = \Phi^{-1}_{\varepsilon(r)}(x, \kappa_{\xi(\varepsilon(r))}(x,r)^{-1}).$$

Since there exist constants $C \ge 1$ and $r_0 \in (0, \min(1, d_X))$ such that

$$1 \le K(x,r) \le Cr^{-Q}$$

for all $x \in X$ and $0 < r \le \min\{r_0, r_1\}$ by Lemma 4.4, we see from (Φ5) and (4.2) that

$$\Phi_{\varepsilon(r)}\left(y, K(x,r)\right) \ge C\Phi_{\varepsilon(r)}\left(x, K(x,r)\right) = C\kappa_{\xi(\varepsilon(r))}(x,r)^{-1}$$

for all $y \in B(x,r)$ and $0 < r \le \min\{r_0, r_1\}$. Therefore, we have by (Φ3)

$$
\begin{aligned}
&\frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap X} g_r(y)\, d\mu(y) \\
&\le K(x,r) + \frac{A_2}{\mu(B(x,r))} \int_{B(x,r)\cap X} g_r(y) \frac{g_r(y)^{-1}\Phi_{\varepsilon(r)}\left(y, g_r(y)\right)}{K(x,r)^{-1}\Phi_{\varepsilon(r)}\left(y, K(x,r)\right)}\, d\mu(y) \\
&\le CK(x,r) \left\{ 1 + \frac{\kappa_{\xi(\varepsilon(r))}(x,r)}{\mu(B(x,r))} \int_{B(x,r)\cap X} \Phi_{\varepsilon(r)}\left(y, g_r(y)\right)\, d\mu(y) \right\} \\
&\le CK(x,r)
\end{aligned}
$$

for $x \in X$ and $0 < r \le \min\{r_0, r_1\}$. Hence, we find by Lemma 4.5

$$
\frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap X} g_r(y)\, d\mu(y) \le C\Phi^{-1}(x, \kappa(x,r)^{-1})
$$

for all $x \in X$ and $0 < r \le \min\{r_0, r_1\}$.

In case $\min\{r_0, r_1\} < r < d_X$, we have by (4.4)

$$
\frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap X} f(y)\, d\mu(y) \le C \le C\Phi^{-1}(x, \kappa(x,r)^{-1})\eta\left((\log(e+1/r))^{-1} \right)^{-1},
$$

as required. □

Set

$$
\Gamma(x,s) = \int_{1/s}^{d_X} \rho^{\alpha}\Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big)\eta\left((\log(e+1/\rho))^{-1} \right)^{-1} \frac{d\rho}{\rho}
$$

for $s \ge 2/d_X$ and $x \in X$. For $0 \le s < 2/d_X$ and $x \in X$, we set $\Gamma(x,s) = \Gamma(x, 2/d_X)(d_X/2)s$. Then note that $\Gamma(x,\cdot)$ is strictly increasing and continuous for each $x \in X$.

LEMMA 4.7 (cf. [14, Lemma 3.5]). *There exists a positive constant C' such that $\Gamma(x, 2/d_X) \ge C'$ for all $x \in X$.*

LEMMA 4.8. *Assume that $\Phi(x,t)$ satisfies (Φ5). Let $\tau > 1$. Then there exists a constant $C > 0$ such that*

$$
\int_{X\setminus B(x,\delta)} \frac{d(x,y)^{\alpha} f(y)}{\mu(B(x, \tau d(x,y)))}\, d\mu(y) \le C\Gamma\left(x, \frac{1}{\delta}\right)
$$

for all $x \in X$, $0 < \delta \le d_X/2$ and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$.

Proof. Let j_0 be the smallest positive integer such that $\tau^{j_0}\delta \ge d_X$. By Lemma 4.6, we have

$$
\begin{aligned}
&\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\\
=\ & \sum_{j=1}^{j_0} \int_{X\cap(B(x,\tau^j\delta)\setminus B(x,\tau^{j-1}\delta))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\\
\le\ & \sum_{j=1}^{j_0} (\tau^j\delta)^\alpha \frac{1}{\mu(B(x,\tau^j\delta))} \int_{X\cap B(x,\tau^j\delta)} f(y)\,d\mu(y)\\
\le\ & C\Bigg(\sum_{j=1}^{j_0-1} (\tau^j\delta)^\alpha \Phi^{-1}(x,\kappa(x,\tau^j\delta)^{-1})\eta\left((\log(e+1/(\tau^j\delta)))^{-1}\right)^{-1}\\
& + d_X^\alpha \Phi^{-1}(x,\kappa(x,d_X)^{-1})\eta\left((\log(e+1/d_X))^{-1}\right)^{-1}\Bigg),
\end{aligned}
$$

where we assume that $\sum_{j=1}^{0} a_j = 0$ for $a_j \in \mathbf{R}$. By (κ2) and (4.1), we have

$$
\begin{aligned}
&\int_{\tau^{j-1}\delta}^{\tau^j\delta} t^\alpha \Phi^{-1}(x,\kappa(x,t)^{-1})\eta\left((\log(e+1/t))^{-1}\right)^{-1}\,\frac{dt}{t}\\
\ge\ & (\tau^{j-1}\delta)^\alpha \Phi^{-1}(x,Q_2^{-1}\kappa(x,\tau^j\delta)^{-1})\eta\left((\log(e+1/(\tau^j\delta)))^{-1}\right)^{-1}\log\tau\\
\ge\ & \frac{(\tau^j\delta)^\alpha\log\tau}{\tau^\alpha A_2 Q_2}\Phi^{-1}(x,\kappa(x,\tau^j\delta)^{-1})\eta\left((\log(e+1/(\tau^j\delta)))^{-1}\right)^{-1}\\
=\ & C(\tau^j\delta)^\alpha\log\tau\ \Phi^{-1}(x,\kappa(x,\tau^j\delta)^{-1})\eta\left((\log(e+1/(\tau^j\delta)))^{-1}\right)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{d_X/2}^{d_X} t^\alpha \Phi^{-1}(x,\kappa(x,t)^{-1})\eta\left((\log(e+1/t))^{-1}\right)^{-1}\,\frac{dt}{t}\\
\ge\ & \frac{d_X^\alpha\log 2}{2^\alpha A_2 Q_2}\Phi^{-1}(x,\kappa(x,d_X)^{-1})\eta\left((\log(e+1/d_X))^{-1}\right)^{-1}\\
=\ & C d_X^\alpha \Phi^{-1}(x,\kappa(x,d_X)^{-1})\eta\left((\log(e+1/d_X))^{-1}\right)^{-1}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\,d\mu(y)\\
\le\ & \frac{C}{\log\tau}\Bigg(\sum_{j=1}^{j_0-1}\int_{\tau^{j-1}\delta}^{\tau^j\delta} t^\alpha \Phi^{-1}(x,\kappa(x,t)^{-1})\eta\left((\log(e+1/t))^{-1}\right)^{-1}\,\frac{dt}{t}\\
& + \int_{d_X/2}^{d_X} t^\alpha \Phi^{-1}(x,\kappa(x,t)^{-1})\eta\left((\log(e+1/t))^{-1}\right)^{-1}\,\frac{dt}{t}\Bigg)\\
\le\ & \frac{C}{\log\tau}\Gamma\left(x,\frac{1}{\delta}\right),
\end{aligned}
$$

as required. □

LEMMA 4.9. *Assume that $\Phi(x,t)$ satisfies ($\Phi5$). Let $\tau > 2$ and $\vartheta > 1$ such that $\tau > (\vartheta+1)/(\vartheta-1)$. Let $\gamma > 0$ and define*

$$\lambda_\gamma(z,r) = \frac{1}{1 + \int_r^{d_X} \rho^\gamma \Phi^{-1}(z, \kappa(z,\rho)^{-1}) \eta\left((\log(e+1/\rho))^{-1} \right)^{-1} \frac{d\rho}{\rho}}$$

for $z \in X$ and $0 < r < d_X$. Then there exists a constant $C_{I,\gamma} > 0$ such that

$$\frac{\lambda_\gamma(z,r)}{\mu(B(z,\vartheta r))} \int_{X\cap B(z,r)} I_{\gamma,\tau} f(x)\, d\mu(x) \le C_{I,\gamma}$$

for all $z \in X$, $0 < r < d_X$ and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$.

Proof. Let $z \in X$ and $0 < r < d_X$. Write

$$\begin{aligned} I_{\gamma,\tau} f(x) &= \int_{X\cap B(z,\vartheta r)} \frac{d(x,y)^\gamma f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) + \int_{X\setminus B(z,\vartheta r)} \frac{d(x,y)^\gamma f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \\ &= I_1(x) + I_2(x) \end{aligned}$$

for $x \in B(z,r)$. By Fubini's theorem,

$$\begin{aligned} &\int_{X\cap B(z,r)} I_1(x)\, d\mu(x) \\ &= \int_{X\cap B(z,\vartheta r)} \left(\int_{X\cap B(z,r)} \frac{d(x,y)^\gamma}{\mu(B(x,\tau d(x,y)))}\, d\mu(x) \right) f(y)\, d\mu(y) \\ &\le \int_{X\cap B(z,\vartheta r)} \left(\int_{X\cap B(y,(\vartheta+1)r)} \frac{d(x,y)^\gamma}{\mu(B(x,\tau d(x,y)))}\, d\mu(x) \right) f(y)\, d\mu(y). \end{aligned}$$

Hence

$$\begin{aligned} &\int_{X\cap B(z,r)} I_1(x)\, d\mu(x) \\ &\le \int_{X\cap B(z,\vartheta r)} \left(\sum_{j=1}^\infty \int_{X\cap(B(y,R_j)\setminus B(y,R_{j+1}))} \frac{d(x,y)^\gamma}{\mu(B(x,\tau d(x,y)))}\, d\mu(x) \right) f(y)\, d\mu(y) \\ &\le \int_{X\cap B(z,\vartheta r)} \left(\sum_{j=1}^\infty \int_{X\cap(B(y,R_j)\setminus B(y,R_{j+1}))} \frac{R_j^\gamma}{\mu(B(x,\tau R_{j+1}))}\, d\mu(x) \right) f(y)\, d\mu(y) \\ &\le \int_{X\cap B(z,\vartheta r)} \left(\sum_{j=1}^\infty \int_{X\cap(B(y,R_j)\setminus B(y,R_{j+1}))} \frac{R_j^\gamma}{\mu(B(y,R_j))}\, d\mu(x) \right) f(y)\, d\mu(y) \\ &\le \int_{X\cap B(z,\vartheta r)} \left(\sum_{j=1}^\infty R_j^\gamma \right) f(y)\, d\mu(y) \\ &= \frac{(\vartheta+1)^\gamma (\tau/2)^\gamma}{(\tau/2)^\gamma - 1} r^\gamma \int_{X\cap B(z,\vartheta r)} f(y)\, d\mu(y), \end{aligned}$$

where $R_j = (\vartheta+1)(\tau/2)^{-j+1}r$. Now, by Lemma 4.6, ($\kappa 2$) and (4.1), we have

$$
\begin{aligned}
& r^\gamma \int_{X\cap B(z,\vartheta r)} f(y)\,d\mu(y) \\
& \le Cr^\gamma \mu(B(z,\vartheta r))\Phi^{-1}(z,\kappa(z,\vartheta r)^{-1})\eta\left((\log(e+1/(\vartheta r)))^{-1}\right)^{-1} \\
& \le \frac{C}{\log\vartheta}\mu(B(z,\vartheta r))\int_r^{\vartheta r}\rho^\gamma\Phi^{-1}(z,\kappa(z,\rho)^{-1})\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}\frac{d\rho}{\rho}
\end{aligned}
$$

if $0 < r \le d_X/\vartheta$ and, by Lemma 4.6 and (4.4), we have

$$
\begin{aligned}
r^\gamma \int_{X\cap B(z,\vartheta r)} f(y)\,d\mu(y) &= r^\gamma \int_{B(z,d_X)} f(y)\,d\mu(y) \\
&\le C{d_X}^\gamma \mu(B(z,d_X))\Phi^{-1}(z,\kappa(z,d_X)^{-1})\eta\left((\log(e+1/d_X))^{-1}\right)^{-1} \\
&\le C\mu(B(z,\vartheta r))
\end{aligned}
$$

if $d_X/\vartheta < r < d_X$. Therefore

$$
\int_{X\cap B(z,r)} I_1(x)\,d\mu(x) \le \frac{C}{((\tau/2)^\gamma-1)\log\vartheta}\frac{\mu(B(z,\vartheta r))}{\lambda_\gamma(z,r)}
$$

for all $0 < r < d_X$.

Set $c = (\tau(\vartheta-1)-1)/\vartheta > 1$. For I_2, first note that $I_2(x) = 0$ if $x \in X$ and $r \ge d_X/\vartheta$. Let $0 < r < d_X/\vartheta$. Let j_0 be the smallest positive integer such that $\vartheta c^{j_0} r \ge d_X$. Here we claim that $x \in B(z,r)$ and $y \in X \setminus B(z,\vartheta r)$ imply that

$$
d(y,z) \le \frac{\vartheta}{\vartheta-1}d(x,y) \tag{4.5}
$$

and

$$
B(z,cd(z,y)) \subset B(x,\tau d(x,y)). \tag{4.6}
$$

Indeed, we have $d(x,z) < r$ and $d(y,z) \ge \vartheta r$. Hence it follows that

$$
d(y,z) \le d(x,y) + d(x,z) \le d(x,y) + \frac{1}{\vartheta}d(y,z),
$$

which yields (4.5). Also observe that when $w \in B(z,cd(z,y))$, we have by (4.5)

$$
d(w,x) \le d(z,x)+d(w,z) \le \frac{1}{\vartheta}d(z,y)+cd(z,y) \le \left(c+\frac{1}{\vartheta}\right)\frac{\vartheta}{\vartheta-1}d(x,y) = \tau d(x,y),
$$

which yields (4.6).

Consequently it follows from (4.6) that

$$
I_2(x) \le C\int_{X\setminus B(z,\vartheta r)} \frac{d(z,y)^\gamma f(y)}{\mu(B(z,cd(z,y)))}\,d\mu(y) \quad \text{for} \quad x \in X\cap B(z,r).
$$

By Lemma 4.6, we have

$$
\begin{aligned}
I_2(x) &\le C\sum_{j=1}^{j_0}\int_{B(z,\vartheta c^j r)\setminus B(z,\vartheta c^{j-1} r)} \frac{d(z,y)^\gamma}{\mu(B(z,cd(z,y)))} f(y)\,d\mu(y)\\
&\le C\sum_{j=1}^{j_0}(\vartheta c^j r)^\gamma \frac{1}{\mu(B(z,\vartheta c^j r))}\int_{X\cap B(z,\vartheta c^j r)} f(y)\,d\mu(y)\\
&\le C\Bigg(\sum_{j=1}^{j_0-1}(\vartheta c^j r)^\gamma \Phi^{-1}(x,\kappa(x,\vartheta c^j r)^{-1})\eta\left((\log(e+1/(\vartheta c^j r)))^{-1}\right)^{-1}\\
&\quad + d_X^\gamma \Phi^{-1}(x,\kappa(x,d_X)^{-1})\eta\left((\log(e+1/d_X))^{-1}\right)^{-1}\Bigg),
\end{aligned}
$$

where we assume that $\sum_{j=1}^{0} a_j = 0$ for $a_j \in \mathbf{R}$. As in the proof of Lemma 4.8, we obtain

$$
\begin{aligned}
I_2(x) &\le \frac{C}{\log c}\Bigg(\sum_{j=1}^{j_0-1}\int_{\vartheta c^{j-1} r}^{\vartheta c^j r} \rho^\gamma \Phi^{-1}(x,\kappa(x,\rho)^{-1})\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}\frac{d\rho}{\rho}\\
&\quad + \int_{d_X/2}^{d_X} \rho^\gamma \Phi^{-1}(x,\kappa(x,\rho)^{-1})\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}\frac{d\rho}{\rho}\Bigg)\\
&\le C\int_r^{d_X} \rho^\gamma \Phi^{-1}(z,\kappa(z,\rho)^{-1})\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}\frac{d\rho}{\rho}\\
&\le \frac{C}{\log c}\frac{1}{\lambda_\gamma(z,r)}
\end{aligned}
$$

for all $x \in X \cap B(z,r)$. Hence

$$
\int_{X\cap B(z,r)} I_2(x)\,d\mu(x) \le \frac{C}{\log c}\frac{\mu(B(z,r))}{\lambda_\gamma(z,r)} \le \frac{C}{\log c}\frac{\mu(B(z,\vartheta r))}{\lambda_\gamma(z,r)}.
$$

Thus this lemma is proved. □

5 Trudinger's inequality for grand Musielak-Orlicz-Morrey spaces

In this section, we deal with the case $\Gamma(x,t)$ satisfies the uniform log-type condition:

($\Gamma_{\log}$) there exists a constant $c_\Gamma > 0$ such that

$$
\Gamma(x,t^2) \le c_\Gamma \Gamma(x,t)
$$

for all $x \in X$ and $t \ge 1$.

By ($\Gamma_{\log}$), together with Lemma 4.7, we see that $\Gamma(x,t)$ satisfies the uniform doubling condition in t:

LEMMA 5.1 (cf. [14, Lemma 4.2]). *Suppose* $\Gamma(x,t)$ *satisfies* ($\Gamma_{\log}$). *For every* $a > 1$, *there exists* $b > 0$ *such that* $\Gamma(x,at) \le b\Gamma(x,t)$ *for all* $x \in X$ *and* $t > 0$.

Theorem 5.2. *Assume that $\Phi(x,t)$ satisfies (Φ5) and $\Gamma(x,t)$ satisfies ($\Gamma_{\log}$). For each $x \in X$, let $\gamma(x) = \sup_{s>0}\Gamma(x,s)$. Suppose $\Psi(x,t) : X \times [0,\infty) \to [0,\infty]$ satisfies the following conditions:*

(Ψ1) *$\Psi(\cdot,t)$ is measurable on X for each $t \in [0,\infty)$ and $\Psi(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in X$;*

(Ψ2) *there is a constant $A_1' \geq 1$ such that $\Psi(x,t) \leq \Psi(x, A_1' s)$ for all $x \in X$ whenever $0 < t < s$;*

(Ψ3) *$\Psi(x, \Gamma(x,t)/A_2') \leq A_3' t$ for all $x \in X$ and $t > 0$ with constants $A_2',\ A_3' \geq 1$ independent of x.*

Let $\tau > 2$ and $\vartheta > 1$ such that $\tau > (\vartheta+1)/(\vartheta-1)$. Then, for $0 < \gamma < \alpha$, there exists a constant $C^* > 0$ such that $I_{\alpha,\tau}f(x)/C^* < \gamma(x)$ for a.e. $x \in X$ and

$$\frac{\lambda_\gamma(z,r)}{\mu(B(z,\vartheta r))} \int_{X\cap B(z,r)} \Psi\left(x, \frac{I_{\alpha,\tau}f(x)}{C^*}\right) d\mu(x) \leq 1$$

for all $z \in X$, $0 < r < d_X$ and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \leq 1$.

Proof. Let $f \geq 0$ and $\|f\|_{\Phi,\kappa;\eta,\xi;X} \leq 1$. Fix $x \in X$. For $0 < \delta \leq d_X/2$, Lemma 4.8 implies

$$\begin{aligned} I_{\alpha,\tau}f(x) &\leq \int_{X\cap B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) + C\Gamma\left(x, \frac{1}{\delta}\right) \\ &= \int_{X\cap B(x,\delta)} d(x,y)^{\alpha-\gamma} \frac{d(x,y)^\gamma f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) + C\Gamma\left(x, \frac{1}{\delta}\right) \\ &\leq C\left\{\delta^{\alpha-\gamma} I_{\gamma,\tau}f(x) + \Gamma\left(x, \frac{1}{\delta}\right)\right\} \end{aligned}$$

with constants $C > 0$ independent of x.

If $I_{\gamma,\tau}f(x) \leq 2/d_X$, then we take $\delta = d_X/2$. Then, by Lemma 4.7

$$I_{\alpha,\tau}f(x) \leq C\Gamma\left(x, \frac{2}{d_X}\right).$$

By Lemma 5.1, there exists $C_1^* > 0$ independent of x such that

$$I_{\alpha,\tau}f(x) \leq C_1^*\Gamma\left(x, \frac{1}{2A_3'}\right) \qquad \text{if } I_{\gamma,\tau}f(x) \leq 2/d_X. \tag{5.1}$$

Next, suppose $2/d_X < I_{\gamma,\tau}f(x) < \infty$. Let $m = \sup_{s\geq 2/d_X, x\in X}\Gamma(x,s)/s$. By ($\Gamma_{\log}$), $m < \infty$. Define δ by

$$\delta^{\alpha-\gamma} = \frac{(d_X/2)^{\alpha-\gamma}}{m}\Gamma(x, I_{\gamma,\tau}f(x))(I_{\gamma,\tau}f(x))^{-1}.$$

Since $\Gamma(x, I_{\gamma,\tau}f(x))(I_{\gamma,\tau}f(x))^{-1} \leq m$, $0 < \delta \leq d_X/2$. Then by Lemma 4.7

$$\begin{aligned} \frac{1}{\delta} &\leq C\Gamma(x, I_{\gamma,\tau}f(x))^{-1/(\alpha-\gamma)}(I_{\gamma,\tau}f(x))^{1/(\alpha-\gamma)} \\ &\leq C\Gamma(x, 2/d_X)^{-1/(\alpha-\gamma)}(I_{\gamma,\tau}f(x))^{1/(\alpha-\gamma)} \leq C(I_{\gamma,\tau}f(x))^{1/(\alpha-\gamma)}. \end{aligned}$$

Hence, using $(\Gamma_{\log})$ and Lemma 5.1, we obtain

$$\Gamma\left(x, \frac{1}{\delta}\right) \le \Gamma\big(x, C(I_{\gamma,\tau}f(x))^{1/(\alpha-\gamma)}\big) \le C\Gamma(x, I_{\gamma,\tau}f(x)).$$

By Lemma 5.1 again, we see that there exists a constant $C_2^* > 0$ independent of x such that

$$I_{\alpha,\tau}f(x) \le C_2^*\Gamma\left(x, \frac{1}{2C_{I,\gamma}A_3'}I_{\gamma,\tau}f(x)\right) \qquad \text{if } 2/d_X < I_{\gamma,\tau}f(x) < \infty, \tag{5.2}$$

where $C_{I,\gamma}$ is the constant given in Lemma 4.9.

Now, let $C^* = A_1'A_2'\max(C_1^*, C_2^*)$. Then, by (5.1) and (5.2),

$$\frac{I_{\alpha,\tau}f(x)}{C^*} \le \frac{1}{A_1'A_2'}\max\left\{\Gamma\left(x, \frac{1}{2A_3'}\right), \Gamma\left(x, \frac{1}{2C_{I,\gamma}A_3'}I_{\gamma,\tau}f(x)\right)\right\}$$

whenever $I_{\gamma,\tau}f(x) < \infty$. Since $I_{\gamma,\tau}f(x) < \infty$ for a.e. $x \in X$ by Lemma 4.9, $I_{\alpha,\tau}f(x)/C^* < \gamma(x)$ a.e. $x \in X$, and by ($\Psi2$) and ($\Psi3$), we have

$$\begin{aligned}
&\Psi\left(x, \frac{I_{\alpha,\tau}f(x)}{C^*}\right)\\
&\le \max\left\{\Psi\left(x, \Gamma\left(x, \frac{1}{2A_3'}\right)/A_2'\right), \Psi\left(x, \Gamma\left(x, \frac{1}{2C_{I,\gamma}A_3'}I_{\gamma,\tau}f(x)\right)/A_2'\right)\right\}\\
&\le \frac{1}{2} + \frac{1}{2C_{I,\gamma}}I_{\gamma,\tau}f(x)
\end{aligned}$$

for a.e. $x \in X$. Thus, noting that $\lambda_\gamma(z,r) \le 1$ and using Lemma 4.9, we have

$$\begin{aligned}
&\frac{\lambda_\gamma(z,r)}{\mu(B(z,\vartheta r))}\int_{X\cap B(z,r)}\Psi\left(x, \frac{I_{\alpha,\tau}f(x)}{C^*}\right)d\mu(x)\\
&\qquad \le \frac{1}{2}\lambda_\gamma(z,r) + \frac{1}{2C_{I,\gamma}}\frac{\lambda_\gamma(z,r)}{\mu(B(z,\vartheta r))}\int_{X\cap B(z,r)}I_{\gamma,\tau}f(x)\,d\mu(x)\\
&\qquad \le \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

for all $z \in X$ and $0 < r < d_X$. □

REMARK 5.3. If $\Gamma(x,s)$ is bounded, that is,

$$\sup_{x\in X}\int_0^{d_X}\rho^\alpha\Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big)\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}d\rho < \infty,$$

then by Lemma 4.8 we see that $I_{\alpha,\tau}|f|$ is bounded for every $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$.

REMARK 5.4. We can not take $\gamma = \alpha$ in Theorem 5.2. For details, see [18, Remark 2.8].

As in the proof of [14, Corollary 4.6], we obtain the following corollary applying Theorem 5.2 to special Φ and κ given in Examples 2.1 and 2.2.

Corollary 5.5. *Let κ be as in Example 2.2 and let $p(x)$ and $q(x)=q_1(x)$ be as in Examples 2.1. Let $\tau>2$ and $\vartheta>1$ such that $\tau>(\vartheta+1)/(\vartheta-1)$. Set $\eta(t)=t^{\theta}$ for $\theta>0$ and $\Phi(x,t)=t^{p(x)}(\log(e+t))^{q(x)}$.*

Assume that

$$\alpha-\nu(x)/p(x)=0 \qquad \text{for all } x\in X.$$

(1) *Suppose*

$$\inf_{x\in X}(-q(x)/p(x)-\beta(x)/p(x)+\theta+1)>0.$$

Then for $0<\gamma<\alpha$ there exist constants $C^*>0$ and $C^{**}>0$ such that

$$\frac{r^{\nu(z)/p(z)-\gamma}}{\mu(B(z,\vartheta r))}\int_{B(z,r)\cap X}\exp\left(\left(\frac{I_{\alpha,\tau}f(x)}{C^*}\right)^{p(x)/(p(x)+\theta p(x)-\beta(x)-q(x))}\right)d\mu(x)\le C^{**}$$

for all $z\in X$, $0<r\le d_X$ and nonnegative $f\in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X}\le 1$.

(2) *If*

$$\sup_{x\in X}(-q(x)/p(x)-\beta(x)/p(x)+\theta+1)\le 0,$$

then for $0<\gamma<\alpha$ there exist constants $C^*>0$ and $C^{**}>0$ such that

$$\frac{r^{\nu(z)/p(z)-\gamma}}{\mu(B(z,\vartheta r))}\int_{B(z,r)\cap X}\exp\left(\exp\left(\frac{I_{\alpha,\tau}f(x)}{C^*}\right)\right)d\mu(x)\le C^{**}$$

for all $z\in X$, $0<r<d_X$ and nonnegative $f\in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X}\le 1$.

6 Continuity for grand Musielak-Orlicz-Morrey spaces

In this section, we discuss the continuity of Riesz potentials $I_{\alpha,\tau}f$ of functions in grand Musielak-Orlicz-Morrey spaces under the condition: there are constants $\theta>0,\iota>1$ and $C_0>0$ such that

$$\left|\frac{d(x,y)^{\alpha}}{\mu(B(x,\tau d(x,y)))}-\frac{d(z,y)^{\alpha}}{\mu(B(z,\tau d(z,y)))}\right|\le C_0\left(\frac{d(x,z)}{d(x,y)}\right)^{\theta}\frac{d(x,y)^{\alpha}}{\mu(B(x,\iota d(x,y)))} \tag{6.1}$$

whenever $d(x,z)\le d(x,y)/2$.

We consider the functions

$$\omega(x,r)=\int_0^r \rho^{\alpha}\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}\frac{d\rho}{\rho}$$

and

$$\omega_\theta(x,r)=r^{\theta}\int_r^{d_X}\rho^{\alpha-\theta}\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}\frac{d\rho}{\rho}$$

for $\theta>0$ and $0<r\le d_X$.

Lemma 6.1 (cf. [14, Lemma 5.1]). *Let $E \subset X$. If $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in E$, then $\omega_\theta(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in E$.*

Lemma 6.2 (cf. [14, Lemma 5.2]). *There exists a constant $C > 0$ such that*

$$\omega(x, 2r) \le C\omega(x,r)$$

for all $x \in X$ and $0 < r \le d_X/2$.

Theorem 6.3. *Assume that $\Phi(x,t)$ satisfies ($\Phi5$). Let $\tau > 1$. Then there exists a constant $C > 0$ such that*

$$|I_{\alpha,\tau} f(x) - I_{\alpha,\tau} f(z)| \le C\{\omega(x, d(x,z)) + \omega(z, d(x,z)) + \omega_\theta(x, d(x,z))\}$$

for all $x, z \in X$ with $d(x,z) \le d_X/4$ and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$.

Before giving a proof of Theorem 6.3, we prepare two more lemmas.

Lemma 6.4. *Assume that $\Phi(x,t)$ satisfies ($\Phi5$). Let $\tau > 1$ and let f be a nonnegative function on X such that $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$. Then there exists a constant $C > 0$ such that*

$$\int_{X\cap B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \le C\omega(x,\delta)$$

for all $x \in X$ and $0 < \delta \le d_X$.

Proof. Let f be a nonnegative function on X with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$. As usual we start by decomposing $B(x,\delta)$ dyadically:

$$\begin{aligned}
&\int_{X\cap B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \\
=& \sum_{j=1}^{\infty} \int_{X\cap(B(x,\tau^{-j+1}\delta)\setminus B(x,\tau^{-j}\delta))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \\
\le& \sum_{j=1}^{\infty} (\tau^{-j+1}\delta)^\alpha \frac{1}{\mu(B(x,\tau^{-j+1}\delta))} \int_{B(x,\tau^{-j+1}\delta)} f(y)\, d\mu(y).
\end{aligned}$$

By Lemma 4.6, we have

$$\begin{aligned}
&\int_{X\cap B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \\
\le& C\sum_{j=1}^{\infty} (\tau^{-j+1}\delta)^\alpha \Phi^{-1}(x, \kappa(x,\tau^{-j+1}\delta)^{-1}) \eta\left((\log(e + 1/(\tau^{-j+1}\delta)))^{-1}\right)^{-1} \\
\le& \frac{C}{\log \tau} \int_0^\delta \rho^\alpha \Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big) \eta\left((\log(e+1/\rho))^{-1}\right)^{-1} \frac{d\rho}{\rho} \\
=& C\omega(x,\delta).
\end{aligned}$$

□

The following lemma can be proved on the same manner as Lemma 4.8.

LEMMA 6.5. *Assume that $\Phi(x,t)$ satisfies ($\Phi5$). Let $\theta \in \mathbf{R}$ and let $\tau > 1$. Let f be a nonnegative function on X such that $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$. Then there exists a constant $C > 0$ such that*

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^{\alpha-\theta} f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \le C\delta^{-\theta}\omega_\theta(x,\delta)$$

for all $x \in X$ and $0 < \delta \le d_X/2$.

Proof of Theorem 6.3. Let f be a nonnegative function on X with $\|f\|_{\Phi,\kappa;\eta,\xi;X} \le 1$ and let $x, z \in X$ with $d(x,z) \le d_X/4$. Write

$$\begin{aligned} & I_{\alpha,\tau}f(x) - I_{\alpha,\tau}f(z) \\ = {} & \int_{X\cap B(x,2d(x,z))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) - \int_{X\cap B(x,2d(x,z))} \frac{d(z,y)^\alpha f(y)}{\mu(B(z,\tau d(z,y)))}\, d\mu(y) \\ & + \int_{X\setminus B(x,2d(x,z))} \left(\frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))} - \frac{d(z,y)^\alpha}{\mu(B(z,\tau d(z,y)))} \right) f(y)\, d\mu(y). \end{aligned}$$

Using Lemmas 6.2 and 6.4, we have

$$\int_{X\cap B(x,2d(x,z))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,\tau d(x,y)))}\, d\mu(y) \le C\omega(x, 2d(x,z)) \le C\omega(x, d(x,z))$$

and

$$\begin{aligned} \int_{X\cap B(x,2d(x,z))} \frac{d(z,y)^\alpha f(y)}{\mu(B(z,\tau d(z,y)))}\, d\mu(y) &\le \int_{X\cap B(z,3d(x,z))} \frac{d(z,y)^\alpha f(y)}{\mu(B(z,\tau d(z,y)))}\, d\mu(y) \\ &\le C\omega(z, 3d(x,z)) \le C\omega(z, d(x,z)). \end{aligned}$$

On the other hand, by (6.1) and Lemma 6.5, we have

$$\begin{aligned} & \int_{X\setminus B(x,2d(x,z))} \left| \frac{d(x,y)^\alpha}{\mu(B(x,\tau d(x,y)))} - \frac{d(z,y)^\alpha}{\mu(B(z,\tau d(z,y)))} \right| f(y)\, d\mu(y) \\ \le {} & Cd(x,z)^\theta \int_{X\setminus B(x,2d(x,z))} \frac{d(x,y)^{\alpha-\theta} f(y)}{\mu(B(x,\iota d(x,y)))}\, d\mu(y) \\ \le {} & C\omega_\theta(x, 2d(x,z)) \le C\omega_\theta(x, d(x,z)). \end{aligned}$$

Then we have the conclusion. □

In view of Lemma 6.1, we obtain the following corollary.

COROLLARY 6.6. *Assume that $\Phi(x,t)$ satisfies ($\Phi5$). Let $\tau > 1$.*

(a) *Let $x_0 \in X$ and suppose $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in X \cap B(x_0,\delta)$ for some $\delta > 0$. Then $I_{\alpha,\tau}f$ is continuous at x_0 for every $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$.*

(b) *Suppose $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in X$. Then $I_{\alpha,\tau}f$ is uniformly continuous on X for every $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(X)$.*

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