Sobolev and Trudinger type inequalities on grand Musielak-Orlicz-Morrey spaces

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October 20, 2014

Abstract

Our aim in this paper is to establish generalizations of Sobolev's inequality and Trudinger's inequality for general potentials of functions in grand Musielak-Orlicz-Morrey spaces.

1 Introduction

Grand Lebesgue spaces were introduced in [15] for the study of Jacobian. They play important roles also in the theory of partial differential equations (see [10], [16] and [29], etc.). The generalized grand Lebesgue spaces appeared in [12], where the existence and uniqueness of the non-homogeneous N-harmonic equations were studied. The boundedness of the maximal operator on the grand Lebesgue spaces was studied in [9]. For variable exponent Lebesgue spaces, see [6] and [7]. In [21] and [17], grand Morrey spaces and generalized grand Morrey spaces were introduced. For Morrey spaces, we refer to [24] and [27]. Further, grand Morrey spaces of variable exponent were considered in [11].

On the other hand, the classical Sobolev's inequality for Riesz potentials of L^p-functions (see, e.g. [2, Theorem 3.1.4 (b)]) has been extended to various function spaces. For Morrey spaces, Sobolev's inequality was studied in [1], [27], [5], [25], etc., for Morrey spaces of variable exponent in [3], [13], [14], [22], [23], etc., for grand Morrey spaces in [21] and [17], and also for grand Morrey spaces of variable exponent in [11]. Recently, Sobolev's inequality has been extended by the authors [19] to an inequality for general potentials of functions in Musielak-Orlicz-Morrey spaces.

The classical Trudinger's inequality for Riesz potentials of L^p-functions (see, e.g. [2, Theorem 3.1.4 (c)]) has been also extended to function spaces as above; see [22], [23] for Morrey spaces of variable exponent, [11] for grand Morrey spaces of variable exponent and [20] for Musielak-Orlicz-Morrey spaces.

2000 Mathematics Subject Classification : Primary 31B15, 46E35

Key words and phrases : Riesz potentials, maximal functions, Sobolev's inequality, Trudinger's inequality, grand Musielak-Orlicz-Morrey spaces

In this paper, we define (generalized) grand Musielak-Orlicz-Morrey space on a bounded open set in $\mathbf{R}^N$ and give a Sobolev type inequality as well as a Trudinger type inequality for general potentials of functions in such spaces.

2 Preliminaries

Let G be a bounded open set in $\mathbf{R}^N$ and let d_G denote the diameter of G.

We consider a function

$$\Phi(x,t) = t\phi(x,t) : G \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions ($\Phi 1$) – ($\Phi 4$):

($\Phi 1$) $\phi(\,\cdot\,,t)$ is measurable on G for each $t \ge 0$ and $\phi(x,\,\cdot\,)$ is continuous on $[0,\infty)$ for each $x \in G$;

($\Phi 2$) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \phi(x,1) \le A_1 \quad \text{for all } x \in G;$$

($\Phi 3$) there exists a constant $\varepsilon_0 > 0$ such that $t \mapsto t^{-\varepsilon_0}\phi(x,t)$ is uniformly almost increasing, namely there exists a constant $A_2 \ge 1$ such that

$$t^{-\varepsilon_0}\phi(x,t) \le A_2 s^{-\varepsilon_0}\phi(x,s)$$

for all $x \in G$ whenever $0 < t < s$;

($\Phi 4$) there exists a constant $A_3 \ge 1$ such that

$$\phi(x,2t) \le A_3\phi(x,t) \quad \text{for all } x \in G \text{ and } t > 0.$$

Note that ($\Phi 3$) implies that

$$t^{-\varepsilon}\phi(x,t) \le A_2 s^{-\varepsilon}\phi(x,s)$$

for all $x \in G$ and $0 < \varepsilon \le \varepsilon_0$ whenever $0 < t < s$.

Also note that ($\Phi 2$), ($\Phi 3$) and ($\Phi 4$) imply

$$0 < \inf_{x \in G} \phi(x,t) \le \sup_{x \in G} \phi(x,t) < \infty$$

for each $t > 0$ and there exists $\omega > 1$ such that

$$(A_1 A_2)^{-1} t^{1+\varepsilon_0} \le \Phi(x,t) \le A_1 A_2 A_3 t^{\omega} \tag{2.1}$$

for $t \ge 1$; in fact we can take $\omega \ge 1 + \log A_3 / \log 2$.

We shall also consider the following condition:

($\Phi 5$) for every $\gamma > 0$, there exists a constant $B_\gamma \ge 1$ such that

$$\phi(x,t) \le B_\gamma \phi(y,t)$$

whenever $|x - y| \le \gamma t^{-1/N}$ and $t \ge 1$.

Let $\bar{\phi}(x,t) = \sup_{0\le s\le t} \phi(x,s)$ and

$$\overline{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)\,dr.$$

Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \le \overline{\Phi}(x,t) \le A_2\Phi(x,t)$$

for all $x \in G$ and $t \ge 0$.

EXAMPLE 2.1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1, \ldots, k$ be measurable functions on G such that

$$1 < p^- := \inf_{x\in G} p(x) \le \sup_{x\in G} p(x) =: p^+ < \infty$$

and

$$-\infty < q_j^- := \inf_{x\in G} q_j(x) \le \sup_{x\in G} q_j(x) =: q_j^+ < \infty \quad j = 1, \ldots k.$$

Set $L(t) := \log(e + t)$, $L^{(1)}(t) = L(t)$ and $L^{(j)}(t) = L(L^{(j-1)}(t))$, $j = 2, \ldots$. Then,

$$\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t) = t^{p(x)} \prod_{j=1}^{k} \left(L^{(j)}(t)\right)^{q_j(x)}$$

satisfies (Φ1), (Φ2), (Φ3) with $0 < \varepsilon_0 < p^- - 1$ and (Φ4). (2.1) holds for any $\omega > p^+$. $\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)$ satisfies (Φ5) if $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{C_p}{L(1/|x-y|)} \quad (x \ne y)$$

and $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely

$$|q_j(x) - q_j(y)| \le \frac{C_{q_j}}{L^{(j+1)}(1/|x-y|)} \quad (x \ne y)$$

for $j = 1, \ldots, k$ (cf. [19, Example 2.1]).

We also consider a function $\kappa(x,r) : G \times (0, d_G) \to (0,\infty)$ satisfying the following conditions:

(κ1) $\kappa(x,\cdot)$ is continuous on $(0, d_G)$ for each $x \in G$ and satisfies the uniform doubling condition: there is a constant $Q_1 \ge 1$ such that

$$Q_1^{-1}\kappa(x,r) \le \kappa(x,r') \le Q_1\kappa(x,r)$$

for all $x \in G$ whenever $0 < r \le r' \le 2r < d_G$;

(κ2) $r \mapsto r^{-\delta}\kappa(x,r)$ is uniformly almost increasing for some $\delta > 0$, namely there is a constant $Q_2 > 0$ such that

$$r^{-\delta}\kappa(x,r) \le Q_2 s^{-\delta}\kappa(x,s)$$

for all $x \in G$ whenever $0 < r < s < d_G$;

(κ3) there is a constant $Q_3 \ge 1$ such that

$$Q_3^{-1} \min(1, r^N) \le \kappa(x,r) \le Q_3$$

for all $x \in G$ and $0 < r < d_G$.

EXAMPLE 2.2. Let $\nu(\cdot)$ and $\beta(\cdot)$ be functions on G such that $\nu^- := \inf_{x\in G} \nu(x) > 0$, $\nu^+ := \sup_{x\in G} \nu(x) \le N$ and $-c(N - \nu(x)) \le \beta(x) \le c$ for all $x \in G$ and some constant $c > 0$. Then $\kappa(x,r) = r^{\nu(x)}(\log(e + 1/r))^{\beta(x)}$ satisfies (κ1), (κ2) and (κ3); we can take any $0 < \delta < \nu^-$ for (κ2).

Given $\Phi(x,t)$ and $\kappa(x,r)$, we define the Musielak-Orlicz-Morrey space $L^{\Phi,\kappa}(G)$ by

$$L^{\Phi,\kappa}(G) = \left\{ f \in L^1_{loc}(G)\,;\, \sup_{x\in G,\, 0<r<d_G} \frac{\kappa(x,r)}{|B(x,r)|} \int_{B(x,r)\cap G} \Phi\big(y, |f(y)|\big)\, dy < \infty \right\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{\Phi,\kappa;G} = \inf \left\{ \lambda > 0\,;\, \sup_{x\in G,\, 0<r<d_G} \frac{\kappa(x,r)}{|B(x,r)|} \int_{B(x,r)\cap G} \overline{\Phi}\big(y, |f(y)|/\lambda\big)\, dy \le 1 \right\}$$

(cf. [26]).

In case $\kappa(x,r) = r^N$, $L^{\Phi,\kappa}(G)$ is the Musielak-Orlicz space

$$L^{\Phi}(G) = \left\{ f \in L^1_{loc}(G)\,;\, \int_G \Phi(y, |f(y)|)\, dy < \infty \right\}$$

with the norm

$$\|f\|_{\Phi;G} = \inf \left\{ \lambda > 0\,;\, \int_G \overline{\Phi}\big(y, |f(y)|/\lambda\big)\, dy \le 1 \right\}.$$

REMARK 2.3. The Musielak-Orlicz spaces $L^{\Phi}(G)$ include

- Orlicz spaces defined by Young functions satisfying the doubling condition;
- variable exponent Lebesgue spaces.

The Musielak-Orlicz-Morrey spaces $L^{\Phi,\kappa}(G)$ include Morrey spaces as well as variable exponent Morrey spaces.

3 Grand Musielak-Orlicz-Morrey space

For $\varepsilon \ge 0$, set $\Phi_\varepsilon(x,t) := t^{-\varepsilon}\Phi(x,t) = t^{1-\varepsilon}\phi(x,t)$. Then, $\Phi_\varepsilon(x,t)$ satisfies (Φ1), (Φ2) with the same A_1 and (Φ4) with the same A_3. If $\Phi(x,t)$ satisfies (Φ5), then so does $\Phi_\varepsilon(x,t)$ with the same $\{B_\gamma\}_{\gamma>0}$.

If $0 \le \varepsilon < \varepsilon_0$, then $\Phi_\varepsilon(x,t)$ satisfies ($\Phi3$) with ε_0 replaced by $\varepsilon_0 - \varepsilon$ and the same A_2. It follows that

$$\frac{1}{2A_3}\Phi_\varepsilon(x,t) \le \overline{\Phi_\varepsilon}(x,t) \le A_2\Phi_\varepsilon(x,t) \tag{3.1}$$

for all $x \in G$, $t \ge 0$ and $0 \le \varepsilon \le \varepsilon_0$.

By ($\Phi3$), we see that for $0 \le \varepsilon \le \varepsilon_0$

$$\Phi_\varepsilon(x,at) \begin{cases} \le A_2 a \Phi_\varepsilon(x,t) & \text{if } 0 \le a \le 1 \\ \ge A_2^{-1} a \Phi_\varepsilon(x,t) & \text{if } a \ge 1. \end{cases} \tag{3.2}$$

Let

$$\tilde{\sigma} = \sup\{\sigma \ge 0 \,:\, r^{N-\sigma}\kappa(x,r)^{-1} \text{ is bounded on } G \times (0, \min(1,d_G))\}.$$

By ($\kappa2$), $0 \le \tilde{\sigma} \le N$. If $\tilde{\sigma} = 0$, then let $\sigma_0 = 0$; otherwise fix any $\sigma_0 \in (0, \tilde{\sigma})$. We also take δ_0 such that $0 < \delta_0 < \delta$ for δ in ($\kappa2$).

For $-\delta_0 \le \sigma \le \sigma_0$, set

$$\kappa_\sigma(x,r) = r^\sigma \kappa(x,r)$$

for $x \in G$ and $0 < r < d_G$. Then $\kappa_\sigma(x,r)$ satisfies ($\kappa1$), ($\kappa2$) and ($\kappa3$) with constants independent of σ.

Lemma 3.1. *For $0 \le \varepsilon \le \varepsilon_0$, let*

$$\Phi_\varepsilon^{-1}(x,s) = \sup\{t > 0 \,:\, \Phi_\varepsilon(x,t) < s\} \quad (x \in G,\ s > 0).$$

Then there exists $r_0 \in (0, \min(1,d_G))$ such that $\kappa_\sigma(x,r) \le 1$ and

$$\Phi_\varepsilon^{-1}\big(x, \kappa_\sigma(x,r)^{-1}\big) \ge 1$$

for all $x \in G$, $0 < r \le r_0$, $-\delta_0 \le \sigma \le \sigma_0$ and $0 < \varepsilon \le \varepsilon_0$.

Proof. By ($\kappa2$) and ($\kappa3$),

$$\kappa_\sigma(x,r) \le Q_2Q_3 \min(1,d_G)^{-\delta} r^{\delta+\sigma} \le Q_2Q_3 \min(1,d_G)^{-\delta} r^{\delta-\delta_0}$$

for $x \in G$, $0 < r < \min(1,d_G)$ and $-\delta_0 \le \sigma \le \sigma_0$. Hence, there is $r' \in (0, \min(1,d_G))$ such that $\kappa_\sigma(x,r) \le 1$ for $x \in G$, $0 < r \le r'$ and $-\delta_0 \le \sigma \le \sigma_0$. By (2.1), we see that

$$\Phi_\varepsilon^{-1}\big(x, \kappa_\sigma(x,r)^{-1}\big) \ge C^{-1}\kappa_\sigma(x,r)^{-1/\omega} \ge C'^{-1} r^{-(\delta-\delta_0)/\omega}$$

whenever $x \in G$, $0 < r \le r'$, $-\delta_0 \le \sigma \le \sigma_0$ and $0 < \varepsilon \le \varepsilon_0$ with constants $C,\ C' > 0$ independent of $x, r, \sigma, \varepsilon$. Hence the assertion of the lemma holds if we take $r_0 \in (0, r']$ satisfying $r_0^{-(\delta-\delta_0)/\omega} \ge C'$. □

Proposition 3.2. *Assume that $\Phi(x,t)$ satisfies ($\Phi5$). If $0 \le \varepsilon_1 \le \varepsilon_2 \le \varepsilon_0$, $-\delta_0 \le \sigma_j \le \sigma_0$, $j = 1, 2$ and*

$$\sigma_1 + \frac{\delta - \delta_0}{\omega}\varepsilon_1 \le \sigma_2 + \frac{\delta - \delta_0}{\omega}\varepsilon_2,$$

then $L^{\Phi_{\varepsilon_1},\kappa_{\sigma_1}}(G) \subset L^{\Phi_{\varepsilon_2},\kappa_{\sigma_2}}(G)$ *and*

$$\|f\|_{\Phi_{\varepsilon_2},\kappa_{\sigma_2};G} \le C\|f\|_{\Phi_{\varepsilon_1},\kappa_{\sigma_1};G}$$

for all $f \in L^{\Phi_{\varepsilon_1},\kappa_{\sigma_1}}(G)$ *with* $C > 0$ *independent of* ε_1, ε_2, σ_1, σ_2.

In particular,

$$L^{\Phi,\kappa}(G) \subset L^{\Phi_\varepsilon,\kappa_\sigma}(G)$$

if $0 \le \varepsilon \le \varepsilon_0$, $-\delta_0 \le \sigma \le \sigma_0$ *and* $\sigma + ((\delta - \delta_0)/\omega)\varepsilon \ge 0$.

Proof. Let $\|f\|_{\Phi_{\varepsilon_1},\kappa_{\sigma_1};G} \le 1$. Then

$$\frac{\kappa_{\sigma_1}(x,r)}{|B(x,r)|}\int_{B(x,r)} \Phi_{\varepsilon_1}\big(y,|f(y)|\big)\,dy \le 1$$

for $x \in G$ and $0 < r < d_G$.

For $x \in G$ and $0 < r < d_G$, let

$$k(x,r) = \Phi_{\varepsilon_1}^{-1}\big(x,\kappa_{\sigma_1}(x,r)^{-1}\big)$$

and

$$I(x,r) = \int_{B(x,r)} \Phi_{\varepsilon_2}\big(y,|f(y)|\big)\,dy.$$

We write $I(x,r) = I_1(x,r) + I_2(x,r)$, where

$$I_1(x,r) = \int_{B(x,r)\cap\{y:|f(y)|\le k(x,r)\}} \Phi_{\varepsilon_2}\big(y,|f(y)|\big)\,dy$$

and

$$I_2(x,r) = \int_{B(x,r)\cap\{y:|f(y)|> k(x,r)\}} \Phi_{\varepsilon_2}\big(y,|f(y)|\big)\,dy.$$

If $|f(y)| \le k(x,r)$, then

$$\Phi_{\varepsilon_2}(y,|f(y)|) \le A_2\Phi_{\varepsilon_2}(y,k(x,r)) = A_2 k(x,r)^{\varepsilon_1-\varepsilon_2}\Phi_{\varepsilon_1}(y,k(x,r)).$$

Let $r_0 \in (0,\min(1,d_G))$ be the number given in Lemma 3.1. Then, (3.2) implies

$$k(x,r) \le C\kappa_{\sigma_1}(x,r)^{-1} \le Cr^{-N}$$

for $0 < r \le r_0$ with constants independent of x, σ_1, ε_1. Hence, by (Φ5), there is a constant $B > 0$ independent of x, σ_1, ε_1, such that

$$\Phi_{\varepsilon_1}(y,k(x,r)) \le B\Phi_{\varepsilon_1}(x,k(x,r))$$

whenever $|x-y| < r \le r_0$. Therefore,

$$I_1(x,r) \le C|B(x,r)|k(x,r)^{\varepsilon_1-\varepsilon_2}\Phi_{\varepsilon_1}(x,k(x,r)) = C|B(x,r)|k(x,r)^{\varepsilon_1-\varepsilon_2}\kappa_{\sigma_1}(x,r)^{-1}$$

for $0 < r \le r_0$.

On the other hand, if $|f(y)| > k(x,r)$, then

$$\Phi_{\varepsilon_2}(y,|f(y)|) = |f(y)|^{\varepsilon_1-\varepsilon_2}\Phi_{\varepsilon_1}(y,|f(y)|) \le k(x,r)^{\varepsilon_1-\varepsilon_2}\Phi_{\varepsilon_1}(y,|f(y)|),$$

so that

$$I_2(x,r) \le k(x,r)^{\varepsilon_1-\varepsilon_2} \int_{B(x,r)} \Phi_{\varepsilon_1}(y,|f(y)|)\,dy$$
$$\le |B(x,r)| k(x,r)^{\varepsilon_1-\varepsilon_2} \kappa_{\sigma_1}(x,r)^{-1}$$

for $0 < r \le r_0$.

Therefore,

$$I(x,r) \le C|B(x,r)| k(x,r)^{\varepsilon_1-\varepsilon_2} \kappa_{\sigma_1}(x,r)^{-1},$$

which implies

$$\frac{\kappa_{\sigma_2}(x,r)}{|B(x,r)|} \int_{B(x,r)} \Phi_{\varepsilon_2}\big(y,|f(y)|\big)\,dy \le C r^{\sigma_2-\sigma_1} k(x,r)^{\varepsilon_1-\varepsilon_2}$$

for $0 < r \le r_0$. Since

$$k(x,r)^{-1} \le C r^{(\delta-\delta_0)/\omega}$$

and $\sigma_2 - \sigma_1 + ((\delta-\delta_0)/\omega)(\varepsilon_2-\varepsilon_1) \ge 0$ by assumption,

$$\frac{\kappa_{\sigma_2}(x,r)}{|B(x,r)|} \int_{B(x,r)} \Phi_{\varepsilon_2}\big(y,|f(y)|\big)\,dy \le C r^{\sigma_2-\sigma_1+((\delta-\delta_0)/\omega)(\varepsilon_2-\varepsilon_1)} \le C$$

for $0 < r \le r_0$ with positive constants C's independent of x, σ_j, ε_j $(j=1,2)$.

In case $r_0 < r < d_G$, we see

$$I(x,r) \le A_2 \int_{B(x,r)} \Phi_{\varepsilon_2}(y,1)\,dy + \int_{B(x,r)} \Phi_{\varepsilon_1}(y,|f(y)|)\,dy$$
$$\le A_1A_2|B(x,r)| + |B(x,r)|\kappa_{\sigma_1}(x,r)^{-1},$$

so that

$$\frac{\kappa_{\sigma_2}(x,r)}{|B(x,r)|} \int_{B(x,r)} \Phi_{\varepsilon_2}\big(y,|f(y)|\big)\,dy \le A_1A_2\kappa_{\sigma_2}(x,r) + r^{\sigma_2-\sigma_1} \le C$$

with C independent of r, x, σ_1 σ_2.

Therefore, $\|f\|_{\Phi_{\varepsilon_2},\kappa_{\sigma_2};G} \le C$ with $C > 0$ independent of ε_1, ε_2, σ_1, σ_2. □

Let $\eta(\varepsilon)$ be an increasing positive function on $(0,\infty)$ such that $\eta(0+) = 0$. Let $\xi(\varepsilon)$ be a function on $(0,\varepsilon_1]$ with some $\varepsilon_1 \in (0,\varepsilon_0/2]$ such that $-\delta_0 \le \xi(\varepsilon) \le \sigma_0$ for $0 < \varepsilon \le \varepsilon_1$, $\xi(0+) = 0$ and $\varepsilon \mapsto \xi(\varepsilon) + ((\delta-\delta_0)/\omega)\varepsilon$ is non-decreasing; in particular, $\xi(\varepsilon) + ((\delta-\delta_0)/\omega)\varepsilon \ge 0$ for $0 < \varepsilon \le \varepsilon_1$.

Given $\Phi(x,t)$, $\kappa(x,r)$, $\eta(\varepsilon)$ and $\xi(\varepsilon)$, the associated (generalized) grand Musielak-Orlicz-Morrey space is defined by (cf. [17] for generalized grand Morrey space)

$$\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G) = \left\{ f \in \bigcap_{0<\varepsilon\le\varepsilon_1} L^{\Phi_\varepsilon,\kappa_{\xi(\varepsilon)}}(G)\,;\ \|f\|_{\Phi,\kappa;\eta,\xi;G} < \infty \right\},$$

where

$$\|f\|_{\Phi,\kappa;\eta,\xi;G} = \sup_{0<\varepsilon\le\varepsilon_1} \eta(\varepsilon)\|f\|_{\Phi_\varepsilon,\kappa_{\xi(\varepsilon)};G}.$$

$\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ is a Banach space with the norm $\|f\|_{\Phi,\kappa;\eta,\xi;G}$. Note that, in view of Proposition 3.2, this space is determined independent of the choice of ε_1.

In case $\xi(\varepsilon) \equiv 0$, the symbol ξ may be omitted. If $\kappa(x,r) = r^N$ and $\xi(\varepsilon) \equiv 0$, then the symbol κ will be also omitted; namely

$$\widetilde{L}^{\Phi}_{\eta}(G) = \left\{ f \in \bigcap_{0<\varepsilon\le\varepsilon_0} L^{\Phi_\varepsilon}(G)\,;\ \|f\|_{\Phi;\eta;G} := \sup_{0<\varepsilon\le\varepsilon_0} \eta(\varepsilon)\|f\|_{\Phi_\varepsilon;G} < \infty \right\}.$$

This space may be called a grand Musielak-Orlicz space.

Remark 3.3. The grand Musielak-Orlicz space $\widetilde{L}^{\Phi}_{\eta}(G)$ include the following spaces:

- generalized grand Lebesgue spaces introduced in [4];
- grand Orlicz spaces introduced in [18] where $\Phi(x,t) = \Phi(t)$ satisfying

  $$\sup_{0<\varepsilon\le\varepsilon_0} \eta(\varepsilon) \int_1^\infty t^{-N-\varepsilon}\Phi(t)\,\frac{dt}{t} < \infty$$

  (see also [8]).

The (generalized) grand Musielak-Orlicz-Morrey space $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ include also the following spaces:

- grand Morrey spaces introduced in [21] where $\xi(\varepsilon) \equiv 0$;
- grand grand Morrey spaces introduced in [28] and generalized grand Morrey spaces introduced in [17] where $\xi(\varepsilon)$ is an increasing positive function on $(0,\infty)$.

4 Boundedness of the maximal operator

Hereafter, *we shall always assume that* $\Phi(x,t)$ *satisfies* (Φ5).

For a nonnegative $f \in L^1_{loc}(G)$, $x \in G$, $0 < r < d_G$ and $\varepsilon > 0$, set

$$I(f;x,r) := \frac{1}{|B(x,r)|}\int_{B(x,r)\cap G} f(y)\,dy$$

and

$$J_\varepsilon(f;x,r) := \frac{1}{|B(x,r)|}\int_{B(x,r)\cap G} \Phi_\varepsilon\big(y, f(y)\big)\,dy.$$

We show a Jensen type inequality for functions in $L^{\Phi_\varepsilon,\kappa_\sigma}(G)$.

Lemma 4.1. *There exists a constant* $C > 0$ (*independent of* ε *and* σ) *such that*

$$\Phi_\varepsilon(x, I(f;x,r)) \le C J_\varepsilon(f;x,r)$$

for all $x \in G$, $0 < r < d_G$, $0 < \varepsilon \le \varepsilon_0$ *and for all nonnegative* $f \in L^1_{loc}(G)$ *such that* $f(y) \ge 1$ *or* $f(y) = 0$ *for each* $y \in G$ *and* $\|f\|_{\Phi_\varepsilon,\kappa_\sigma;G} \le 1$ *with* $-\delta_0 \le \sigma \le \sigma_0$.

Proof. Let f be as in the statement of the lemma and let $I = I(f; x, r)$ and $J_\varepsilon = J_\varepsilon(f; x, r)$ for $x \in G$, $0 < r < d_G$ and $0 < \varepsilon \le \varepsilon_0$. Note that $\|f\|_{\Phi_\varepsilon,\kappa_\sigma;G} \le 1$ implies $J_\varepsilon \le 2A_3\kappa_\sigma(x,r)^{-1}$ by (3.1).

By (Φ2) and (3.2), $\Phi_\varepsilon\big(y, f(y)\big) \ge (A_1A_2)^{-1} f(y)$, since $f(y) \ge 1$ or $f(y) = 0$. Hence $I \le A_1A_2J_\varepsilon$. Thus, if $J_\varepsilon \le 1$, then

$$\Phi_\varepsilon(x, I) \le (A_1A_2J_\varepsilon)A_2\phi(x, A_1A_2) \le CJ_\varepsilon.$$

Next, suppose $J_\varepsilon > 1$. Since $\Phi_\varepsilon(x,t) \to \infty$ as $t \to \infty$, there exists $K_\varepsilon \ge 1$ such that

$$\Phi_\varepsilon(x, K_\varepsilon) = \Phi_\varepsilon(x, 1)J_\varepsilon.$$

Then $K_\varepsilon \le A_2J_\varepsilon$ by (3.2). With this K_ε, we have

$$\int_{B(x,r)\cap G} f(y)\,dy \le K_\varepsilon|B(x,r)| + A_2\int_{B(x,r)\cap G} f(y)\frac{f(y)^{-\varepsilon}\phi\big(y, f(y)\big)}{K_\varepsilon^{-\varepsilon}\phi(y, K_\varepsilon)}\,dy.$$

Since $\kappa_\sigma(x,r)J_\varepsilon \le 2A_3$,

$$1 \le K_\varepsilon \le A_2J_\varepsilon \le 2A_2A_3\kappa_\sigma(x,r)^{-1} \le Cr^{-N}$$

with a constant $C > 0$ independent of ε and σ. Hence, by (Φ5) there is $\beta \ge 1$, independent of f, x, r, ε and σ such that

$$\phi(x, K_\varepsilon) \le \beta\phi(y, K_\varepsilon)$$

for all $y \in B(x,r)$. Thus, we have

$$\begin{aligned}\int_{B(x,r)\cap G} f(y)\,dy &\le K_\varepsilon|B(x,r)| + \frac{A_2\beta}{K_\varepsilon^{-\varepsilon}\phi(x,K_\varepsilon)}\int_{B(x,r)\cap G}\Phi_\varepsilon\big(y, f(y)\big)\,dy \\ &= K_\varepsilon|B(x,r)| + A_2\beta|B(x,r)|\frac{J_\varepsilon}{K_\varepsilon^{-\varepsilon}\phi(x,K_\varepsilon)}.\end{aligned}$$

Since

$$K_\varepsilon^{-\varepsilon}\phi(x,K_\varepsilon) = K_\varepsilon^{-1}\Phi_\varepsilon(x,K_\varepsilon) = K_\varepsilon^{-1}J_\varepsilon\Phi_\varepsilon(x,1) \ge A_1^{-1}K_\varepsilon^{-1}J_\varepsilon,$$

it follows that

$$I \le (1 + A_1A_2\beta)K_\varepsilon,$$

so that by (Φ2), (Φ3) and (Φ4)

$$\Phi_\varepsilon(x, I) \le C\Phi_\varepsilon(x, K_\varepsilon) \le CJ_\varepsilon$$

with constants $C > 0$ independent of f, x, r, ε and σ as required. □

For a locally integrable function f on G, the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)\cap G}|f(y)|\,dy.$$

The following lemma can be proved in a way similar to the proof of [25, Theorem 1]:

Lemma 4.2. *Let $p_0 > 1$ and $-\delta_0 \le \sigma \le \sigma_0$. Then there exists a constant $C > 0$ independent of σ for which the following holds: If f is a measurable function such that*

$$\int_{B(x,r)\cap G} |f(y)|^{p_0}\, dy \le |B(x,r)|\kappa_\sigma(x,r)^{-1}$$

for all $x \in G$ and $0 < r < d_G$, then

$$\int_{B(x,r)\cap G} [Mf(y)]^{p_0}\, dy \le C|B(x,r)|\kappa_\sigma(x,r)^{-1}$$

for all $x \in G$ and $0 < r < d_G$.

Lemma 4.3. *There is a constant $C > 0$ (independent of ε and σ) such that*

$$\|Mf\|_{\Phi_\varepsilon,\kappa_\sigma;G} \le C\|f\|_{\Phi_\varepsilon,\kappa_\sigma;G}$$

for all $f \in L^{\Phi_\varepsilon,\kappa_\sigma}(G)$ whenever $0 < \varepsilon \le \varepsilon_0/2$ and $-\delta_0 \le \sigma \le \sigma_0$.

Proof. Set $p_0 = 1 + \varepsilon_0/2$ and consider the function

$$\tilde{\Phi}(x,t) = \Phi(x,t)^{1/p_0}.$$

Then $\tilde{\Phi}(x,t)$ also satisfies all the conditions (Φj), $j = 1, 2, \ldots, 5$ with ε_0 replaced by $\varepsilon_0' = \varepsilon_0/(2+\varepsilon_0)$. In fact, it trivially satisfies (Φj) for $j = 1, 2, 4, 5$. Since

$$t^{-\varepsilon_0'}t^{-1}\tilde{\Phi}(x,t) = [t^{-\varepsilon_0}\phi(x,t)]^{1/p_0},$$

condition $(\Phi 3)$ implies that $\tilde{\Phi}(x,t)$ satisfies $(\Phi 3)$ with ε_0 replaced by ε_0'.

Let $0 < \varepsilon \le \varepsilon_0/2$, $-\delta_0 \le \sigma \le \sigma_0$ and $f \ge 0$ and $\|f\|_{\Phi_\varepsilon,\kappa_\sigma;G} \le 1$. Let $f_1 = f\chi_{\{z:f(z)\ge 1\}}$ and $f_2 = f - f_1$, where χ_E is the characteristic function of E.

Since $\Phi_\varepsilon(x,t) \ge 1/(A_1A_2)$ for $t \ge 1$, we see that

$$\tilde{\Phi}_{\varepsilon/p_0}(x,t) = \Phi_\varepsilon(x,t)^{1/p_0} \le (A_1A_2)^{1-1/p_0}\Phi_\varepsilon(x,t)$$

if $t \ge 1$, so that

$$\int_{B(x,r)\cap G} \tilde{\Phi}_{\varepsilon/p_0}\left(y, f_1(y)\right)\, dy \le 2(A_1A_2)^{1-1/p_0}A_3|B(x,r)|\kappa_\sigma(x,r)^{-1}$$

for every $x \in G$ and $0 < r < d_G$. Hence $\|f_1\|_{\tilde{\Phi}_{\varepsilon/p_0},\kappa_\sigma;G} \le c_0$ with $c_0 > 0$ independent of ε and σ.

Let $F_\varepsilon(x) = \Phi_\varepsilon(x, f(x))$. Then $\tilde{\Phi}_{\varepsilon/p_0}(x, f(x)) = F_\varepsilon(x)^{1/p_0}$. Applying Lemma 4.1 to $\tilde{\Phi}_{\varepsilon/p_0}$ and f_1/c_0, we have

$$\Phi_\varepsilon(x, Mf_1(x)) = \left[\tilde{\Phi}_{\varepsilon/p_0}\big(x, Mf_1(x)\big)\right]^{p_0} \le C[M(F_\varepsilon^{1/p_0})(x)]^{p_0}.$$

On the other hand, since $Mf_2 \le 1$, we have by $(\Phi 2)$ and $(\Phi 3)$

$$\Phi_\varepsilon(x, Mf_2(x)) \le A_1A_2.$$

Thus, we obtain

$$\Phi_\varepsilon\big(x, Mf(x)\big) \le C\left\{\left[M(F_\varepsilon^{1/p_0})(x)\right]^{p_0} + 1\right\}$$

for $x \in G$ with a constant $C > 0$ independent of f and ε. Hence

$$\int_{B(x,r)\cap G} \Phi_\varepsilon\big(y, Mf(y)\big)\,dy \le C\left\{\int_{B(x,r)\cap G}\left[M(F_\varepsilon^{1/p_0})(y)\right]^{p_0} dy + |B(x,r)|\right\}$$

for $x \in G$ and $0 < r < d_G$. Since $\|f\|_{\Phi_\varepsilon,\kappa_\sigma;G} \le 1$ and $\Phi_\varepsilon(y, f(y)) = F_\varepsilon(y) = (F_\varepsilon^{1/p_0}(y))^{p_0}$, Lemma 4.2 implies

$$\int_{B(x,r)\cap G}\left[M(F_\varepsilon^{1/p_0})(y)\right]^{p_0} dy \le C|B(x,r)|\kappa_\sigma(x,r)^{-1}$$

with a constant $C > 0$ independent of x, r, ε, σ. Hence,

$$\int_{B(x,r)\cap G} \Phi_\varepsilon\big(y, Mf(y)\big)\,dy \le C|B(x,r)|\kappa_\sigma(x,r)^{-1},$$

which shows

$$\|Mf\|_{\Phi_\varepsilon,\kappa_\sigma;G} \le C\|f\|_{\Phi_\varepsilon,\kappa_\sigma;G}$$

with a constant $C > 0$ independent of ε and σ. □

From this lemma we obtain the boundedness of the maximal operator on $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$.

THEOREM 4.4. *The maximal operator M is bounded from $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ into itself; namely there exists a constant $C > 0$ such that*

$$\|Mf\|_{\Phi,\kappa;\eta,\xi;G} \le C\|f\|_{\Phi,\kappa;\eta,\xi;G}$$

for all $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$.

COROLLARY 4.5. *If $\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)$ satisfies the conditions in Example 2.1, then the maximal operator M is bounded from $\widetilde{L}_\eta^{\Phi_{p(\cdot),\{q_j(\cdot)\}}}(G)$ into itself.*

5 Sobolev type inequality

LEMMA 5.1 ([19, Lemma 5.1]). *Let $F(x,t)$ be a positive function on $G \times (0,\infty)$ satisfying the following conditions:*

(F1) *$F(x,\cdot)$ is continuous on $(0,\infty)$ for each $x \in G$;*

(F2) *there exists a constant $K_1 \ge 1$ such that*

$$K_1^{-1} \le F(x,1) \le K_1 \quad \text{for all } x \in G.$$

(F3) $t \mapsto t^{-\varepsilon'} F(x,t)$ *is uniformly almost increasing for* $\varepsilon' > 0$*; namely there exists a constant* $K_2 \ge 1$ *such that*

$$t^{-\varepsilon'} F(x,t) \le K_2 s^{-\varepsilon'} F(x,s) \quad \text{for all } x \in G \quad \text{whenever } 0 < t < s;$$

Set

$$F^{-1}(x,s) = \sup\{t > 0\,;\, F(x,t) < s\}$$

for $x \in G$ *and* $s > 0$*. Then:*

(1) $F^{-1}(x,\cdot)$ *is non-decreasing.*

(2)

$$F^{-1}(x,\lambda s) \le (K_2 \lambda)^{1/\varepsilon'} F^{-1}(x,s)$$

for all $x \in G$*,* $s > 0$ *and* $\lambda \ge 1$*.*

(3)

$$F(x, F^{-1}(x,t)) = t$$

for all $x \in G$ *and* $t > 0$*.*

(4)

$$K_2^{-1/\varepsilon'} t \le F^{-1}(x, F(x,t)) \le K_2^{2/\varepsilon'} t$$

for all $x \in G$ *and* $t > 0$*.*

(5)

$$\min\left\{1, \left(\frac{s}{K_1 K_2}\right)^{1/\varepsilon'}\right\} \le F^{-1}(x,s) \le \max\{1, (K_1 K_2 s)^{1/\varepsilon'}\}$$

for all $x \in G$ *and* $s > 0$*.*

Remark 5.2. $F(x,t) = \Phi_\varepsilon(x,t)$ ($0 < \varepsilon \le \varepsilon_0$) satisfies (F1), (F2) and (F3) with $\varepsilon' = 1$, $K_1 = A_1$ and $K_2 = A_2$ and $F(x,t) = \kappa_\sigma(x,t)$ ($-\delta_0 \le \sigma \le \sigma_0$) satisfies (F1), (F2) and (F3) with $\varepsilon' = \delta - \delta_0$, $K_1 = Q_1$ and $K_2 = Q_2$.

Lemma 5.3. *There exists a constant* $C > 0$ *such that*

$$\frac{\eta(\varepsilon)}{|B(x,r)|} \int_{B(x,r)\cap G} f(y)\,dy \le C \Phi_\varepsilon^{-1}(x, \kappa_{\xi(\varepsilon)}(x,r)^{-1})$$

for all $x \in G$*,* $0 < r < d_G$*,* $0 < \varepsilon \le \varepsilon_1$ *and nonnegative functions* f *on* G *such that* $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$*.*

Proof. Let f be a nonnegative function on G such that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$. Then we have by (3.1)

$$\frac{\kappa_{\xi(\varepsilon)}(x,r)}{|B(x,r)|} \int_{B(x,r)\cap G} \Phi_\varepsilon(y, \eta(\varepsilon) f(y))\,dy \le 2A_3$$

for all $x \in G$, $0 < r < d_G$ and $0 < \varepsilon \le \varepsilon_1$. Fix ε and let $f_1 = f\chi_{\{x : \eta(\varepsilon) f(x) \ge 1\}}$ and $f_2 = f - f_1$. By Lemma 4.1,

$$\Phi_\varepsilon\left(x, \frac{1}{|B(x,r)|} \int_{B(x,r)\cap G} \eta(\varepsilon) f_1(y)\,dy\right) \le C \kappa_{\xi(\varepsilon)}(x,r)^{-1}$$

for all $x \in G$, $0 < r < d_G$ and $0 < \varepsilon \le \varepsilon_0/2$ with a constant $C > 0$ independent of x, r, ε. Since

$$\Phi_\varepsilon\left(x, \frac{1}{|B(x,r)|}\int_{B(x,r)\cap G} \eta(\varepsilon) f_2(y)\,dy\right) \le A_2\Phi_\varepsilon(x,1) \le A_1A_2,$$

we have

$$\Phi_\varepsilon\left(x, \frac{1}{|B(x,r)|}\int_{B(x,r)\cap G} \eta(\varepsilon) f(y)\,dy\right) \le C_1\kappa_{\xi(\varepsilon)}(x,r)^{-1}$$

with a constant $C_1 \ge 1$ independent of x, r, ε. Hence, we find by Lemma 5.1 with $F = \Phi_\varepsilon$ and $\varepsilon' = 1$

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)\cap G} \eta(\varepsilon) f(y)\,dy &\le A_2\Phi_\varepsilon^{-1}\left(x, C_1\kappa_{\xi(\varepsilon)}(x,r)^{-1}\right)\\ &\le C_1A_2^2\Phi_\varepsilon^{-1}\left(x, \kappa_{\xi(\varepsilon)}(x,r)^{-1}\right),\end{aligned}$$

as required. □

As a potential kernel, we consider a function

$$J(x,r) : G \times (0, d_G) \to [0,\infty)$$

satisfying the following conditions:

(J1) $J(\cdot, r)$ is measurable on G for each $r \in (0, d_G)$;

(J2) $J(x, \cdot)$ is non-increasing on $(0, d_G)$ for each $x \in G$;

(J3) $\int_0^{d_G} J(x,r)r^{N-1}dr \le J_0 < \infty$ for every $x \in G$.

Example 5.4. Let $\alpha(\cdot)$ be a measurable function on G such that

$$0 < \alpha^- := \inf_{x\in G} \alpha(x) \le \sup_{x\in G} \alpha(x) =: \alpha^+ < N.$$

Then, $J(x,r) = r^{\alpha(x)-N}$ satisfies (J1), (J2) and (J3).

For a nonnegative measurable function f on G, its J-potential Jf is defined by

$$Jf(x) = \int_G J(x, |x-y|) f(y)\,dy \quad (x \in G).$$

Set

$$\overline{J}(x,r) = \frac{N}{r^N}\int_0^r J(x,\rho)\rho^{N-1}d\rho$$

for $x \in G$ and $0 < r < d_G$. Then $J(x,r) \le \overline{J}(x,r)$. Further, $\overline{J}(x,\cdot)$ is non-increasing and continuous on $(0, d_G)$ for each $x \in G$. Also, set

$$Y_J(x,r) = r^N\overline{J}(x,r)$$

for $x \in G$ and $0 < r < d_G$.

We consider the following condition:

$(\Phi\kappa J)$ there exist constants $\delta' > 0$ and $A_4 \ge 1$ such that

$$s^{\delta'} Y_J(x,s)\Phi_\varepsilon^{-1}(x,\kappa_\sigma(x,s)^{-1}) \le A_4 t^{\delta'} Y_J(x,t)\Phi_\varepsilon^{-1}(x,\kappa_\sigma(x,t)^{-1})$$

for all $x \in G$ whenever $0 < t < s < d_G$, $0 < \varepsilon \le \varepsilon_0/2$, $-\delta_0 \le \sigma \le \sigma_0$ and $\sigma + ((\delta-\delta_0)/\omega)\varepsilon \ge 0$.

LEMMA 5.5. *Assume $(\Phi\kappa J)$. Then there exists a constant $C > 0$ such that*

$$\int_r^{d_G} \rho^N \Phi_\varepsilon^{-1}(x,\kappa_\sigma(x,\rho)^{-1})\, d(-\overline{J}(x,\cdot))(\rho) \le C Y_J(x,r)\Phi_\varepsilon^{-1}(x,\kappa_\sigma(x,r)^{-1})$$

for all $x \in G$, $0 < r \le d_G/2$, $0 < \varepsilon \le \varepsilon_0/2$ and $-\min(\delta_0, ((\delta-\delta_0)/\omega)\varepsilon) \le \sigma \le \sigma_0$.

Proof. We follow the proof of [19, Lemma 6.2], noting that the constants are independent of ε and σ. □

LEMMA 5.6. *Assume $(\Phi\kappa J)$. Then there exists a constant $C > 0$ such that*

$$\eta(\varepsilon)\int_{G\setminus B(x,r)} J(x,|x-y|)f(y)\,dy \le C Y_J(x,r)\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,r)^{-1})$$

for all $x \in G$, $0 < r \le d_G/2$, $0 < \varepsilon \le \varepsilon_1$ and $f \ge 0$ satisfying $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

Proof. By the integration by parts, we have

$$\begin{aligned}&\int_{G\setminus B(x,r)} J(x,|x-y|)f(y)\,dy \\ &\qquad \le J(x,d_G-0)\int_G f(y)\,dy + \int_r^{d_G}\left(\int_{B(x,\rho)\cap G} f(y)\,dy\right) d(-J(x,\cdot))(\rho),\end{aligned}$$

where $J(x,d_G-0) = \lim_{\rho\to d_G-0} J(x,\rho)$. Hence, by Lemma 5.3, we have

$$\begin{aligned}&\eta(\varepsilon)\int_{G\setminus B(x,r)} J(x,|x-y|)f(y)\,dy \\ &\qquad \le C\Big\{Y_J(x,d_G)\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,d_G)^{-1}) \\ &\qquad\qquad + \int_r^{d_G} |B(x,\rho)|\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,\rho)^{-1})\, d(-J(x,\cdot))(\rho)\Big\}.\end{aligned}$$

Hence by $(\Phi\kappa J)$ and the previous lemma we obtain the required result. □

LEMMA 5.7. *Assume $(\Phi\kappa J)$. Then there exists a constant $C > 0$ such that*

$$\eta(\varepsilon)Jf(x) \le C\left\{\eta(\varepsilon)Mf(x)Y_J\Big(x,\kappa_{\xi(\varepsilon)}^{-1}\big(x,\Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))^{-1}\big)\Big) + 1\right\}$$

for all $x \in G$, $0 < \varepsilon \le \varepsilon_1$ and $f \ge 0$ satisfying $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

Proof. Let f be a nonnegative function on G such that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$. For $0 < r \le d_G/2$, we write

$$\begin{aligned} Jf(x) &= \int_{B(x,r)\cap G} J(x,|x-y|)f(y)\,dy + \int_{G\setminus B(x,r)} J(x,|x-y|)f(y)\,dy \\ &= J_1(x) + J_2(x). \end{aligned}$$

First note that

$$J_1(x) \le CY_J(x,r)Mf(x)$$

(see, e.g., [30, p. 63, (16)]). By Lemma 5.6, we have

$$\eta(\varepsilon)J_2(x) \le CY_J(x,r)\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,r)^{-1}).$$

Hence

$$\eta(\varepsilon)Jf(x) \le CY_J(x,r)\left\{\eta(\varepsilon)Mf(x) + \Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,r)^{-1})\right\} \tag{5.1}$$

for $x \in G$, $0 < r \le d_G/2$ and $0 < \varepsilon \le \varepsilon_1$.

We consider two cases.

Case 1: $d_G/2 < \kappa_{\xi(\varepsilon)}^{-1}\big(x, \Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))^{-1}\big)$. In this case, let $r = d_G/2$. Since

$$\Phi_\varepsilon(x,\eta(\varepsilon)Mf(x)) \le Q_2\kappa_{\xi(\varepsilon)}(x,d_G/2)^{-1} \le Q_2Q_3\max\big(1,(d_G/2)^{-N}\big),$$

it follows that $\eta(\varepsilon)Mf(x) \le C_1$ with a constant $C_1 > 0$ independent of x and ε. Also,

$$\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,r)^{-1}) = \Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,d_G/2)^{-1}) \le C_2$$

with a constant $C_2 > 0$ independent of x and ε. Hence, by (5.1) and (J3),

$$\eta(\varepsilon)Jf(x) \le C$$

with a constant $C > 0$ independent of x and ε.

Case 2: $d_G/2 \ge \kappa_{\xi(\varepsilon)}^{-1}\big(x, \Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))^{-1}\big)$. In this case, take

$$r = \kappa_{\xi(\varepsilon)}^{-1}\big(x, \Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))^{-1}\big).$$

Then $\kappa_{\xi(\varepsilon)}(x,r)^{-1} = \Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))$, so that by Lemma 5.1(4)

$$\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,r)^{-1}) \le C\eta(\varepsilon)Mf(x)$$

with a constant $C > 0$ independent of x and ε. Hence, by (5.1)

$$\begin{aligned} \eta(\varepsilon)Jf(x) &\le CY_J(x,r)\eta(\varepsilon)Mf(x) \\ &= C\eta(\varepsilon)Mf(x)Y_J\Big(x, \kappa_{\xi(\varepsilon)}^{-1}\big(x,\Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))^{-1}\big)\Big) \end{aligned}$$

with a constant $C > 0$ independent of x and ε. □

The following theorem gives a Sobolev type inequality for potentials Jf of $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$. Example 5.9 below shows that this theorem includes known Sobolev type inequalities as special cases.

THEOREM 5.8. *Assume* ($\Phi\kappa J$). *Suppose a function*

$$\Psi(x,t) : G \times [0,\infty) \to [0,\infty)$$

satisfies (Φ1) – (Φ4) *with* ε_0 *replaced by some* ε_0' *in* (Φ3) *and*

($\Psi\Phi$) *there exist a constant* $A' \ge 1$ *and a strictly increasing continuous function* $\zeta(\varepsilon)$ *on* $[0,\varepsilon_1]$ *such that* $\zeta(0) = 0$, $\varepsilon \mapsto \xi(\varepsilon) + ((\delta - \delta_0)/\omega^*)\zeta(\varepsilon)$ *is nondecreasing with* $\omega^* > 1$ *such that* $\Psi(x,t) \le Ct^{\omega^*}$ *for* $t \ge 1$, *and*

$$\Psi_{\zeta(\varepsilon)}\Big(x, tY_J\Big(x, \kappa^{-1}_{\xi(\varepsilon)}\big(x, \Phi_\varepsilon(x,t)^{-1}\big)\Big)\Big) \le A'\Phi_\varepsilon(x,t)$$

for all $x \in G$, $t \ge 1$ *and* $0 < \varepsilon \le \varepsilon_1$.

Then there exists a constant $C > 0$ *such that*

$$\|Jf\|_{\Psi,\kappa;\eta\circ\zeta^{-1},\xi\circ\zeta^{-1};G} \le C\|f\|_{\Phi,\kappa;\eta,\xi;G}$$

for all $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$.

Proof. Let f be a nonnegative function on G such that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$. Choose $\varepsilon_1' \in (0,\varepsilon_1]$ such that $\zeta(\varepsilon_1') \le \varepsilon_0'$. Let $x \in G$, $0 < r < d_G$ and $0 < \varepsilon \le \varepsilon_1'$. By Lemma 5.7 and ($\Psi\Phi$) we have

$$\begin{aligned}
&\Psi_{\zeta(\varepsilon)}\big(x, \eta(\varepsilon)Jf(x)\big)\\
&\le C\left\{\Psi_{\zeta(\varepsilon)}\Big(x, \eta(\varepsilon)Mf(x)Y_J\Big(x, \kappa^{-1}_{\xi(\varepsilon)}\big(x, \Phi_\varepsilon(x, \eta(\varepsilon)Mf(x))^{-1}\big)\Big)\Big) + 1\right\}\\
&\le C\left\{\Phi_\varepsilon\big(x, \eta(\varepsilon)Mf(x)\big) + 1\right\}.
\end{aligned}$$

Note that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$ implies $\|Mf\|_{\Phi,\kappa;\eta,\xi;G} \le C$ by Theorem 4.4. Hence there is a constant $C_1' > 0$ such that

$$\frac{\kappa_{\xi(\varepsilon)}(x,r)}{|B(x,r)|}\int_{B(x,r)\cap G} \Phi_\varepsilon\big(y, \eta(\varepsilon)Mf(y)\big)\,dy \le C_1'$$

for all $x \in G$, $0 < r < d_G$ and $0 < \varepsilon \le \varepsilon_1'$. Therefore, there is another constant $C_2' > 0$ such that

$$\frac{\kappa_{\xi(\varepsilon)}(x,r)}{|B(x,r)|}\int_{B(x,r)\cap G} \Psi_{\zeta(\varepsilon)}\big(y, \eta(\varepsilon)Jf(y)\big)\,dy \le C_2'$$

for all $x \in G$, $0 < r < d_G$ and $0 < \varepsilon \le \varepsilon_1'$, so that

$$\frac{\kappa_{(\xi\circ\zeta^{-1})(\varepsilon')}(x,r)}{|B(x,r)|}\int_{B(x,r)\cap G} \Psi_{\varepsilon'}\big(y, (\eta\circ\zeta^{-1})(\varepsilon')Jf(y)\big)\,dy \le C_2'$$

for all $x \in G$, $0 < r < d_G$ and $0 < \varepsilon' \le \zeta(\varepsilon_1')$, which implies the required result. □

Example 5.9. Let $\Phi(x,t) = \Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)$ be as in Example 2.1, $\kappa(x,r) = r^{\nu(x)}(\log(e+1/r))^{\beta(x)}$ be as in Example 2.2 and $J(x,r) = r^{\alpha(x)-N}$ be as in Example 5.4.

Note that $\sigma_0 = 0$ if $\nu^+ := \sup_{x\in G}\nu(x) = N$ and $0 < \sigma_0 < N - \nu^+$ if $\nu^+ < N$. We may take $0 < \delta_0 < \delta < \nu^-$ and $\omega > p^+$. Then,

$$\sigma + \frac{\delta-\delta_0}{\omega}\varepsilon < \sigma + \frac{\nu^-}{p^+}\varepsilon \le \sigma + \frac{\nu(x)}{p(x)}\varepsilon.$$

Hence, if $\sigma + ((\delta-\delta_0)/\omega)\varepsilon \ge 0$, then

$$\frac{\nu(x)+\sigma}{p(x)-\varepsilon} \ge \frac{\nu(x)}{p(x)}. \tag{5.2}$$

Since

$$Y_J(x,r)\Phi_\varepsilon^{-1}(x,\kappa_\sigma(x,r)^{-1}) \sim r^{\alpha(x)-(\nu(x)+\sigma)/(p(x)-\varepsilon)}\big[Q(x,1/r)(\log(e+1/r))^{\beta(x)}\big]^{-1/(p(x)-\varepsilon)},$$

where $Q(x,t) = \prod_{j=1}^k \big(L^{(j)}(t)\big)^{q_j(x)}$, we see that condition $(\Phi\kappa J)$ holds if

$$\inf_{x\in G}\left(\frac{\nu(x)}{p(x)} - \alpha(x)\right) > 0.$$

Set

$$\Psi(x,t) = \big[\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)\big]^{p^*(x)/p(x)}(\log(e+t))^{p^*(x)\alpha(x)\beta(x)/\nu(x)},$$

where $1/p^*(x) = 1/p(x) - \alpha(x)/\nu(x)$.

We see

$$tY_J\big(x,\kappa_\sigma^{-1}(x,\Phi_\varepsilon(x,t)^{-1})\big) \sim t^{p(x)/p^*_\sigma(x)+\varepsilon\alpha(x)/(\nu(x)+\sigma)}\Big[Q(x,t)(\log(e+t))^{\beta(x)}\Big]^{-\alpha(x)/(\nu(x)+\sigma)},$$

where $1/p^*_\sigma(x) = 1/p(x) - \alpha(x)/(\nu(x)+\sigma)$. Hence

$$\begin{aligned}
&\Psi\Big(x, tY_J\big(x,\kappa_\sigma^{-1}(x,\Phi_\varepsilon(x,t)^{-1})\big)\Big)\\
&\quad\sim t^{p(x)p^*(x)/p^*_\sigma(x)+\varepsilon p^*(x)\alpha(x)/(\nu(x)+\sigma)}\big[Q(x,t)(\log(e+t))^{\beta(x)}\big]^{-p^*(x)\alpha(x)/(\nu(x)+\sigma)}\\
&\qquad\times Q(x,t)^{p^*(x)/p(x)}(\log(e+t))^{p^*(x)\alpha(x)\beta(x)/\nu(x)}\\
&\quad= \Phi_\varepsilon(x,t)t^{\sigma(p^*(x)-p(x))/(\nu(x)+\sigma)+\varepsilon[p^*(x)\alpha(x)/(\nu(x)+\sigma)+1]}\\
&\qquad\times Q(x,t)^{\sigma(p^*(x)-p(x))/[p(x)(\nu(x)+\sigma)]}(\log(e+t))^{\sigma p^*(x)\alpha(x)\beta(x)/[\nu(x)(\nu(x)+\sigma)]}.
\end{aligned}$$

Here, note that $\xi(\varepsilon) + (\nu^-/p^+)\varepsilon \ge 0$ implies $\nu(x)+\xi(\varepsilon) > \nu(x)/2$ if $0 < \varepsilon \le 1/2$. Let $0 < \varepsilon \le \min(1/2,\varepsilon_1)$. Let $\theta = (\delta-\delta_0)/\omega$. Since

$$\frac{\xi(\varepsilon)}{\nu(x)+\xi(\varepsilon)} \le \frac{\xi(\varepsilon)+\theta\varepsilon}{\nu(x)} \quad \text{and} \quad \frac{p^*(x)\alpha(x)}{\nu(x)+\xi(\varepsilon)} + 1 \le 2\frac{p^*(x)}{p(x)},$$

$$\begin{aligned}
&\Psi\Big(x, tY_J\big(x,\kappa_{\xi(\varepsilon)}^{-1}(x,\Phi_\varepsilon(x,t)^{-1})\big)\Big)\\
&\quad\lesssim \Phi_\varepsilon(x,t)t^{(\xi(\varepsilon)+\theta\varepsilon)(p^*(x)-p(x))/\nu(x)+2\varepsilon p^*(x)/p(x)}[\log(e+t)]^{m_1(\xi(\varepsilon)+\theta\varepsilon)}
\end{aligned}$$

for $t \geq 1$ with a constant $m_1 \geq 0$. In view of (5.2), we also see that

$$tY_J\big(x, \kappa_{\xi(\varepsilon)}^{-1}(x, \Phi_\varepsilon(x,t)^{-1})\big) \gtrsim t^{p(x)/p^*(x)}[\log(e+t)]^{-m_2}$$

with a constant $m_2 \geq 0$, which implies

$$\begin{aligned}
&\Psi_{\zeta(\varepsilon)}\Big(x,\, tY_J\big(x, \kappa_{\xi(\varepsilon)}^{-1}(x, \Phi_\varepsilon(x,t)^{-1})\big)\Big) \\
&\quad \lesssim \Phi_\varepsilon(x,t)\left\{t^{p(x)/p^*(x)}[\log(e+t)]^{-m_2}\right\}^{-\zeta(\varepsilon)} \\
&\qquad \times t^{2p^*(x)/p(x)\varepsilon}\left\{t^{(p^*(x)-p(x))/\nu(x)}[\log(e+t)]^{m_1}\right\}^{(\xi(\varepsilon)+\theta\varepsilon)}
\end{aligned}$$

for $t \geq 1$.

Now, let $\zeta(\varepsilon) = a\varepsilon + b(\xi(\varepsilon)+\theta\varepsilon)$ $(a, b > 0)$. If $a > 2\sup_{x\in G}(p^*(x)/p(x))^2$, then

$$\sup_{x\in G,\ t\geq 1}\left\{t^{p(x)/p^*(x)}[\log(e+t)]^{-m_2}\right\}^{-a} t^{2p^*(x)/p(x)} < \infty$$

and if $b > \sup_{x\in G} p^*(x)(p^*(x)-p(x))/(p(x)\nu(x))$, then

$$\sup_{x\in G,\ t\geq 1}\left\{t^{p(x)/p^*(x)}[\log(e+t)]^{-m_2}\right\}^{-b}\left\{t^{(p^*(x)-p(x))/\nu(x)}[\log(e+t)]^{m_1}\right\} < \infty,$$

so that $\Psi(x,t)$ satisfies condition $(\Psi\Phi)$ with $\zeta(\varepsilon) = a\varepsilon + b(\xi(\varepsilon)+\theta\varepsilon)$ $(0 < \varepsilon \leq \min(1/2, \varepsilon_1))$.

6 Trudinger type inequality

In this section, we consider Trudinger type inequality on $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$.

LEMMA 6.1. *Let $t_1,\ t_2 > 0$. If*

$$\Phi(x, t_1) \leq K\Phi(x, t_2)$$

for some $x \in G$ with $K \geq A_2^{-1}$, then $t_1 \leq A_2 K t_2$.

Proof. Assume $t_1 > A_2 K t_2$. Note that $t_1 > t_2$. Using $(\Phi3)$, we have

$$\Phi(x, t_1) = t_1\phi(x, t_1) > Kt_2\phi(x, t_2) = K\Phi(x, t_2),$$

which contradicts the assumption. □

In this section, we assume:

(Ξ) $\xi(\varepsilon) \leq a\varepsilon$ for $0 < \varepsilon \leq \varepsilon_1$ with some $a \geq 0$.

Recall that $\xi(\varepsilon) \geq -((\delta-\delta_0)/\omega)\varepsilon$ by assumption.

Let

$$\varepsilon(r) = (\log(e+1/r))^{-1}$$

for $r > 0$ and let $r_1 \in (0, \min(1, d_G))$ be such that $\varepsilon(r) \leq \varepsilon_1$ for $0 < r \leq r_1$.

Lemma 6.2. *There exists a constant $C \ge 1$ such that*

$$C^{-1}\Phi^{-1}(x,\kappa(x,r)^{-1}) \le \Phi_{\varepsilon(r)}^{-1}(x,\kappa_{\xi(\varepsilon(r))}(x,r)^{-1}) \le C\Phi^{-1}(x,\kappa(x,r)^{-1})$$

for all $x \in G$ and $0 < r \le r_1$.

Proof. Fix $x \in G$ and set

$$t_0(r) = \Phi^{-1}(x,\kappa(x,r)^{-1}) \quad \text{and} \quad t(r) = \Phi_{\varepsilon(r)}^{-1}(x,\kappa_{\xi(\varepsilon(r))}(x,r)^{-1})$$

for $0 < r \le r_1$. Then

$$\begin{aligned}\Phi(x,t_0(r)) &= \kappa(x,r)^{-1} = r^{\xi(\varepsilon(r))}\kappa_{\xi(\varepsilon(r))}(x,r)^{-1} \\ &= r^{\xi(\varepsilon(r))}\Phi_{\varepsilon(r)}(x,t(r)) = r^{\xi(\varepsilon(r))}t(r)^{-\varepsilon(r)}\Phi(x,t(r)).\end{aligned} \tag{6.1}$$

Thus, in view of Lemma 6.1, it is enough to show that there exists a constant $K \ge 1$ independent of x such that

$$K^{-1} \le r^{\xi(\varepsilon(r))}t(r)^{-\varepsilon(r)} \le K \tag{6.2}$$

for all $0 < r \le r_1$.

Note that

$$e^{-a} \le r^{a\varepsilon(r)} \le r^{\xi(\varepsilon(r))} \le r^{-((\delta-\delta_0)/\omega)\varepsilon(r)} \le e^{(\delta-\delta_0)/\omega} \tag{6.3}$$

for $0 < r \le r_1$ and that

$$Q_3^{-1} \le \kappa(x,r)^{-1} \le Q_3\left(1+\frac{1}{r}\right)^N$$

by (κ3).

If $t(r) \le 1$, then by (6.1) and (6.3)

$$\begin{aligned}Q_3^{-1} &\le \kappa(x,r)^{-1} = r^{\xi(\varepsilon(r))}t(r)^{-\varepsilon(r)}\Phi(x,t(r)) \\ &\le e^{(\delta-\delta_0)/\omega}t(r)^{1-\varepsilon(r)}\phi(x,t(r)) \le e^{(\delta-\delta_0)/\omega}A_1A_2t(r)^{1-\varepsilon(d_G)},\end{aligned}$$

so that $t(r) \ge C_1^{-1}$ with a constant $C_1 \ge 1$ independent of x. Thus

$$C_1^{-\varepsilon(d_G)} \le t(r)^{\varepsilon(r)} \le 1$$

if $t(r) \le 1$.

If $t(r) \ge 1$, then by (6.1) and (6.3) again

$$Q_3\left(1+\frac{1}{r}\right)^N \ge \kappa(x,r)^{-1} \ge e^{-a}t(r)^{1-\varepsilon(r)}\phi(x,t(r)) \ge e^{-a}(A_1A_2)^{-1}t(r)^{1-\varepsilon(d_G)},$$

so that $t(r) \le C_2[(1+1/r)^N]^{1/(1-\varepsilon(d_G))}$ with $C_2 \ge 1$ independent of x. Since $(1+1/r)^{\varepsilon(r)}$ is bounded for $r > 0$, it follows that

$$1 \le t(r)^{\varepsilon(r)} \le C_2^{\varepsilon(d_G)}\left[\left(1+\frac{1}{r}\right)^N\right]^{\varepsilon(r)/(1-\varepsilon(d_G))} \le C_3$$

if $t(r) \ge 1$, with a constant $C_3 \ge 1$ independent of x.

Therefore, (6.2) holds with $K = \max\{e^{(\delta-\delta_0)/\omega}C_1^{\varepsilon(d_G)}, e^aC_3\}$. □

Lemma 6.3. *There exists a constant $C > 0$ such that*

$$\frac{1}{|B(x,r)|}\int_{B(x,r)\cap G} f(y)\,dy \le C\Phi^{-1}(x,\kappa(x,r)^{-1})\eta\left((\log(e+1/r))^{-1}\right)^{-1} \tag{6.4}$$

for all $x \in G$, $0 < r < d_G$ and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

Proof. Let f be a nonnegative measurable function on G such that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

If $0 < r \le r_1$, then by Lemma 5.3

$$\frac{1}{|B(x,r)|}\int_{B(x,r)\cap G} f(y)\,dy \le C\Phi^{-1}_{\varepsilon(r)}(x,\kappa_{\xi(\varepsilon(r))}(x,r)^{-1})\eta(\varepsilon(r))^{-1}$$

for all $x \in G$. Hence, using the above lemma we obtain (6.4).

In case $r_1 < r < d_G$, note that

$$\Phi^{-1}_{\varepsilon(r_1)}\left(x,\kappa_{\xi(\varepsilon(r_1))}(x,r_1)^{-1}\right) \le C\Phi^{-1}(x,\kappa(x,r)^{-1})$$

by (κ3) and Lemma 5.1(5). Hence, by Lemma 5.3 with $\varepsilon = \varepsilon(r_1)$, we obtain (6.4) in this case, too. □

In this section, we also assume that

(J3′) $J(x,r) \le C_J r^{-\varsigma}$ for $x \in G$ and $0 < r \le d_G$ with constants $0 \le \varsigma < N$ and $C_J > 0$;

(J4) there is $r_0 \in (0, d_G)$ such that

$$\inf_{x\in G} J(x,r_0) > 0 \quad \text{and} \quad \inf_{x\in G}\frac{\overline{J}(x,r_0)}{\overline{J}(x,d_G)} > 1.$$

Here note that (J3′) implies (J3).

Example 6.4. Let $\alpha(\cdot)$ and $J(x,r)$ be as in Example 5.4. Then, $J(x,r)$ satisfies (J3′) and (J4) (with $\varsigma = N - \alpha^-$). In particular, it satisfies (J4) with any $r_0 \in (0, d_G)$.

We consider the function

$$\Gamma(x,s) = \begin{cases} \displaystyle\int_{1/s}^{d_G} \rho^N\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\eta\left((\log(e+1/\rho))^{-1}\right)^{-1} d(-\overline{J}(x,\cdot))(\rho) & \text{if } s \ge 1/r_0, \\ \Gamma(x,1/r_0)r_0 s & \text{if } 0 \le s \le 1/r_0 \end{cases}$$

for every $x \in G$, where r_0 is the number given in (J4). $\Gamma(x,\cdot)$ is strictly increasing and continuous for each $x \in G$.

Lemma 6.5. *There exist positive constants C' and C'' such that*

(a) *$\Gamma(x,s) \le C' s^{\varsigma}\eta\left((\log(e+s))^{-1}\right)^{-1}$ for all $x \in G$ and $s \ge 1/r_0$ with ς in condition* (J3′)*;*

(b) $\Gamma(x,1/r_0) \ge C'' > 0$ *for all* $x \in G$.

Proof. First note from (κ3) and Lemma 5.1(5) that

$$C^{-1} \le \Phi^{-1}(x,\kappa(x,r)^{-1}) \le Cr^{-N}. \tag{6.5}$$

By (6.5) and (J3′),

$$\begin{aligned}\Gamma(x,s) &\le C\eta\left((\log(e+s))^{-1}\right)^{-1}\int_{1/s}^{d_G} d(-\overline{J}(x,\cdot))(\rho)\\ &\le C\eta\left((\log(e+s))^{-1}\right)^{-1}\overline{J}(x,1/s)\\ &\le C's^{\varsigma}\eta\left((\log(e+s))^{-1}\right)^{-1}\end{aligned}$$

for all $x \in G$ and $s \ge 1/r_0$; and

$$\begin{aligned}\Gamma(x,1/r_0) &\ge C^{-1}\int_{r_0}^{d_G}\rho^N d(-\overline{J}(x,\cdot))(\rho) \ge C^{-1}r_0^N\int_{r_0}^{d_G} d(-\overline{J}(x,\cdot))(\rho)\\ &= C^{-1}r_0^N\left(\overline{J}(x,r_0)-\overline{J}(x,d_G)\right) \ge C'' > 0,\end{aligned}$$

where we used (J4) to obtain the inequalities in the last line. □

Lemma 6.6. *There exists a constant* $C > 0$ *such that*

$$\int_{G\setminus B(x,\delta)} J(x,|x-y|)f(y)\,dy \le C\Gamma\left(x,\frac{1}{\delta}\right)$$

for all $x \in G$, $0 < \delta \le r_0$ *and nonnegative* $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ *with* $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

Proof. By integration by parts, Lemma 6.3, (6.5), (J3′) and Lemma 6.5(b), we have

$$\begin{aligned}\int_{G\setminus B(x,\delta)} J(x,|x-y|)f(y)\,dy &\le \int_{G\setminus B(x,\delta)} \overline{J}(x,|x-y|)f(y)\,dy\\ &\le C\Big\{d_G^N\overline{J}(x,d_G)\Phi^{-1}\big(x,\kappa(x,d_G)^{-1}\big)\eta\left((\log(e+1/d_G))^{-1}\right)^{-1}\\ &\quad+\int_{\delta}^{d_G}\rho^N\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}d(-\overline{J}(x,\cdot))(\rho)\Big\}\\ &\le C\big\{\Gamma(x,1/r_0)+\Gamma(x,1/\delta)\big\} \le C\Gamma(x,1/\delta).\end{aligned}$$

□

Lemma 6.7. *Let* $0 < \lambda < N$ *and define*

$$I_\lambda f(x) = \int_G |x-y|^{\lambda-N} f(y)\,dy$$

for a nonnegative measurable function f *on* G *and*

$$\omega_\lambda(z,r) = \frac{1}{1+\displaystyle\int_r^{d_G}\rho^\lambda\Phi^{-1}(z,\kappa(z,\rho)^{-1})\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}\frac{d\rho}{\rho}}$$

for $z \in G$. Then there exists a constant $C_{I,\lambda} > 0$ such that

$$\frac{\omega_\lambda(z,r)}{|B(z,r)|} \int_{B(z,r)\cap G} I_\lambda f(x)\, dx \le C_{I,\lambda}$$

for all $z \in G$, $0 < r < d_G$ and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

Proof. Let $z \in G$. Let $f(x) = 0$ for $x \in \mathbf{R}^N \setminus G$ and write

$$\begin{aligned} I_\lambda f(x) &= \int_{B(z,2r)} |x-y|^{\lambda-N} f(y)\, dy + \int_{G\setminus B(z,2r)} |x-y|^{\lambda-N} f(y)\, dy \\ &= I_1(x) + I_2(x) \end{aligned}$$

for $x \in G$. By Fubini's theorem,

$$\begin{aligned} \int_{B(z,r)\cap G} I_1(x)\, dx &= \int_{B(z,2r)} \left(\int_{B(z,r)\cap G} |x-y|^{\lambda-N}\, dx \right) f(y)\, dy \\ &\le \int_{B(z,2r)} \left(\int_{B(y,3r)} |x-y|^{\lambda-N}\, dx \right) f(y)\, dy \\ &\le C \int_{B(z,2r)} \left(\int_0^{3r} t^\lambda \frac{dt}{t} \right) f(y)\, dy \\ &\le \frac{C}{\lambda} r^\lambda \int_{B(z,2r)} f(y)\, dy. \end{aligned}$$

Now, by Lemma 6.3, (κ2) and Lemma 5.1(2), we have

$$\begin{aligned} r^\lambda \int_{B(z,2r)} f(y)\, dy &\le C r^\lambda |B(z,2r)| \Phi^{-1}(z, \kappa(z,2r)^{-1}) \eta \left((\log(e+1/(2r)))^{-1} \right)^{-1} \\ &\le C|B(z,r)| \int_r^{2r} \rho^\lambda \Phi^{-1}(z, \kappa(z,\rho)^{-1}) \eta \left((\log(e+1/\rho))^{-1} \right)^{-1} \frac{d\rho}{\rho} \end{aligned}$$

if $0 < r < d_G/2$ and, by Lemma 6.3 and (6.5), we have

$$\begin{aligned} r^\lambda \int_{B(z,2r)} f(y)\, dy &= r^\lambda \int_{B(z,d_G)} f(y)\, dy \\ &\le C {d_G}^\lambda |B(z,d_G)| \Phi^{-1}(z, \kappa(z,d_G)^{-1}) \eta \left((\log(e+1/d_G))^{-1} \right)^{-1} \le C|B(z,r)| \end{aligned}$$

if $d_G/2 \le r < d_G$. Therefore

$$\int_{B(z,r)\cap G} I_1(x)\, dx \le \frac{C}{\lambda} \frac{|B(z,r)|}{\omega_\lambda(z,r)}$$

for all $0 < r < d_G$.

For I_2, first note that $I_2(x) = 0$ if $x \in G$ and $r \ge d_G/2$. Let $0 < r < d_G/2$. Since

$$I_2(x) \le C \int_{G\setminus B(z,2r)} |z-y|^{\lambda-N} f(y)\, dy \quad \text{for} \quad x \in B(z,r) \cap G,$$

by integration by parts and Lemma 6.3, we have

$$\begin{aligned} I_2(x) &\le C\Big\{ {d_G}^\lambda \Phi^{-1}(z,\kappa(z,d_G)^{-1})\eta\left((\log(e+1/d_G))^{-1}\right)^{-1} \\ &\quad + \int_{2r}^{d_G} \rho^\lambda \Phi^{-1}(z,\kappa(z,\rho)^{-1})\eta\left((\log(e+1/\rho))^{-1}\right)^{-1} \frac{d\rho}{\rho}\Big\} \\ &\le \frac{C}{\omega_\lambda(z,r)} \end{aligned}$$

for all $x \in B(z,r) \cap G$. Hence

$$\int_{B(z,r)\cap G} I_2(x)\,dx \le C\frac{|B(z,r)|}{\omega_\lambda(z,r)}.$$

Thus this lemma is proved. □

From now on, we deal with the case $\Gamma(x,r)$ satisfies the uniform log-type condition:

$(\Gamma_{\log})$ there exists a constant $c_\Gamma > 0$ such that

$$\Gamma(x,s^2) \le c_\Gamma \Gamma(x,s)$$

for all $x \in G$ and $s \ge 1$.

By $(\Gamma_{\log})$, together with Lemma 6.5, we see that $\Gamma(x,s)$ satisfies the uniform doubling condition in s:

Lemma 6.8 ([20, Lemma 4.2]). *For every $a > 1$, there exists $b > 0$ such that $\Gamma(x,as) \le b\Gamma(x,s)$ for all $x \in G$ and $s > 0$.*

Now we consider the following condition (J5):

(J5) there exists $0 < \lambda < N - \varsigma$ such that $r \mapsto r^{N-\lambda}J(x,r)$ is uniformly almost increasing on $(0,d_G)$ for ς in condition (J3′).

Example 6.9. Let J be as in Example 5.4. It satisfies (J5) with $0 < \lambda < \alpha^-$.

Theorem 6.10. *Assume that Γ satisfies $(\Gamma_{\log})$ and J satisfies (J5). For each $x \in G$, let $\gamma(x) = \sup_{s>0}\Gamma(x,s)$. Suppose $\Lambda(x,t) : G \times [0,\infty) \to [0,\infty]$ satisfies the following conditions:*

(Λ1) *$\Lambda(\cdot,t)$ is measurable on G for each $t \in [0,\infty)$; $\Lambda(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in G$;*

(Λ2) *there is a constant $A_1' \ge 1$ such that $\Lambda(x,t) \le \Lambda(x,A_1's)$ for all $x \in G$ whenever $0 < t < s$;*

(Λ3) *$\Lambda(x,\Gamma(x,s)/A_2') \le A_3's$ for all $x \in G$ and $s > 0$ with constants $A_2',\ A_3' \ge 1$ independent of x.*

Then, for λ given in (J5)*, there exists a constant $C^* > 0$ such that $Jf(x)/C^* \le \gamma(x)$ for a.e. $x \in G$ and*

$$\frac{\omega_\lambda(z,r)}{|B(z,r)|}\int_{B(z,r)\cap G}\Lambda\left(x,\frac{Jf(x)}{C^*}\right)\,dx \le 1$$

for all $z \in G$, $0 < r < d_G$ and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

By ($\Gamma_{\log}$) and (Λ3), the assertion of this theorem can be considered as exponential integrability of Jf; cf. Corollary 6.12 below.

Proof. Let f be a nonnegative measurable function on G such that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$. Fix $x \in G$. For $0 < \delta \le r_0$, Lemma 6.6, (J5) and (J3$'$) imply

$$\begin{aligned}
Jf(x) &\le \int_{B(x,\delta)} J(x,|x-y|)f(y)\,dy + C\Gamma\left(x,\frac{1}{\delta}\right)\\
&= \int_{B(x,\delta)} |x-y|^{N-\lambda}J(x,|x-y|)|x-y|^{\lambda-N}f(y)\,dy + C\Gamma\left(x,\frac{1}{\delta}\right)\\
&\le C\left\{\delta^{N-\lambda}J(x,\delta)I_\lambda f(x) + \Gamma\left(x,\frac{1}{\delta}\right)\right\}\\
&\le C\left\{\delta^{N-\varsigma-\lambda}I_\lambda f(x) + \Gamma\left(x,\frac{1}{\delta}\right)\right\}
\end{aligned}$$

with constants $C > 0$ independent of x.

If $I_\lambda f(x) \le 1/r_0$, then we take $\delta = r_0$. Then, by Lemma 6.5(b)

$$Jf(x) \le C\Gamma\left(x,\frac{1}{r_0}\right).$$

By Lemma 6.8, there exists $C_1^* > 0$ independent of x such that

$$Jf(x) \le C_1^*\Gamma\left(x,\frac{1}{2A_3'}\right) \qquad \text{if } I_\lambda f(x) \le 1/r_0. \tag{6.6}$$

Next, suppose $1/r_0 < I_\lambda f(x) < \infty$. Let $m = \sup_{s\ge 1/r_0, x\in G}\Gamma(x,s)/s$. By ($\Gamma_{\log}$), $m < \infty$. Define δ by

$$\delta^{N-\varsigma-\lambda} = \frac{r_0^{N-\varsigma-\lambda}}{m}\Gamma(x,I_\lambda f(x))(I_\lambda f(x))^{-1}.$$

Since $\Gamma(x,I_\lambda f(x))(I_\lambda f(x))^{-1} \le m$, $0 < \delta \le r_0$. Then by Lemma 6.5(b)

$$\begin{aligned}
\frac{1}{\delta} &\le C\Gamma(x,I_\lambda f(x))^{-1/(N-\varsigma-\lambda)}(I_\lambda f(x))^{1/(N-\varsigma-\lambda)}\\
&\le C\Gamma(x,1/r_0)^{-1/(N-\varsigma-\lambda)}(I_\lambda f(x))^{1/(N-\varsigma-\lambda)} \le C(I_\lambda f(x))^{1/(N-\varsigma-\lambda)}.
\end{aligned}$$

Hence, using ($\Gamma_{\log}$) and Lemma 6.8, we obtain

$$\Gamma\left(x,\frac{1}{\delta}\right) \le \Gamma\big(x, C(I_\lambda f(x))^{1/(N-\varsigma-\lambda)}\big) \le C\Gamma(x,I_\lambda f(x)).$$

By Lemma 6.8 again, we see that there exists a constant $C_2^* > 0$ independent of x such that

$$Jf(x) \le C_2^* \Gamma\left(x, \frac{1}{2C_{I,\lambda}A_3'} I_\lambda f(x)\right) \qquad \text{if } 1/r_0 < I_\lambda f(x) < \infty, \tag{6.7}$$

where $C_{I,\lambda}$ is the constant given in Lemma 6.7.

Now, let $C^* = A_1' A_2' \max(C_1^*, C_2^*)$. Then, by (6.6) and (6.7),

$$\frac{Jf(x)}{C^*} \le \frac{1}{A_1'A_2'} \max\left\{ \Gamma\left(x, \frac{1}{2A_3'}\right), \Gamma\left(x, \frac{1}{2C_{I,\lambda}A_3'} I_\lambda f(x)\right)\right\}$$

whenever $I_\lambda f(x) < \infty$. Since $I_\lambda f(x) < \infty$ for a.e. $x \in G$ by Lemma 6.7, $Jf(x)/C^* \le \gamma(x)$ a.e. $x \in G$, and by (Λ2) and (Λ3), we have

$$\begin{aligned} &\Lambda\left(x, \frac{Jf(x)}{C^*}\right) \\ &\le \max\left\{ \Lambda\left(x, \Gamma\left(x, \frac{1}{2A_3'}\right)/A_2'\right), \Lambda\left(x, \Gamma\left(x, \frac{1}{2C_{I,\lambda}A_3'} I_\lambda f(x)\right)/A_2'\right)\right\} \\ &\le \frac{1}{2} + \frac{1}{2C_{I,\lambda}} I_\lambda f(x) \end{aligned}$$

for a.e. $x \in G$. Thus, noting that $\omega_\lambda(z,r) \le 1$ and using Lemma 6.7, we have

$$\begin{aligned} &\frac{\omega_\lambda(z,r)}{|B(z,r)|} \int_{B(z,r)\cap G} \Lambda\left(x, \frac{Jf(x)}{C^*}\right) dx \\ &\qquad \le \frac{1}{2}\omega_\lambda(z,r) + \frac{1}{2C_{I,\lambda}} \frac{\omega_\lambda(z,r)}{|B(z,r)|} \int_{B(z,r)\cap G} I_\lambda f(x)\, dx \\ &\qquad \le \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

for all $z \in G$ and $0 < r < d_G$. □

Remark 6.11. If $\Gamma(x,s)$ is bounded, that is,

$$\sup_{x\in G} \int_0^{d_G} \rho^N \Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big) \eta\left((\log(e+1/\rho))^{-1}\right)^{-1} d(-\overline{J}(x,\cdot))(\rho) < \infty,$$

then by Lemma 6.6 we see that $J|f|$ is bounded for every $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$. In particular, if $\omega_{N-\varsigma}(x,r)^{-1}$ is bounded, that is,

$$\sup_{x\in G} \int_0^{d_G} \rho^{N-\varsigma} \Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big) \eta\left((\log(e+1/\rho))^{-1}\right)^{-1} \frac{d\rho}{\rho} < \infty,$$

then $\Gamma(x,s)$ is bounded by (J3′), and hence $J|f|$ is bounded for every $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$.

If we further assume a continuity of the potential kernel J like condition (J5) in our paper [20], then we can show a continuity of Jf for $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$, as in [20, Theorem 5.3].

Applying Theorem 6.10 to special Φ, κ and J, we obtain the following corollary;

COROLLARY 6.12. *Let $\kappa(x,r)$ and $\alpha(x)$ be as in Examples 2.2 and 5.4 and let $p(x)$ and $q(x)$ be as in Examples 2.1. Set $\eta(t) = t^\theta$ for $\theta > 0$, $\Phi(x,t) = t^{p(x)}(\log(e+t))^{q(x)}$ and*

$$I_{\alpha(\cdot)} f(x) = \int_G |x-y|^{\alpha(x)-N} f(y)\, dy$$

for a nonnegative locally integrable function f on G.

Assume that

$$\alpha(x) - \nu(x)/p(x) = 0 \qquad \text{for all } x \in G.$$

(1) *Suppose that*

$$\inf_{x\in G}(-q(x)/p(x) - \beta(x)/p(x) + \theta + 1) > 0.$$

Then for $0 < \lambda < \alpha^-$ there exist constants $C^* > 0$ and $C^{**} > 0$ such that

$$\frac{r^{\nu(z)/p(z)-\lambda}}{|B(z,r)|} \int_{B(z,r)\cap G} \exp\left(\left(\frac{I_{\alpha(\cdot)} f(x)}{C^*} \right)^{p(x)/(p(x)+\theta p(x)-\beta(x)-q(x))} \right) dx \le C^{**}$$

for all $z \in G$, $0 < r < d_G$ and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

(2) *If*

$$\sup_{x\in G}(-q(x)/p(x) - \beta(x)/p(x) + \theta + 1) \le 0.$$

then for $0 < \lambda < \alpha^-$ there exist constants $C^* > 0$ and $C^{**} > 0$ such that

$$\frac{r^{\nu(z)/p(z)-\lambda}}{|B(z,r)|} \int_{B(z,r)\cap G} \exp\left(\exp\left(\frac{I_{\alpha(\cdot)} f(x)}{C^*} \right) \right) dx \le C^{**}$$

for all $z \in G$, $0 < r < d_G$ and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

Proof. In the present situation, we see that

$$\Gamma(x,s) \sim \begin{cases} (\log(e+s))^{-q(x)/p(x)-\beta(x)/p(x)+\theta+1} & \text{in case (1),} \\ \log(\log(e+s)) & \text{in case (2)} \end{cases}$$

for all $x \in G$ and $s \ge 1/r_0 = 2/d_G$. Hence, we may take

$$\Lambda(x,t) = \begin{cases} \exp\left(t^{p(x)/(p(x)+\theta p(x)-q(x)-\beta(x))}\right) & \text{in case (1),} \\ \exp(\exp t) & \text{in case (2).} \end{cases}$$

On the other hand,

$$\omega_{\lambda'}(z,r) \sim r^{\nu(z)/p(z)-\lambda'} (\log(e+1/r))^{-q(x)/p(x)-\beta(x)/p(x)+\theta}$$

for all $z \in G, 0 < s < d_G$ and $0 < \lambda' < \alpha^-$, so that

$$r^{\nu(z)/p(z)-\lambda} \le C\omega_{\lambda'}(z,r)$$

if $0 < \lambda < \lambda' < \alpha^-$. Thus, given $0 < \lambda < \alpha^-$, Theorem 6.10 implies the required results. □

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