Continuity properties for logarithmic potentials of functions in Morrey spaces of variable exponent

Dedicated to Professor Yoshihiro Mizuta on the occasion of his sixtieth birthday

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Abstract

Our aim in this paper is to deal with continuity properties for logarithmic potentials of functions in Morrey spaces of variable exponent. Our exponent approaches 1 on some part of the domain, and hence continuity properties depend on the shape of that part and the speed of the exponent approaching 1.

1 Introduction

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space. Following Orlicz [7] and Kováčik-Rákosník [1], for a continuous function $p(\cdot) : \mathbf{R}^n \to [1, \infty)$, which is called a variable exponent, we consider the generalized Lebesgue space $L^{p(\cdot)}(\mathbf{R}^n)$. In recent years, the generalized Lebesgue spaces have attracted more and more attention, in connection with the study of elasticity, fluid mechanics and differential equations with $p(\cdot)$-growth; see Růžička [9].

In the present paper we are concerned with generalized Morrey spaces of variable exponent $p(\cdot)$. We use the notation $B(x,r)$ to denote the open ball centered at x with radius r. For $0 \le \nu \le n$ and a positive function $\varphi : (0,\infty) \to (0,\infty)$, we define the $L^{p(\cdot),\nu,\varphi}$-norm of a locally integrable function f on $\mathbf{R}^n$ by

$$\|f\|_{p(\cdot),\nu,\varphi} = \inf\left\{\lambda > 0 : \sup_{x\in\mathbf{R}^n, r>0} r^{-\nu}\varphi(r)\int_{B(x,r)}\left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \le 1\right\}.$$

We denote by $L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ the space of all measurable functions f on $\mathbf{R}^n$ with $\|f\|_{p(\cdot),\nu,\varphi} < \infty$. This space $L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ is referred to as a generalized Morrey

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space of variable exponent. In particular, $L^{p(\cdot),0,1}(\mathbf{R}^n)$ is equal to the generalized Lebesgue space $L^{p(\cdot)}(\mathbf{R}^n)$.

In this paper, we consider a variable exponent $p(\cdot)$ satisfying the following special condition. For a compact set K in $\mathbf{R}^n$, we define

$$K(r) = \{x \in \mathbf{R}^n : \delta_K(x) < r\},$$

where $\delta_K(x)$ denotes the distance of x from K. For a nonincreasing function $k(\cdot) : (0, \infty] \to (0, \infty)$, we consider a function ω satisfying a generalized log-Hölder condition such that $\omega(0) = 0$,

$$\omega(r) = \frac{\log k(r)}{\log(1/r)}$$

for $0 < r < r_0$ and $\omega(r) = \omega(r_0)$ for $r \geq r_0$, where the number r_0 is chosen so that $\omega(r)$ is nondecreasing on $(0, r_0)$ (see Lemma 2.3 below). Now we define a variable exponent $p(\cdot)$ by

$$p(x) = p_0 + \omega(\delta_K(x))$$

for $p_0 \geq 1$; set $p(x) = p_0$ on K.

For $0 \leq \alpha \leq n$, we say that the $(n-\alpha)$-dimensional upper Minkowski content of K is finite (see Mattila [2]) if there exists a constant $C > 0$ such that

$$|K(r)| \leq C r^\alpha \qquad \text{for small } r > 0,$$

where $|E|$ denotes the Lebesgue measure of a set E. Note here that if K is a singleton, then its 0-dimensional upper Minkowski content is finite, and if K is a spherical surface, then its $(n-1)$-dimensional upper Minkowski content is finite. Moreover, as examples of K, we may consider fractal type sets like Cantor sets or Koch curves.

In view of [4, Lemma 2.4], we know that if $\nu = 0, \varphi(\cdot) \equiv 1$ and the $(n-\alpha)$-dimensional upper Minkowski content of K is finite, then for each bounded open set $G \subset \mathbf{R}^n$,

$$\int_G |f(x)|^{p_0} k(|f(x)|^{-1})^{\alpha/p_0}\, dx < \infty$$

for all $f \in L^{p(\cdot)}(\mathbf{R}^n)$. Our first aim in this paper is to obtain the following theorem which gives an extension of the above fact to the generalized Morrey space of variable exponent. For this purpose we need several conditions on k and φ, which are stated in Section 2.

Theorem A (cf. [4, Lemma 2.4]). *Suppose $0 \leq \nu \leq \alpha \leq n$ and the $(n-\alpha)$-dimensional upper Minkowski content of K is finite. Then for each bounded open set $G \subset \mathbf{R}^n$ there exists a constant $C > 0$ such that*

$$\int_{G\cap B(x,r)} |f(y)|^{p_0} k(|f(y)|^{-1})^{(\alpha-\nu)/p_0} dy \leq C r^\nu \varphi(r)^{-1}$$

for all $x \in \mathbf{R}^n$, $r > 0$ *and* $f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ *with* $\|f\|_{p(\cdot),\nu,\varphi} \le 1$.

In Section 3, we consider the logarithmic potential of a locally integrable function f on $\mathbf{R}^n$, which is defined by

$$Lf(x) = \int_{\mathbf{R}^n} (\log(1/|x-y|))\, f(y)dy.$$

Here it is natural to assume that

$$\int_{\mathbf{R}^n} (\log(2+|y|))|f(y)|dy < \infty, \tag{1.1}$$

which is equivalent to the condition that $-\infty < Lf \not\equiv \infty$ (see [3, Section 2.6]). If f is a locally integrable function on $\mathbf{R}^n$ satisfying (1.1) and

$$\int_{\mathbf{R}^n} |f(y)|(\log(2+|f(y)|))dy < \infty,$$

then it is known that Lf is continuous on $\mathbf{R}^n$ (see [3, Theorem 9.1, Section 5.9]). Our second aim is to show the following theorem which deals with the continuity of logarithmic potentials of functions in Morrey spaces (see Section 3 for the definition of φ_1 and Φ).

THEOREM B. *Let* $f \in L^{1,\nu,\varphi}(\mathbf{R}^n)$ *satisfy* (1.1).

(1) *If* $0 \le \nu \le 1$ *and* $\varphi_1(1/2) < \infty$ *when* $\nu = 0$, *then*

$$|Lf(x) - Lf(z)| \le C|x-z|^{\nu}\Phi(|x-z|)$$

whenever $0 < |x-z| < 1/2$, *where the constant* C *may depend on the* L^1*-norm and* $L^{1,\nu,\varphi}$*-norm of* f.

(2) *If* $\nu > 1$, *then*

$$|Lf(x) - Lf(z)| \le C|x-z|$$

whenever $0 < |x-z| < 1/2$, *where the constant* C *may depend on the* L^1*-norm and* $L^{1,\nu,\varphi}$*-norm of* f.

In the final section, our aim is to show the following theorem which deals with the continuity of logarithmic potentials of functions in Morrey spaces of variable exponent by use of Theorems A and B (see Section 4 for the definition of φ_k).

THEOREM C. *Assume that* $p_0 = 1, 0 \le \nu \le \alpha \le n$ *and the* $(n-\alpha)$*-dimensional upper Minkowski content of* K *is finite. Let* $f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ *satisfy* (1.1).

(1) *If* $0 \le \nu \le 1$ *and*

$$\int_0^{1/2} \varphi(t)^{-1}k(t)^{-\alpha}\frac{dt}{t} < \infty$$

when $\nu = 0$, *then*

$$|Lf(x) - Lf(z)| \le C|x - z|^\nu \varphi_k(|x - z|)$$

whenever $0 < |x - z| < 1/2$, *where the constant* C *may depend on the* L^1*-norm and* $L^{p(\cdot),\nu,\varphi}$*-norm of* f.

(2) *If* $\nu > 1$, *then*

$$|Lf(x) - Lf(z)| \le C|x - z|$$

whenever $0 < |x - z| < 1/2$, *where the constant* C *may depend on the* L^1*-norm and* $L^{p(\cdot),\nu,\varphi}$*-norm of* f.

2 Morrey spaces of variable exponent

Throughout this paper, let C denote various positive constants independent of the variables in question.

We say that a positive function φ on $(0, \infty)$ is quasi-increasing if there exists a constant $C > 1$ such that

$$\varphi(s) \le C\varphi(t) \qquad \text{whenever } 0 < s \le t.$$

A positive function φ on $(0, \infty)$ is called quasi-decreasing if $\varphi(t)^{-1}$ is quasi-increasing.

From now on we consider a positive function φ on $(0, \infty)$ for which there exists a constant $C_1 > 1$ such that

(φ1) $\quad C_1^{-1}\varphi(r) \le \varphi(t) \le C_1\varphi(r) \quad$ whenever $0 < r \le t \le 2r^2$ or $0 < r^2 \le t \le 2r$,

which implies the doubling condition:

(φ2) $\quad C_2^{-1}\varphi(r) \le \varphi(t) \le C_2\varphi(r) \quad$ whenever $0 < r \le t \le 2r$

for some constant $C_2 > 1$. Our typical example of φ is of the form

$$\varphi(r) = a(\log_{(1)}(1/r))^b(\log_{(2)}(1/r))^c$$

for $r > 0$, where $a > 0$ and $b, c \in \mathbf{R}$ and $\log_{(0)} t = e$, $\log_{(1)} t = \log(e + t)$ and $\log_{(m+1)} t = \log(e + \log_{(m)} t)$ for $m = 1, 2, \ldots$.

Lemma 2.1 ([3, Lemma 3.1, Section 5.3]). *If* $\gamma > 0$, *then* $t^\gamma\varphi(t)$ *is quasi-increasing on* $(0, \infty)$.

LEMMA 2.2 (cf. [4, Lemma 2.3]). *For $0 \le \alpha \le n$ suppose the $(n-\alpha)$-dimensional upper Minkowski content of K is finite. If $\psi(t)$ is a positive quasi-increasing measurable function on $(0,\infty)$ satisfying the doubling condition, then for each bounded open set $G \subset \mathbf{R}^n$ there exists a constant $C > 0$ such that*

$$\int_{(G\cap K(\rho)\cap B(x,r))\setminus K} \psi(\delta_K(y))^{-1} dy \le C \int_0^{\min\{r,\rho\}} t^\alpha \psi(t)^{-1} \frac{dt}{t}$$

for all $x \in \mathbf{R}^n, r > 0$ and $\rho > 0$.

Proof. Since G is bounded, we have

$$|G \cap K(r)| \le C r^\alpha$$

for all $r > 0$. First consider the case $K \cap B(x, 2r) \neq \emptyset$. For each integer j, set $K_j = \{y \in G \cap B(x,r) : 2^{-j-1} \min\{r,\rho\} \le \delta_K(y) < 2^{-j} \min\{r,\rho\}\}$. Then we have by the doubling condition on ψ

$$\begin{aligned}
\int_{(G\cap K(\rho)\cap B(x,r))\setminus K} \psi(\delta_K(y))^{-1} dy &= \sum_{j=-2}^{\infty} \int_{K_j} \psi(\delta_K(y))^{-1}\, dy \\
&\le C \sum_{j=-2}^{\infty} \psi(2^{-j}\min\{r,\rho\})^{-1} |G \cap K(2^{-j}\min\{r,\rho\})| \\
&\le C \sum_{j=0}^{\infty} \psi(2^{-j}\min\{r,\rho\})^{-1} (2^{-j}\min\{r,\rho\})^\alpha \\
&\le C \int_0^{\min\{r,\rho\}} t^\alpha \psi(t)^{-1} \frac{dt}{t}
\end{aligned}$$

for all $r > 0$ and $\rho > 0$.

Next consider the case $K \cap B(x, 2r) = \emptyset$. Then note that $r < \delta_K(y) \le \rho$ if $y \in G \cap K(\rho) \cap B(x,r)$. It follows from the doubling condition on ψ that

$$\begin{aligned}
\int_{(G\cap K(\rho)\cap B(x,r))\setminus K} \psi(\delta_K(y))^{-1} dy &\le C\psi(r)^{-1} \int_{G\cap B(x,r)} dy \\
&\le C\psi(r)^{-1} \begin{cases} r^n & \text{if } r < 1, \\ |G| & \text{if } r \ge 1 \end{cases} \\
&\le C r^\alpha \psi(r)^{-1} \le C \int_0^{\min\{r,\rho\}} t^\alpha \psi(t)^{-1} \frac{dt}{t}
\end{aligned}$$

for all $r > 0$ and $\rho > 0$. Thus the required assertion is now proved. □

Consider a positive continuous nonincreasing function k on $(0,\infty)$ for which there exist $\varepsilon_0 \ge 0$ and $0 < r_0 < 1$ such that

(k) $(\log(1/r))^{-\varepsilon_0} k(r)$ is nondecreasing on $(0, r_0)$ and $k(r_0) \geq e^{\varepsilon_0}$;

(k') $k(r) > 1$ for all $r > 0$.

By (k) and (k'), we find (see [4]) that

$$C^{-1}k(r) \leq k(r^2) \leq Ck(r) \qquad \text{whenever } r > 0, \tag{2.1}$$

which implies the doubling condition on k for $r > 0$. Our typical example of k is of the form

$$k(r) = a(\log_{(1)}(1/r))^b (\log_{(2)}(1/r))^c$$

for $r \in (0, r_0)$, where $a > 0, b \geq 0$ and $c \in \mathbf{R}$ are numbers for which r_0 can be chosen so that $k(r)$ is nonincreasing on $(0, r_0)$ and satisfies (k).

In view of (k), we have the following lemma.

Lemma 2.3 ([4, Lemma 2.1]). *There exists* $0 < r^* < r_0$ *such that* $\omega(r) = \log k(r)/\log(1/r)$ *is nondecreasing on* $(0, r^*)$.

In view of this lemma, we retake the above $r_0 > 0$ so that $\log k(r)/\log(1/r)$ is nondecreasing on $(0, r_0)$.

In what follows, we set

$$\omega(r) = \omega(r_0) \qquad \text{for } r \geq r_0$$

and consider a positive continuous function $p(\cdot)$ such that $p(x) = p_0$ on K and

$$p(x) = p_0 + \omega(\delta_K(x))$$

for $\delta_K(x) > 0$, where $p_0 \geq 1$.

For $0 \leq \nu \leq n$ and a locally integrable function f on $\mathbf{R}^n$, we define its $L^{p(\cdot),\nu,\varphi}$-norm by

$$\|f\|_{p(\cdot),\nu,\varphi} = \inf \left\{ \lambda > 0 : \sup_{x \in \mathbf{R}^n, r > 0} r^{-\nu} \varphi(r) \int_{B(x,r)} \left| \frac{f(y)}{\lambda} \right|^{p(y)} dy \leq 1 \right\}.$$

We denote by $L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ the space of all locally integrable functions f on $\mathbf{R}^n$ with $\|f\|_{p(\cdot),\nu,\varphi} < \infty$. Hereafter it is natural to assume further that φ is measurable,

(φ3) $\varphi(r)$ is quasi-decreasing on $(0, \infty)$ when $\nu = 0$ and

(φ4) $\limsup_{r \to 0} \varphi(r) < \infty$ when $\nu = n$.

It is worth to note the following results.

Lemma 2.4. *Suppose $\nu > 0$. Then*

$$\int_0^r t^\nu \varphi(t)^{-1} \frac{dt}{t} \le C r^\nu \varphi(r)^{-1}$$

for all $r > 0$.

Proof. Taking $0 < \nu' < \nu$, we see by Lemma 2.1 that

$$\int_0^r t^\nu \varphi(t)^{-1} \frac{dt}{t} \le C r^{\nu-\nu'} \varphi(r)^{-1} \int_0^r t^{\nu'} \frac{dt}{t} \le C r^\nu \varphi(r)^{-1}$$

for all $r > 0$, as required. □

Lemma 2.5. *Let $0 \le \nu \le n$. If G is a bounded open set in $\mathbf{R}^n$, then there exists a constant $C > 0$ such that*

$$\int_{G \cap B(x,r)} dy \le C r^\nu \varphi(r)^{-1}$$

for all $x \in \mathbf{R}^n$ and $r > 0$.

Proof. Since $r^{n-\nu}\varphi(r)$ is quasi-increasing on $(0, 1)$ by Lemma 2.1 and (φ4), we see that

$$\int_{G \cap B(x,r)} dy \le C r^n \le C r^\nu \varphi(r)^{-1}$$

when $0 < r < 1$. If $r \ge 1$, then

$$\int_{G \cap B(x,r)} dy \le |G| \le C r^\nu \varphi(r)^{-1}$$

since $r^\nu \varphi(r)^{-1}$ is quasi-increasing on $(0, \infty)$ by Lemma 2.1 and (φ3). □

Now we prove Theorem A.

Proof of Theorem A. Let $f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ with $\|f\|_{p(\cdot),\nu,\varphi} \le 1$.

First consider the case $\nu = \alpha$. In this case, by Lemma 2.5, we have

$$\int_{G \cap B(x,r)} |f(y)|^{p_0} dy \le \int_{G \cap B(x,r)} dy + \int_{G \cap B(x,r)} |f(y)|^{p(y)} dy \le C r^\nu \varphi(r)^{-1}$$

for $x \in \mathbf{R}^n$ and $r > 0$.

Next consider the case $0 \le \nu < \alpha$. Setting $G' = \{y \in G : |f(y)| \le e\}$, we note that

$$\int_{G' \cap B(x,r)} |f(y)|^{p_0} k(|f(y)|^{-1})^{(\alpha-\nu)/p_0} dy \le C \int_{G \cap B(x,r)} dy \le C r^\nu \varphi(r)^{-1}$$

by Lemma 2.5. Consider

$$N(t) = t^{-(\alpha-\nu)/p_0}(\log(1/t))^{-\varepsilon_0(\alpha-\nu)/p_0^2-2/p_0}\varphi(t)^{-1/p_0}$$

and

$$G'' = \{y \in (K(r_0) \cap B(x,r)) \setminus K : |f(y)| < N(\delta(y))\},$$

where we set $\delta(y) = \delta_K(y)$ for simplicity; here recall that ε_0 is the constant in (k). Since $t^{p_0}k(t^{-1})^{(\alpha-\nu)/p_0}$ is nondecreasing on $(0,\infty)$, using condition (k), we have for $y \in G''$,

$$\begin{aligned} |f(y)|^{p_0}k(|f(y)|^{-1})^{(\alpha-\nu)/p_0} &\le CN(\delta(y))^{p_0}k(N(\delta(y))^{-1})^{(\alpha-\nu)/p_0} \\ &\le CN(\delta(y))^{p_0}k(\delta(y))^{(\alpha-\nu)/p_0} \\ &\le C\delta(y)^{-(\alpha-\nu)}(\log(1/\delta(y)))^{-2}\varphi(\delta(y))^{-1}. \end{aligned}$$

Hence it follows from Lemma 2.2 that

$$\begin{aligned} \int_{G''} |f(y)|^{p_0}k(|f(y)|^{-1})^{(\alpha-\nu)/p_0}dy &\le C\int_{G''} \delta(y)^{-(\alpha-\nu)}(\log(1/\delta(y)))^{-2}\varphi(\delta(y))^{-1}dy \\ &\le C\int_0^{\min\{r,r_0\}} t^{\nu}(\log(1/t))^{-2}\varphi(t)^{-1}\frac{dt}{t} \\ &\le Cr^{\nu}\varphi(r)^{-1}\int_0^{r_0}(\log(1/t))^{-2}\frac{dt}{t} \\ &\le Cr^{\nu}\varphi(r)^{-1} \end{aligned}$$

since $t^{\nu}\varphi(t)^{-1}$ is quasi-increasing on $(0,\infty)$.

If $y \in (K(r_0) \cap B(x,r)) \setminus (G' \cup G'' \cup K)$, then

$$|f(y)| \ge N(\delta(y)),$$

so that, by Lemma 2.1 and (φ1), we have

$$\begin{aligned} &|f(y)|^{-p_0/(\alpha-\nu)}(\log|f(y)|)^{-\varepsilon_0/p_0-2/(\alpha-\nu)}\varphi(|f(y)|^{-1})^{-1/(\alpha-\nu)} \\ &\le CN(\delta(y))^{-p_0/(\alpha-\nu)}(\log(1/\delta(y)))^{-\varepsilon_0/p_0-2/(\alpha-\nu)}\varphi(\delta(y))^{-1/(\alpha-\nu)} \\ &\le C\delta(y). \end{aligned}$$

Set $M(t) = t^{-p_0/(\alpha-\nu)}(\log t)^{-\varepsilon_0/p_0-2/(\alpha-\nu)}\varphi(t^{-1})^{-1/(\alpha-\nu)}$ for simplicity. Then, in view of Lemma 2.3, we see that

$$\frac{\log k(\delta(y))}{\log(1/\delta(y))} \ge \frac{\log k(CM(|f(y)|))}{\log(1/(CM(|f(y)|)))}.$$

Noting that

$$k(CM(|f(y)|)) \ge Ck(|f(y)|^{-1})$$

by (2.1) and

$$\log(1/(CM(|f(y)|))) \le (p_0/(\alpha-\nu))\log(|f(y)|) + \varepsilon(|f(y)|)$$

with $\varepsilon(r) \le C(\log(\log r) + \max\{0, \log \varphi(r^{-1})\})$, we establish

$$\begin{aligned} \frac{\log k(\delta(y))}{\log(1/\delta(y))} \log(|f(y)|) &\ge \frac{\log(Ck(|f(y)|^{-1}))}{\frac{p_0}{\alpha-\nu}\log(|f(y)|) + \varepsilon(|f(y)|)} \log(|f(y)|) \\ &\ge \frac{\alpha-\nu}{p_0} \log k(|f(y)|^{-1}) - C \end{aligned}$$

since $|f(y)| \ge e$ and $(\log k(r^{-1}))\varepsilon(r)/(\log r + \varepsilon(r))$ is bounded for $r \ge e$. Hence we have

$$\begin{aligned} |f(y)|^{p(y)-p_0} &= \exp\left(\frac{\log k(\delta(y))}{\log(1/\delta(y))} \log |f(y)|\right) \\ &\ge \exp\left(\frac{\alpha-\nu}{p_0} \log k(|f(y)|^{-1}) - C\right) \\ &= Ck(|f(y)|^{-1})^{(\alpha-\nu)/p_0}. \end{aligned}$$

Thus it follows that

$$\int_{(K(r_0)\cap B(x,r))\setminus(G'\cup G'')} |f(y)|^{p_0} k(|f(y)|^{-1})^{(\alpha-\nu)/p_0} dy \le C \int_{B(x,r)} |f(y)|^{p(y)}\, dy \le Cr^\nu \varphi(r)^{-1}$$

since $|K| = 0$ for $\alpha > 0$.

Finally, since $p(y) = p_0 + \omega(r_0) > p_0$ when $\delta(y) \ge r_0$, we find by Lemma 2.5

$$\begin{aligned} &\int_{(G\cap B(x,r))\setminus K(r_0)} |f(y)|^{p_0} k(|f(y)|^{-1})^{(\alpha-\nu)/p_0} dy \\ &\qquad \le C \int_{B(x,r)} |f(y)|^{p(y)}\, dy + C \int_{G\cap B(x,r)} dy \le Cr^\nu \varphi(r)^{-1}. \end{aligned}$$

The required assertion is now proved. □

REMARK 2.6. We set $\Psi_k(t) = t^{p_0} k(t^{-1})^{(\alpha-\nu)/p_0}$ for $t > 0$, which satisfies the doubling condition by (2.1). For $0 \le \nu \le n$ and a locally integrable function f on $\mathbf{R}^n$, we define its quasi-norm by

$$\|f\|_{\Psi_k,\nu,\varphi} = \inf\left\{\lambda > 0 : \sup_{x\in\mathbf{R}^n, r>0} r^{-\nu}\varphi(r) \int_{B(x,r)} \Psi_k\left(\left|\frac{f(y)}{\lambda}\right|\right) dy \le 1\right\}$$

(see [5]). We denote by $L^{\Psi_k,\nu,\varphi}(\mathbf{R}^n)$ the space of all locally integrable functions f on $\mathbf{R}^n$ with $\|f\|_{\Psi_k,\nu,\varphi} < \infty$. Then it follows from Theorem A that for each bounded open set G,

$$\|f\|_{\Psi_k,\nu,\varphi} \le C\|f\|_{p(\cdot),\nu,\varphi} \qquad \text{whenever } f \in L^{p(\cdot),\nu,\varphi}(G),$$

where $L^{p(\cdot),\nu,\varphi}(G) = \{f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n) : f = 0 \text{ outside } G\}$.

Remark 2.7. Set $K = \{0\}$. Let

$$k(t) = (\log_{(m)}(1/t))^a$$

for $a > 0$ and an integer $m \geq 0$ and

$$\varphi(t) = (\log_{(\ell)}(1/t))^b$$

for an integer $\ell \geq 1$ and $b > 0$. Then

$$p(x) = p_0 + \frac{a \log(\log_{(m)}(1/|x|))}{\log(1/|x|)}$$

for $x \in B(0, r_0) \setminus \{0\}$. Then we can find $f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ satisfying

$$\int_{B(0,r)} |f(y)|^{p_0} (\log_{(m)} |f(y)|)^{a(n-\nu)/p_0} dy \geq C r^\nu (\log_{(\ell)}(1/r))^{-b}$$

for all $0 < r < r_0$. This shows that the conclusion of Theorem A is sharp when $K = \{0\}$ and k, φ are as above.

For this purpose, in case $0 < \nu \leq n$, we consider the function

$$f(y) = |y|^{-(n-\nu)/p_0} (\log_{(\ell)}(1/|y|))^{-b/p_0} (\log_{(m)}(1/|y|))^{-a(n-\nu)/p_0^2}$$

for $y \in B(0, r_0)$; set $f(y) = 0$ when $|y| \geq r_0$. Then, as in the proof of Theorem A, we note that

$$\begin{aligned} f(y)^{p(y)-p_0} &= \exp\left(\frac{a \log(\log_{(m)}(1/|y|))}{\log(1/|y|)} \log f(y) \right) \\ &\leq \exp\left(\frac{a(n-\nu)}{p_0} \log(\log_{(m)}(1/|y|)) + C \right) \\ &\leq C(\log_{(m)}(1/|y|))^{a(n-\nu)/p_0}. \end{aligned}$$

We see by Lemma 2.4 that

$$\begin{aligned} \int_{B(x,r)} f(y)^{p(y)}\, dy &\leq C \int_{B(0,r_0)\cap B(x,r)} |y|^{-(n-\nu)} (\log_{(\ell)}(1/|y|))^{-b} dy \\ &\leq C \int_{B(0,r)} |y|^{-(n-\nu)} (\log_{(\ell)}(1/|y|))^{-b} dy \\ &\leq C \int_0^r t^\nu (\log_{(\ell)}(1/t))^{-b} \frac{dt}{t} \\ &\leq C r^\nu (\log_{(\ell)}(1/r))^{-b} \end{aligned}$$

for all $x \in \mathbf{R}^n$ and $r > 0$, which implies that $f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$. Further, we have

$$\begin{aligned} \int_{B(0,r)} f(y)^{p_0} (\log_{(m)} f(y))^{a(n-\nu)/p_0} dy &= C \int_0^r t^\nu (\log_{(\ell)}(1/t))^{-b} \frac{dt}{t} \\ &\geq C r^\nu (\log_{(\ell)}(1/r))^{-b} \end{aligned}$$

for all $0 < r < r_0$.

In case $\nu = 0$, since

$$C^{-1}(\log_{(\ell)}(1/r))^{-b} \le \int_0^r (\log_{(\ell)}(1/t))^{-(b+1)} \prod_{j=1}^{\ell-1} (\log_{(j)}(1/t))^{-1} \frac{dt}{t} \le C(\log_{(\ell)}(1/r))^{-b}$$

for $0 < r < r_0$, we have only to replace f by

$$f(y) = |y|^{-n/p_0} (\log_{(\ell)}(1/|y|))^{-(b+1)/p_0} (\log_{(m)}(1/|y|))^{-an/p_0^2} \prod_{j=1}^{\ell-1} (\log_{(j)}(1/|y|))^{-1/p_0}$$

for $y \in B(0, r_0)$. Here we used the convention $\prod_{j=1}^{0} a_j = 1$.

We show another imbedding from $L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ to the Morrey space $L^{p_0,\nu,\Phi_k}(G)$, where $\Phi_k(r) = \varphi(r)k(r)^{(\alpha-\nu)/p_0}$ and G is a bounded open set in $\mathbf{R}^n$.

Theorem 2.8. *Suppose $0 \le \nu \le \alpha \le n, 0 \le \kappa \le (\alpha - \nu)/p_0$ and the $(n - \alpha)$-dimensional upper Minkowski content of K is finite. For each bounded open set $G \subset \mathbf{R}^n$ there exists a constant $C > 0$ such that*

$$\int_{G \cap B(x,r)} |f(y)|^{p_0} k(|f(y)|^{-1})^{(\alpha-\nu)/p_0 - \kappa} dy \le C r^{\nu} \varphi(r)^{-1} k(r)^{-\kappa}$$

for all $x \in \mathbf{R}^n$, $r > 0$ and $f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ with $\|f\|_{p(\cdot),\nu,\varphi} \le 1$.

Proof. Since the case $\nu = \alpha$ follows readily from Theorem A, we consider the case $0 \le \nu < \alpha$. Take $\sigma > 0$ such that $\alpha - p_0\sigma > \nu$. Then, since k is nonincreasing, we have

$$\begin{aligned}
&\int_{G \cap B(x,r)} |f(y)|^{p_0} k(|f(y)|^{-1})^{(\alpha-\nu)/p_0 - \kappa} dy \\
&= \int_{\{y \in G \cap B(x,r) : |f(y)| \le r^{-\sigma}\}} |f(y)|^{p_0} k(|f(y)|^{-1})^{(\alpha-\nu)/p_0 - \kappa} dy \\
&\qquad + \int_{\{y \in G \cap B(x,r) : |f(y)| > r^{-\sigma}\}} |f(y)|^{p_0} k(|f(y)|^{-1})^{(\alpha-\nu)/p_0 - \kappa} dy \\
&\le r^{-\sigma p_0} k(r^{\sigma})^{(\alpha-\nu)/p_0 - \kappa} \int_{G \cap B(x,r)} dy \\
&\qquad + k(r^{\sigma})^{-\kappa} \int_{G \cap B(x,r)} |f(y)|^{p_0} k(|f(y)|^{-1})^{(\alpha-\nu)/p_0} dy.
\end{aligned}$$

By Lemma 2.5 with $r^{\nu}\varphi(r)^{-1}$ replaced by $r^{\nu+\sigma p_0}\varphi(r)^{-1}k(r)^{-(\alpha-\nu)/p_0}$, Theorem A and (2.1), we have

$$\int_{G \cap B(x,r)} |f(y)|^{p_0} k(|f(y)|^{-1})^{(\alpha-\nu)/p_0 - \kappa} dy \le C r^{\nu} \varphi(r)^{-1} k(r)^{-\kappa}$$

for all $x \in \mathbf{R}^n$ and $r > 0$, as required. □

REMARK 2.9. Let $0 \le \nu \le \alpha \le n$. Set $\Phi_k(t) = \varphi(t)k(t)^{(\alpha-\nu)/p_0}$ for $t > 0$. Then Theorem 2.8 implies that

$$\|f\|_{p_0,\nu,\Phi_k} \le C\|f\|_{p(\cdot),\nu,\varphi} \qquad \text{whenever } f \in L^{p(\cdot),\nu,\varphi}(G)$$

for each bounded open set $G \subset \mathbf{R}^n$.

Here we recall that

$$\|f\|_{p_0,\nu,\Phi_k} = \sup_{x\in\mathbf{R}^n, r>0} \left(r^{-\nu}\Phi_k(r) \int_{B(x,r)} |f(y)|^{p_0} dy \right)^{1/p_0}.$$

3 Continuity of logarithmic potentials of functions in Morrey spaces

For the function φ as above, we consider a function φ_1 on $(0, 1/2]$ and a nonincreasing function φ_2 on $(0, 1/2]$ such that

$$\varphi_1(r) = \int_0^r \varphi(t)^{-1}\frac{dt}{t} \qquad \text{and} \qquad \varphi_2(r) = \int_r^1 \varphi(t)^{-1}\frac{dt}{t}.$$

We set

$$\Phi(r) = \begin{cases} \varphi_1(r) & \text{if } \nu = 0, \\ \varphi(r)^{-1} & \text{if } 0 < \nu < 1, \\ \varphi_2(r) & \text{if } \nu = 1 \end{cases}$$

for $0 < r \le 1/2$. In view of (φ2), note that

$$\varphi_1(r) \ge C\varphi(r)^{-1} \text{ and } \varphi_2(r) \ge C\varphi(r)^{-1}$$

for $0 < r \le 1/2$.

REMARK 3.1. Let $\varphi(t) = (\log_{(1)}(1/t))^{\beta}$ for $\beta \in \mathbf{R}$. Then

$$\varphi_1(r) \le C(\log_{(1)}(1/r))^{-\beta+1} \qquad \text{if } \beta > 1$$

and

$$\varphi_2(r) \le C \begin{cases} (\log_{(1)}(1/r))^{-\beta+1} & \text{if } \beta < 1, \\ \log_{(2)}(1/r) & \text{if } \beta = 1, \\ 1 & \text{if } \beta > 1 \end{cases}$$

for $0 < r \le 1/2$.

Our aim in this section is to give a proof of Theorem B, which deals with the continuity of logarithmic potentials of functions in Morrey spaces of constant exponent. For the proof we prepare the following two lemmas.

LEMMA 3.2. *Let $0 \le \nu \le n$. If $f \in L^{1,\nu,\varphi}(\mathbf{R}^n)$, then there exists a constant $C > 0$ such that*

$$\int_{B(x,\delta)} (\log(\delta/|x-y|))\,|f(y)|dy \le C \begin{cases} \delta^\nu \Phi(\delta) & \text{if } 0 \le \nu \le 1, \\ \delta & \text{if } \nu > 1 \end{cases}$$

for all $x \in \mathbf{R}^n$ and $0 < \delta < 1/2$, where the constant C may depend on the $L^{1,\nu,\varphi}$-norm of f.

Proof. Let $f \in L^{1,\nu,\varphi}(\mathbf{R}^n)$. By Lemma 2.4, we have

$$\begin{aligned} \int_{B(x,\delta)} (\log(\delta/|x-y|))\,|f(y)|dy &= \int_0^\delta (\log(\delta/t)) \left(\int_{\partial B(x,t)} |f(y)|dS(y) \right) dt \\ &\le \int_0^\delta \left(\int_{B(x,t)} |f(y)|dy \right) \frac{dt}{t} \\ &\le C \int_0^\delta t^\nu \varphi(t)^{-1} \frac{dt}{t} \\ &\le C \begin{cases} \delta^\nu \Phi(\delta) & \text{if } 0 \le \nu \le 1, \\ \delta & \text{if } \nu > 1 \end{cases} \end{aligned}$$

for all $x \in \mathbf{R}^n$ and $0 < \delta < 1/2$, as required. □

LEMMA 3.3. *Let $0 \le \nu \le n$. If $f \in L^{1,\nu,\varphi}(\mathbf{R}^n)$ satisfies (1.1), then*

$$\int_{\mathbf{R}^n \setminus B(x,\delta)} |x-y|^{-1} |f(y)|dy \le C \begin{cases} \delta^{\nu-1} \Phi(\delta) & \text{if } 0 \le \nu \le 1, \\ 1 & \text{if } \nu > 1 \end{cases}$$

for all $x \in \mathbf{R}^n$ and $0 < \delta < 1/2$, where the constant C may depend on the L^1-norm and $L^{1,\nu,\varphi}$-norm of f.

Proof. Let $f \in L^{1,\nu,\varphi}(\mathbf{R}^n)$ satisfy (1.1). For $x \in \mathbf{R}^n$ and $0 < \delta < 1/2$, we find

$$\begin{aligned} \int_{\mathbf{R}^n \setminus B(x,\delta)} |x-y|^{-1} |f(y)|dy &= \int_\delta^\infty t^{-1} \left(\int_{\partial B(x,t)} |f(y)|dS(y) \right) dt \\ &\le \int_\delta^\infty t^{-1} \left(\int_{B(x,t)} |f(y)|dy \right) \frac{dt}{t} \\ &\le C \int_\delta^1 t^{\nu-1} \varphi(t)^{-1} \frac{dt}{t} + \int_{\mathbf{R}^n} |f(y)|dy \int_1^\infty t^{-1} \frac{dt}{t} \\ &\le C \int_\delta^1 t^{\nu-1} \varphi(t)^{-1} \frac{dt}{t} + \int_{\mathbf{R}^n} |f(y)|dy \\ &\le C \begin{cases} \delta^{\nu-1} \Phi(\delta) & \text{if } 0 \le \nu \le 1, \\ 1 & \text{if } \nu > 1 \end{cases} \end{aligned}$$

since $f \in L^1(\mathbf{R}^n)$ by (1.1). □

Now we are ready to prove Theorem B.

Proof of Theorem B. Let $f \in L^{1,\nu,\varphi}(\mathbf{R}^n)$ satisfy (1.1). By Lemma 3.2 and (φ2), we have

$$
\begin{aligned}
&\int_{B(x,2|x-z|)} |\log(1/|x-y|) - \log(1/|z-y|)|\,|f(y)|dy \\
\le\ & \int_{B(x,2|x-z|)} (\log(3|x-z|/|x-y|))\,|f(y)|dy \\
&\qquad + \int_{B(z,3|x-z|)} (\log(3|x-z|/|z-y|))\,|f(y)|dy \\
\le\ & C \begin{cases} |x-z|^{\nu}\Phi(|x-z|) & \text{if } 0 \le \nu \le 1, \\ |x-z| & \text{if } \nu > 1 \end{cases} \qquad (3.1)
\end{aligned}
$$

for $|x-z| < 1/6$.

On the other hand, we see from the mean value theorem for analysis, Lemma 3.3 and (φ2) that

$$
\begin{aligned}
&\int_{\mathbf{R}^n \setminus B(x,2|x-z|)} |\log(1/|x-y|) - \log(1/|z-y|)|\,|f(y)|dy \\
\le\ & C|x-z| \int_{\mathbf{R}^n \setminus B(x,2|x-z|)} |x-y|^{-1}|f(y)|dy \\
\le\ & C \begin{cases} |x-z|^{\nu}\Phi(|x-z|) & \text{if } 0 \le \nu \le 1, \\ |x-z| & \text{if } \nu > 1 \end{cases} \qquad (3.2)
\end{aligned}
$$

for $|x-z| < 1/6$.

Hence it follows from (3.1) and (3.2) that

$$
|Lf(x) - Lf(z)| \le C \begin{cases} |x-z|^{\nu}\Phi(|x-z|) & \text{if } 0 \le \nu \le 1, \\ |x-z| & \text{if } \nu > 1 \end{cases}
$$

for $|x-z| < 1/6$, which proves the theorem. □

For $p_0 > 1$, we set $\nu_{p_0} = \nu/p_0 + n/p_0'$ and

$$
\Phi_{p_0}(r) = \begin{cases} \varphi(r)^{-1/p_0} & \text{if } \nu_{p_0} < 1, \\ \int_r^1 \varphi(t)^{-1/p_0} \frac{dt}{t} & \text{if } \nu_{p_0} = 1 \end{cases}
$$

for $0 < r \le 1/2$.

Corollary 3.4. *Suppose $p_0 > 1$. Let $f \in L^{p_0,\nu,\varphi}(\mathbf{R}^n)$ satisfy (1.1).*

(1) *If $\nu_{p_0} \le 1$, then*

$$
|Lf(x) - Lf(z)| \le C|x-z|^{\nu_{p_0}}\Phi_{p_0}(|x-z|)
$$

whenever $0 < |x-z| < 1/2$, where the constant C may depend on the L^1-norm and $L^{p_0,\nu,\varphi}$-norm of f.

(2) *If* $\nu_{p_0} > 1$, *then*

$$|Lf(x) - Lf(z)| \le C|x - z|$$

whenever $0 < |x - z| < 1/2$, *where the constant* C *may depend on the* L^1*-norm and* $L^{p_0,\nu,\varphi}$*-norm of* f.

Proof. Let $f \in L^{p_0,\nu,\varphi}(\mathbf{R}^n)$ satisfy (1.1). Then Jensen's theorem gives

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy \le \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|^{p_0}dy\right)^{1/p_0} \le Cr^{-(n-\nu)/p_0}\varphi(r)^{-1/p_0}$$

for all $x \in \mathbf{R}^n$ and $r > 0$, so that $f \in L^{1,\nu_{p_0},\varphi^{1/p_0}}(\mathbf{R}^n)$. Hence, applying Theorem B with ν and $\varphi(r)$ replaced by ν_{p_0} and $\varphi(r)^{1/p_0}$, we obtain the required assertion. □

REMARK 3.5. In the case $\nu = 0$, we need the condition $\varphi_1(1/2) < \infty$ for the continuity of Lf.

For this, consider the functions

$$\varphi(t) = (\log_{(1)}(1/t))^a$$

and

$$f(y) = |y|^{-n}(\log(1/|y|))^{-2}\chi_{B(0,1/2)}(y),$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbf{R}^n$. If $a \le 1$, then we see that $\varphi_1(1/2) = \infty$,

(1) $Lf(0) = \int (\log(1/|y|))f(y)dy = \infty$; and

(2) $\int_{B(x,r)} f(y)dy \le C(\log_{(1)}(1/r))^{-1} \le C\varphi(r)^{-1}$ for all $x \in \mathbf{R}^n$ and $r > 0$.

This implies that $f \in L^{1,0,\varphi}(\mathbf{R}^n)$, but Lf is not continuous at the origin.

REMARK 3.6. We show that Theorem B is sharp. In fact, if $0 < \nu \le 1$, then, letting $\varphi(t) = (\log_{(m)}(1/t))^a$ for an integer $m \ge 0$ and $a \in \mathbf{R}$, we can find $f \in L^{1,\nu,\varphi}(\mathbf{R}^n)$ satisfying

$$|Lf(0) - Lf(x^{(i)})| \ge C|x^{(i)}|^\nu\Phi(|x^{(i)}|)$$

for some sequence $\{x^{(i)}\}$ which tends to the origin.

To show this, we consider the sequence $x^{(i)} = (0, 0, \ldots, -1/i)$ and the function

$$f(y) = |y|^{-(n-\nu)}\varphi(|y|)^{-1}\chi_{\Gamma^+}(y),$$

where $\Gamma^+ = \{y = (y', y_n) \in B(0, 1/2) : |y'| < y_n/2\}$. Then, by Lemma 2.4, we have

$$\int_{B(x,r)} f(y)dy \le C \int_0^r t^\nu \varphi(t)^{-1} \frac{dt}{t} \le C r^\nu \varphi(r)^{-1}$$

for all $x \in \mathbf{R}^n$ and $r > 0$, so that $f \in L^{1,\nu,\varphi}(\mathbf{R}^n)$. Further, we have

$$\begin{aligned}
&|Lf(0) - Lf(x^{(i)})| \\
= &\int_{\Gamma^+} \left(\log(1/|y|) - \log(1/|x^{(i)} - y|)\right) f(y)dy \\
\ge &\int_{\Gamma^+ \cap B(0,|x^{(i)}|/2)} \left(\log(1/|y|) - \log(2/|x^{(i)}|)\right) f(y)dy \\
&\qquad + \int_{\Gamma^+ \setminus B(0,|x^{(i)}|/2)} \left(\log(1/|y|) - \log(1/|x^{(i)} - y|)\right) f(y)dy \\
\ge &\ C \int_{\Gamma^+ \cap B(0,|x^{(i)}|/3)} f(y)dy + C|x^{(i)}| \int_{\Gamma^+ \setminus B(0,|x^{(i)}|/2)} |y|^{-1} f(y)dy \\
\ge &\ C \int_0^{|x^{(i)}|/3} t^\nu \varphi(t)^{-1} \frac{dt}{t} + C|x^{(i)}| \int_{|x^{(i)}|/2}^{1/2} t^{\nu-1} \varphi(t)^{-1} \frac{dt}{t} \\
\ge &\ C|x^{(i)}|^\nu \Phi(|x^{(i)}|).
\end{aligned}$$

If $\nu = 0$, then, letting

$$\varphi(t) = (\log_{(m)}(1/t))^a \prod_{j=1}^{m-1} \log_{(j)}(1/t)$$

for $a > 1$ and an integer $m \ge 1$, we have only to consider the function

$$f(y) = |y|^{-n} (\log(1/|y|))^{-1} \varphi(|y|)^{-1} \chi_{\Gamma^+}(y).$$

Then we can show as above that $f \in L^{1,0,\varphi}(\mathbf{R}^n)$ and

$$\begin{aligned}
|Lf(0) - Lf(x^{(i)})| &\ge C \int_{\Gamma^+ \cap B(0,|x^{(i)}|^2/2)} (\log(1/|y|)) f(y)dy \\
&\ge C(\log_{(m)}(1/|x^{(i)}|))^{-a+1}
\end{aligned}$$

for $x^{(i)} = (0, 0, \ldots, -1/i)$.

4 Continuity of logarithmic potentials of functions in Morrey spaces of variable exponent

We set

$$\varphi_k(r) = \begin{cases} \int_0^r \varphi(t)^{-1} k(t)^{-\alpha} \frac{dt}{t} & \text{if } \nu = 0, \\ \varphi(r)^{-1} k(r)^{-(\alpha-\nu)} & \text{if } 0 < \nu < 1, \\ \int_r^1 \varphi(t)^{-1} k(t)^{-(\alpha-1)} \frac{dt}{t} & \text{if } \nu = 1 \end{cases}$$

for $0 < r \le 1/2$.

Our final goal is to give a proof of Theorem C, which deals with the continuity of logarithmic potentials of functions in Morrey spaces of variable exponent.

Proof of Theorem C. Let $f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ satisfy (1.1). We set

$$f = f\chi_{K(r_0)} + f\chi_{\mathbf{R}^n \setminus K(r_0)} = f_1 + f_2.$$

Since $K(r_0)$ is a bounded open set, we have by Theorem 2.8 with $\kappa = \alpha - \nu$

$$\int_{B(x,r)} |f_1(y)| dy \le C r^{\nu} \varphi(r)^{-1} k(r)^{-(\alpha-\nu)}$$

for all $x \in \mathbf{R}^n$ and $r > 0$. Applying Theorem B with $\varphi(r)$ replaced by $\varphi(r)k(r)^{\alpha-\nu}$, we have

$$|Lf_1(x) - Lf_1(z)| \le C \begin{cases} |x-z|^{\nu}\varphi_k(|x-z|) & \text{if } 0 \le \nu \le 1, \\ |x-z| & \text{if } \nu > 1 \end{cases}$$

for $0 < |x-z| < 1/2$. On the other hand, since $p(y) = p_1 := 1 + \omega(r_0)$ for $y \in \mathbf{R}^n \setminus K(r_0)$, we have

$$\int_{B(x,r)} |f_2(y)|^{p_1} dy \le C r^{\nu} \varphi(r)^{-1}$$

for all $x \in \mathbf{R}^n$ and $r > 0$. Then, by Corollary 3.4, we have

$$|Lf_2(x) - Lf_2(z)| \le C \begin{cases} |x-z|^{\nu_{p_1}}\Phi_{p_1}(|x-z|) & \text{if } \nu_{p_1} \le 1, \\ |x-z| & \text{if } \nu_{p_1} > 1 \end{cases}$$

for $0 < |x-z| < 1/2$. Hence, we obtain that

$$|Lf(x) - Lf(z)| \le C \begin{cases} |x-z|^{\nu}\varphi_k(|x-z|) & \text{if } 0 \le \nu \le 1, \\ |x-z| & \text{if } \nu > 1 \end{cases}$$

for $0 < |x-z| < 1/2$ since $\nu_{p_1} \ge \nu$ and $r^{\nu_{p_1}-\nu}\Phi_{p_1}(r)\varphi_k(r)^{-1}$ is quasi-increasing on (0,1/2) for $\nu_{p_1} \le 1$. $\square$

For $p_0 > 1$, we set

$$\varphi_{k,p_0}(r) = \begin{cases} \varphi(r)^{-1/p_0} k(r)^{-(\alpha-\nu)/p_0^2} & \text{if } \nu_{p_0} < 1, \\ \int_r^1 \varphi(t)^{-1/p_0} k(t)^{-(\alpha-1)/p_0^2} \frac{dt}{t} & \text{if } \nu_{p_0} = 1 \end{cases}$$

for $0 < r \le 1/2$.

Using Corollary 3.4 instead of Theorem B, we can similarly show the following corollary.

Corollary 4.1. *Suppose* $p_0 > 1$. *Assume that* $0 \le \nu \le \alpha \le n$ *and the* $(n-\alpha)$-*dimensional upper Minkowski content of* K *is finite. Let* $f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ *satisfy* (1.1).

(1) *If* $\nu_{p_0} \le 1$, *then*

$$|Lf(x) - Lf(z)| \le C|x-z|^{\nu_{p_0}} \varphi_{k,p_0}(|x-z|)$$

whenever $0 < |x-z| < 1/2$, *where the constant* C *may depend on the* L^1-*norm and* $L^{p(\cdot),\nu,\varphi}$-*norm of* f.

(2) *If* $\nu_{p_0} > 1$, *then*

$$|Lf(x) - Lf(z)| \le C|x-z|$$

whenever $0 < |x-z| < 1/2$, *where the constant* C *may depend on the* L^1-*norm and* $L^{p(\cdot),\nu,\varphi}$-*norm of* f.

From now on we consider

$$k(r) = e^b (\log_{(1)}(1/r))^a$$

and

$$\varphi(r) = (\log_{(1)}(1/r))^\beta$$

for $a \ge 0, b \ge 0, \beta \in \mathbf{R}$ and $r > 0$, where $\beta \ge 0$ when $\nu = 0$ and $\beta \le 0$ when $\nu = n$. Then, letting $A = a(n-\nu) + \beta$, we see that

$$\varphi_k(r) \le C\Psi(r),$$

where

$$\Psi(r) = \begin{cases} (\log_{(1)}(1/r))^{-A+1} & \text{if } \nu = 0, \\ (\log_{(1)}(1/r))^{-A} & \text{if } 0 < \nu < 1, \\ (\log_{(1)}(1/r))^{-A+1} & \text{if } \nu = 1 \text{ and } A < 1, \\ \log_{(2)}(1/r) & \text{if } \nu = 1 \text{ and } A = 1, \\ 1 & \text{if } \nu = 1 \text{ and } A > 1 \end{cases}$$

for $0 < r \le 1/2$.

By Theorem C with $K = \{x_0\}$ and Remark 3.1, we have the following result.

Corollary 4.2. *Let* $\omega_{a,b}(0) = 0$,

$$\omega_{a,b}(r) = \frac{a \log(\log_{(1)}(1/r))}{\log(1/r)} + \frac{b}{\log(1/r)}$$

for $0 < r < r_0$ *and* $\omega_{a,b}(r) = \omega_{a,b}(r_0)$ *for* $r \ge r_0$, *where the number* r_0 *is chosen so that* $\omega_{a,b}(r)$ *is nondecreasing on* $(0, r_0)$ *and satisfies (k). Set*

$$p(x) = 1 + \omega_{a,b}(|x_0 - x|).$$

Let $f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ *satisfy* (1.1). *If* $0 \le \nu \le 1$ *and* $A > 1$ *when* $\nu = 0$, *then*

$$|Lf(x) - Lf(z)| \le C|x-z|^\nu \Psi(|x-z|)$$

whenever $0 < |x-z| < 1/2$, *where the constant* C *may depend on the* L^1*-norm and* $L^{p(\cdot),\nu,\varphi}$*-norm of* f.

We have three remarks for Corollary 4.2.

Remark 4.3. When $\nu = \beta = 0$, we showed that

$$\int_G |f(y)|(\log(1+|f(y)|))^{an} dy < \infty$$

for all $f \in L^{p(\cdot)}(\mathbf{R}^n)$ (see Theorem A). It follows from [3, Theorem 9.1, Section 5.9] that Lf is continuous on $\mathbf{R}^n$ even when $\nu = 0$ and $A = an = 1$, in case $\varphi(r) = 1$ for which $\varphi_k(r) = \infty$.

Remark 4.4. In case $\nu = 0$ and $an < 1$, we need the condition $A > 1$ for the continuity of Lf.

For this, set $x_0 = 0$ and consider the function

$$f(y) = |y|^{-n}(\log(1/|y|))^{-2}\chi_{B(0,1/2)}(y).$$

Note that $A = an + \beta$. Thus, if $A \le 1$, then as in Remark 2.7, we see that

(1) $Lf(0) = \displaystyle\int (\log(1/|y|)) f(y) dy = \infty$; and

(2) $\displaystyle\int_{B(x,r)} f(y)^{p(y)} dy \le C \int_{B(x,r)\cap B(0,1/2)} |y|^{-n}(\log(1/|y|))^{an-2} dy \le C(\log(1/r))^{an-1} \le C(\log_{(1)}(1/r))^{-\beta}$ for all $x \in \mathbf{R}^n$ and $0 < r < 1/2$.

This implies that $f \in L^{p(\cdot),0,\varphi}(\mathbf{R}^n)$, but Lf is not continuous at the origin.

Remark 4.5. Corollary 4.2 is seen to be sharp in the following sense: in case $x_0 = 0$ and $0 < \nu \le 1$, we can find $f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ satisfying

$$|Lf(0) - Lf(x^{(i)})| \ge C|x^{(i)}|^\nu \Psi(|x^{(i)}|)$$

for some sequence $\{x^{(i)}\}$ which tends to the origin.

For this purpose, we consider the sequence $x^{(i)} = (0, 0, \ldots, -1/i)$ and the function

$$f(y) = |y|^{-(n-\nu)}(\log(1/|y|))^{-A}\chi_{\Gamma^+}(y),$$

where Γ^+ is as in Remark 3.6. Then, as in Remark 2.7, we have

$$\int_{B(x,r)} f(y)^{p(y)} dy \le C \int_0^r t^{\nu} (\log(1/t))^{-\beta} \frac{dt}{t} \le C r^{\nu} (\log(1/r))^{-\beta}$$

for all $x \in \mathbf{R}^n$ and $0 < r < 1/2$, which implies that $f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$. Further, we can show the required property as in Remark 3.6.

Similarly, for $\nu = 0$, we can find $f \in L^{p(\cdot),0,\varphi}(\mathbf{R}^n)$ satisfying

$$|Lf(0) - Lf(x^{(i)})| \ge C(\log(1/|x^{(i)}|))^{-A+1}$$

for some sequence $\{x^{(i)}\}$ which tends to the origin.

By Theorem C with $K = \partial B(0,1)$, we have the following corollary.

Corollary 4.6. Let

$$p(x) = 1 + \omega_{a,b}(1 - |x|),$$

where $\omega_{a,b}(\cdot)$ is as in Corollary 4.2. Set $A_S = a(1-\nu) + \beta$ and

$$\Psi_S(r) = \begin{cases} (\log_{(1)}(1/r))^{-A_S+1} & \text{if } \nu = 0, \\ (\log_{(1)}(1/r))^{-A_S} & \text{if } 0 < \nu < 1, \\ (\log_{(1)}(1/r))^{-A_S+1} & \text{if } \nu = 1 \text{ and } A_S < 1, \\ \log_{(2)}(1/r) & \text{if } \nu = 1 \text{ and } A_S = 1, \\ 1 & \text{if } \nu = 1 \text{ and } A_S > 1 \end{cases}$$

for $0 < r \le 1/2$.

Let $f \in L^{p(\cdot),\nu,\varphi}(\mathbf{R}^n)$ satisfy (1.1). If $0 \le \nu \le 1$ and $A_S > 1$ when $\nu = 0$, then

$$|Lf(x) - Lf(z)| \le C|x - z|^{\nu} \Psi_S(|x - z|)$$

whenever $0 < |x - z| < 1/2$, where the constant C may depend on the L^1-norm and $L^{p(\cdot),\nu,\varphi}$-norm of f.

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