Compact embeddings in the generalized Sobolev space $W_0^{1,p(\cdot)}(G)$ and existence of solutions for nonlinear elliptic problems

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Abstract

Our aim in this paper is to deal with the compact embedding in the generalized Sobolev space $W_0^{1,p(\cdot)}(G)$ with a variable exponent satisfying the log-Hölder condition. As an application, we find a nontrivial weak solution of the nonlinear elliptic problem

$$-\mathrm{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right) = f(x,u(x)) \quad \text{in } G, \qquad u(x)=0 \quad \text{on } \partial G,$$

which is an extension of Fan-Zhang [3, Theorem 4.7].

1 Introduction

Let $\mathbf{R}^N$ be the N-dimensional Euclidean space and let G be an open bounded set in $\mathbf{R}^N$. Following Orlicz [10] and Kováčik-Rákosník [7], for a function $p(\cdot) : G \to [1,\infty)$, which is called a variable exponent, we define the $L^{p(\cdot)}$-norm of a measurable function u on G by

$$\|u\|_{L^{p(\cdot)}(G)} = \inf\left\{\lambda > 0 : \int_G \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}$$

and denote by $L^{p(\cdot)}(G)$ the family of all measurable functions u with $\|u\|_{L^{p(\cdot)}(G)} < \infty$. In recent years, the generalized Lebesgue spaces have attracted more and more attention, in connection with the study of elasticity, fluid mechanics and differential equations with $p(\cdot)$-growth; see Růžička [11]. Further we denote by $W^{1,p(\cdot)}(G)$ the family of all measurable functions u on G such that

$$\|u\|_{W^{1,p(\cdot)}(G)} = \|u\|_{L^{p(\cdot)}(G)} + \|\nabla u\|_{L^{p(\cdot)}(G)} < \infty$$

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and denote by $W_0^{1,p(\cdot)}(G)$ the closure of $C_0^\infty(G)$ in $W^{1,p(\cdot)}(G)$.

We consider a positive nondecreasing continuous function φ on $(0,\infty)$ for which there exist $\varepsilon_0 \ge 0$ and $0 < r_0 < 1$ such that

(φ) $(\log(1/r))^{-\varepsilon_0}\varphi(1/r)$ is nondecreasing on $(0, r_0)$ and $\varphi(1/r_0) \ge e^{\varepsilon_0}$;

(φ') $\varphi(r) > 1$ for all $r > 0$.

Our typical example of φ is of the form

$$\varphi(r) = a(\log(\beta_0 + r))^b(\log(\beta_0 + \log(\beta_0 + r)))^c$$

for $r \ge 0$, where the constants $a > 0$, $b \ge 0$, $c \in \mathbf{R}$ and $\beta_0 \ge e$ are chosen so that $\varphi(r)$ is nondecreasing on $(0,\infty)$.

Throughout this paper, let us assume that our variable exponent $p(\cdot)$ is a positive continuous function on G satisfying :

(p1) $1 < p_- = \inf_{x\in G} p(x) \le \sup_{x\in G} p(x) = p_+ < N$;

(p2) $|p(x) - p(y)| \le \dfrac{\log \varphi(|x-y|^{-1})}{\log(e + |x-y|^{-1})}$ whenever $x, y \in G$.

When $\varphi(\cdot)$ is a bounded function on $(0,\infty)$, we say that $p(\cdot)$ satisfies the log-Hölder condition on G, that is,

$$|p(x) - p(y)| \le \frac{c}{\log(e + |x-y|^{-1})}$$

for all $x, y \in G$, where c is a positive constant. We know the fact that if $p(\cdot)$ satisfies the log-Hölder condition on G, then the embedding from $W_0^{1,p(\cdot)}(G)$ to $L^{p^*(\cdot)}(G)$ is bounded, where

$$1/p^*(x) = 1/p(x) - 1/N$$

(see Diening [1, Theorem 5.2]). Further, we know that the embedding from $W_0^{1,p(\cdot)}(G)$ to $L^{q(\cdot)}(G)$ is compact for the variable exponent $q(\cdot)$ satisfying $\operatorname{ess\,inf}_{x\in G}(p^*(x) - q(x)) > 0$ (see Fan-Shen-Zhao [2, Theorem 1.3]).

On the other hand, in the case $\varphi(\cdot)$ is an unbounded function on $(0,\infty)$, that is, $\lim_{r\to\infty}\varphi(r) = \infty$, Mizuta and Shimomura [9] showed that the embedding from $W_0^{1,p(\cdot)}(G)$ to $L^{\Phi_A(\cdot,\cdot)}(G)$ is bounded for $A > N$, where

$$L^{\Phi_A(\cdot,\cdot)}(G) = \left\{ u : \|u\|_{L^{\Phi_A(\cdot,\cdot)}(G)} = \inf\left\{ \lambda > 0 : \int_G \Phi_A(x, |u(x)|/\lambda)dx \le 1 \right\} < \infty \right\}$$

with $\Phi_A(x,t) = \{t\varphi(t)^{-A/p(x)^2}\}^{p^*(x)}$ (see also Futamura-Mizuta-Shimomura [5] and Mizuta-Ohno-Shimomura [8]).

In connection with the above facts, our first aim in this paper is to show that the embedding from $W_0^{1,p(\cdot)}(G)$ to $L^{\Phi_A(\cdot,\cdot)}(G)$ is compact for $A > N$.

As an application, we show the existence of a nontrivial weak solution to the nonlinear elliptic problem

$$\begin{cases} -\mathrm{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right) = f(x,u(x)) & \text{in } G, \\ u(x) = 0 & \text{on } \partial G, \end{cases} \tag{1.1}$$

where $f : G \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function satisfying the conditions given in Section 4. This extends Fan-Zhang [3, Theorem 4.7]. Here, we say that u is a weak solution of (1.1) if $u \in W_0^{1,p(\cdot)}(G)$ and

$$\int_G \left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x) - f(x,u(x))v(x)\right) dx = 0$$

for all $v \in W_0^{1,p(\cdot)}(G)$.

2 Preliminaries

Throughout this paper, let C denote various positive constants independent of the variables in question and let $C(a, b, \cdots)$ be a constant which may depend on $a, b, \ldots$. For a measurable subset E of $\mathbf{R}^N$, we denote by $|E|$ the Lebesgue measure of E.

The next result follows readily from the definition of the $L^{q(\cdot)}$-norm.

LEMMA 2.1 ([4, Theorem 1.3]). *If $q(\cdot)$ is a variable exponent on G satisfying $q_+ < \infty$, then*

$$\min\left\{\|u\|^{q_-}_{L^{q(\cdot)}(G)}, \|u\|^{q_+}_{L^{q(\cdot)}(G)}\right\} \le \int_G |u(x)|^{q(x)}dx \le \max\left\{\|u\|^{q_-}_{L^{q(\cdot)}(G)}, \|u\|^{q_+}_{L^{q(\cdot)}(G)}\right\}.$$

We know the following Poincaré inequality for functions in $W_0^{1,q(\cdot)}(G)$.

LEMMA 2.2 ([6, Theorem 4.3]). *If $q(\cdot)$ is a uniform continuous variable exponent on G satisfying $q_+ < \infty$, then there exists a constant $C > 0$ such that*

$$\|u\|_{W^{1,q(\cdot)}(G)} \le C\|\nabla u\|_{L^{q(\cdot)}(G)}$$

for all $u \in W_0^{1,q(\cdot)}(G)$.

It is worth noting the next result; see [8, Section 2].

LEMMA 2.3. *Let $q(\cdot)$ be a variable exponent on G satisfying $q_+ < \infty$ and let $r(\cdot)$ be a measurable function on G satisfying $-\infty < r_- \le r_+ < \infty$. Then $\Phi(x,t) = t^{q(x)}\varphi(t)^{r(x)}$ satisfies the doubling condition; more pricisely,*

$$C^{-1}\Phi(x,t) \le \Phi(x,2t) \le C\Phi(x,t) \tag{2.1}$$

for all $t > 0$ and $x \in G$. Further, there exists a constant $C > 0$ such that

$$\Phi(x,t) \le C\Phi(x,s) \tag{2.2}$$

whenever $0 \le t \le s$ and $x \in G$.

Lemma 2.4 ([8, Lemma 2.5]). *$\|\cdot\|_{L^{\Phi_A(\cdot,\cdot)}(G)}$ is a quasi-norm; more pricisely, for $u, v \in L^{\Phi_A(\cdot,\cdot)}(G)$ and a real number k,*

(i) $\|u\|_{L^{\Phi_A(\cdot,\cdot)}(G)} = 0$ *if and only if* $u = 0$;

(ii) $\|ku\|_{L^{\Phi_A(\cdot,\cdot)}(G)} = |k|\|u\|_{L^{\Phi_A(\cdot,\cdot)}(G)}$;

(iii) $\|u+v\|_{L^{\Phi_A(\cdot,\cdot)}(G)} \le C\left(\|u\|_{L^{\Phi_A(\cdot,\cdot)}(G)} + \|v\|_{L^{\Phi_A(\cdot,\cdot)}(G)}\right)$.

Lemma 2.5. *$L^{\Phi_A(\cdot,\cdot)}(G)$ is a Banach space.*

Proof. First note from Lemma 2.4 (iii) that there exists a constant $c \ge 1$ such that

$$\|u+v\|_{L^{\Phi_A(\cdot,\cdot)}(G)} \le c\left(\|u\|_{L^{\Phi_A(\cdot,\cdot)}(G)} + \|v\|_{L^{\Phi_A(\cdot,\cdot)}(G)}\right)$$

for all $u, v \in L^{\Phi_A(\cdot,\cdot)}(G)$.

Let $\{u_n\}$ be a Cauchy sequence in $L^{\Phi_A(\cdot,\cdot)}(G)$. Then we can take a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that

$$\|u_{n_{j+1}} - u_{n_j}\|_{L^{\Phi_A(\cdot,\cdot)}(G)} < \frac{1}{(4c)^j}.$$

Setting $E_j = \{x \in G : |u_{n_{j+1}}(x) - u_{n_j}(x)| > 1/2^j\}$, we have by (2.2)

$$|E_j| \le C \int_{E_j} \frac{\Phi_A(x, (4c)^j |u_{n_{j+1}}(x) - u_{n_j}(x)|)}{\Phi_A(x, 2^j)} dx \le C 2^{-j},$$

so that $|E| = 0$, where $E = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} E_j$. Hence we see that $u_{n_j} = u_{n_1} + \sum_{k=1}^{j-1}(u_{n_{k+1}} - u_{n_k})$ converges to a function u almost everywhere on G. Since

$$\|u\|_{L^{\Phi_A(\cdot,\cdot)}(G)} \le c\|u_{n_1}\|_{L^{\Phi_A(\cdot,\cdot)}(G)} + \sum_{j=1}^{\infty} c^j \|u_{n_{j+1}} - u_{n_j}\|_{L^{\Phi_A(\cdot,\cdot)}(G)} \le c(\|u_{n_1}\|_{L^{\Phi_A(\cdot,\cdot)}(G)} + 1),$$

we have $u \in L^{\Phi_A(\cdot,\cdot)}(G)$. Fatou's lemma implies that

$$\int_G \Phi_A(x, (4c)^j |u(x) - u_{n_j}(x)|) dx \le \liminf_{j\to\infty} \int_G \Phi_A(x, (4c)^j |u_{n_{j+1}}(x) - u_{n_j}(x)|) dx \le 1,$$

so that we have $\|u - u_{n_j}\|_{L^{\Phi_A(\cdot,\cdot)}(G)} < 1/(4c)^j$, as required. □

We know the following Sobolev inequality for functions in $W_0^{1,p(\cdot)}(G)$.

Lemma 2.6 ([9, Theorem 3.5]). *There exists a constant $C > 0$ such that*

$$\|u\|_{L^{\Phi_A(\cdot,\cdot)}(G)} \le C\|\nabla u\|_{L^{p(\cdot)}(G)}$$

for all $u \in W_0^{1,p(\cdot)}(G)$.

LEMMA 2.7 (cf. [8, Corollary 2.11]). *Let p_0 satisfy $1 \le p_0 < p^*_-$. Then there exists a constant $C > 0$ such that*

$$\int_G \Phi_A(x, |u(x)|)dx \le C\|\nabla u\|^{p_0}_{L^{p(\cdot)}(G)}$$

for all measurable functions $u \in W_0^{1,p(\cdot)}(G)$ with $\|\nabla u\|_{L^{p(\cdot)}(G)} \le 1$.

Proof. If $\|\nabla u\|_{L^{p(\cdot)}(G)} \le 1$, then we can find $\lambda > 0$ such that $\|\nabla u\|_{L^{p(\cdot)}(G)} \le \lambda < 2$ and

$$\int_G \Phi_A(x, |u(x)|/\lambda)dx \le C$$

by Lemmas 2.3 and 2.6, where C is independent of λ. In view of [8, Corollary 2.3], we see that

$$\begin{aligned} \int_G \Phi_A(x, |u(x)|)dx &\le \sup_{x\in G} \tau(x,\lambda)^{p^*(x)} \int_G \Phi_A(x, |u(x)|/\lambda)dx \\ &\le C \sup_{x\in G} \tau(x,\lambda)^{p^*(x)}, \end{aligned}$$

where $\tau(x,\lambda) = \lambda\varphi(\lambda^{-1})^{A/p(x)^2}$. Since $\lambda < 2$, we have

$$\int_G \Phi_A(x, |u(x)|)dx \le C\lambda^{p_0}.$$

Letting $\lambda \to \|\nabla u\|_{L^{p(\cdot)}(G)}$ yields the required inequality. □

3 Compact embeddings

From now on, we assume that $\varphi(\cdot)$ satisfies $\lim_{r\to\infty} \varphi(r) = \infty$.

Now we show our compactness result on the embedding of $W_0^{1,p(\cdot)}(G)$ to $L^{\Phi_A(\cdot,\cdot)}(G)$.

THEOREM 3.1. *The embedding from $W_0^{1,p(\cdot)}(G)$ to $L^{\Phi_A(\cdot,\cdot)}(G)$ is compact for $A > N$.*

Proof. Let $\{u_n\}$ be a bounded sequence in $W_0^{1,p(\cdot)}(G)$ and let $A > B > N$. For $\varepsilon > 0$, we set $v_{j,k} = (u_j - u_k)/\varepsilon$. Clearly $\{v_{j,k}\}$ is bounded in $L^{\Phi_B(\cdot,\cdot)}(G)$ from Lemma 2.6; say $\|v_{j,k}\|_{L^{\Phi_B(\cdot,\cdot)}(G)} \le C(\varepsilon)$. Now there exists a constant $t_0 > e$ such that if $t > t_0$, then

$$\Phi_A(x,t) \le \Phi_B\left(x, \frac{t}{C(\varepsilon)}\right)$$

for all $x \in G$. Let $\delta = (t_0\varphi(t_0)^{-A/p_+^2})^{-p_+^*}$ and set

$$G_{j,k} = \left\{x \in G : |v_{j,k}(x)| \ge \Phi_A^{-1}\left(x, \frac{1}{|G|}\right)\right\},$$

where

$$\Phi_A^{-1}(x,s) = \inf\{t \ge 0 : \Phi_A(x,t) > s\}.$$

Then note that $\Phi_A(x, \Phi_A^{-1}(x,s)) \le s$ for all $x \in G$. On the other hand, we see that $\{u_n\}$ converges in measure since the embedding from $W_0^{1,1}(G)$ to $L^1(G)$ is compact. Therefore, there exists an integer $n_0 > 0$ such that if $j,k \ge n_0$, then $|G_{j,k}| \le \delta$ since $\Phi_A^{-1}(x, 1/|G|) \ge c_0$, where c_0 is independent of x. Set

$$G'_{j,k} = \{x \in G_{j,k} : |v_{j,k}(x)| \ge t_0\} \quad \text{and} \quad G''_{j,k} = G_{j,k} - G'_{j,k}.$$

For $j,k \ge n_0$, we have by (2.2)

$$\begin{aligned}
&\int_G \Phi_A(x, |v_{j,k}(x)|)dx \\
&= \int_{G-G_{j,k}} \Phi_A(x, |v_{j,k}(x)|)dx + \int_{G'_{j,k}} \Phi_A(x, |v_{j,k}(x)|)dx + \int_{G''_{j,k}} \Phi_A(x, |v_{j,k}(x)|)dx \\
&\le C\frac{|G|}{|G|} + \int_{G'_{j,k}} \Phi_B\left(x, \frac{|v_{j,k}(x)|}{C(\varepsilon)}\right) dx + C\delta(t_0\varphi(t_0)^{-A/p_+^2})^{p_+^*} \le C.
\end{aligned}$$

Hence $\|u_j - u_k\|_{L^{\Phi_A(\cdot,\cdot)}(G)} \le C\varepsilon$ and so $\{u_n\}$ converges in $L^{\Phi_A(\cdot,\cdot)}(G)$. □

4 Applications

In this section, as an application of Theorem 3.1, we show the existence result of nontrivial weak solutions to (1.1), which is an extension of [3, Theorem 4.7]. Because we treat the case that the embedding from $W_0^{1,p(\cdot)}(G)$ into $L^{\Phi_A(\cdot,\cdot)}(G)$ is compact, the proof in [3] also works in our case with minor changes (see also [12] in the constant exponent case). However, for the reader's convenience, we give a proof of our theorem.

We set

$$\widetilde{\Phi}_A(x,t) = \left\{t\varphi(t)^{A/p(x)^2}\right\}^{(p^*)'(x)}$$

for $A > N$, where

$$1/(p^*)'(x) = 1 - 1/p^*(x).$$

Then, by Lemma 2.3, there exists a constant $C > 0$ such that

$$\widetilde{\Phi}_A(x, s+t) \le C\left(\widetilde{\Phi}_A(x,s) + \widetilde{\Phi}_A(x,t)\right) \tag{4.1}$$

whenever $s,t \ge 0$ and $x \in G$. Further there exists a constant $C > 0$ such that

$$st \le C\left(\Phi_A(x,s) + \widetilde{\Phi}_A(x,t)\right) \tag{4.2}$$

for all $s,t > 0$. In fact, in the case $st \le \Phi_A(x,s)$, the inequality (4.2) is obvious; in the case $st > \Phi_A(x,s)$, we have $s < Ct^{1/(p^*(x)-1)}\varphi(t)^{A(p^*)'(x)/p(x)^2}$ with the aid of (2.2), so that

$$st < Ct^{1/(p^*(x)-1)+1}\varphi(t)^{A(p^*)'(x)/p(x)^2} = C\widetilde{\Phi}_A(x,t).$$

Hence we obtain inequality (4.2).

We define a functional $L : W_0^{1,p(\cdot)}(G) \to \mathbf{R}$ by

$$L(u) = \int_G \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx \quad \text{for } u \in W_0^{1,p(\cdot)}(G).$$

We note that the Gâteaux derivative $L'(u)$ of L at $u \in W_0^{1,p(\cdot)}(G)$ is given by

$$\langle L'(u), v\rangle = \lim_{t\to 0} \frac{L(u+tv) - L(u)}{t} = \int_G |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x) dx$$

for each $v \in W_0^{1,p(\cdot)}(G)$. By the Vitali convergence theorem, we insist that L' is continuous from $W_0^{1,p(\cdot)}(G)$ to its dual space $(W_0^{1,p(\cdot)}(G))'$ and hence $L \in C^1(W_0^{1,p(\cdot)}(G); \mathbf{R})$.

Lemma 4.1 ([3, Theorem 3.1]). *$L' : W_0^{1,p(\cdot)}(G) \to (W_0^{1,p(\cdot)}(G))'$ is a homeomorphism.*

Let $f : G \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function; more precisely, $f(\cdot, t)$ is measurable for all $t \in \mathbf{R}$ and $f(x, \cdot)$ is continuous for almost every $x \in G$. We consider a functional $J : W_0^{1,p(\cdot)}(G) \to \mathbf{R}$ defined by

$$J(u) = \int_G F(x, u(x)) dx \quad \text{for } u \in W_0^{1,p(\cdot)}(G),$$

where $F(x, t) = \int_0^t f(x, s) ds$.

Lemma 4.2. *Let $f : G \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function satisfying:*

(f1) $|f(x,t)| \le C(1 + |t|^{p^*(x)-1} \varphi(|t|)^{-Ap^*(x)/p(x)^2})$ *for all $x \in G$ and $t \in \mathbf{R}$.*

Then the Fréchet derivative $J'(u)$ of J at $u \in W_0^{1,p(\cdot)}(G)$ exists and

$$\langle J'(u), v\rangle = \int_G f(x, u(x)) v(x) dx$$

for each $v \in W_0^{1,p(\cdot)}(G)$.

Proof. For simplicity, we set $g(x,t) = |t|^{p^*(x)-1} \varphi(|t|)^{-Ap^*(x)/p(x)^2}$. For $u, v \in W_0^{1,p(\cdot)}(G)$

and $t \in (-1,1)$, we have by (f1) and (4.2)

$$\begin{aligned}
&\left|\frac{F(x,u(x)+tv(x))-F(x,u(x))}{t}\right| \\
\leq\ & \frac{1}{|t|}\int_0^{|t|}|f(x,u(x)+sv(x))||v(x)|ds \\
\leq\ & C\left(|v(x)|+\frac{1}{|t|}\int_0^{|t|}g(x,u(x)+sv(x))|v(x)|ds\right) \\
\leq\ & C\left(|v(x)|+\Phi_A(x,|v(x)|)+\frac{1}{|t|}\int_0^{|t|}\widetilde{\Phi}_A(x,g(x,u(x)+sv(x)))ds\right) \\
\leq\ & C\left(|v(x)|+\Phi_A(x,|v(x)|)+\frac{1}{|t|}\int_0^{|t|}\Phi_A(x,|u(x)+sv(x)|)ds\right) \\
\leq\ & C\left(|v(x)|+\Phi_A(x,|v(x)|)+\Phi_A(x,|u(x)|)\right).
\end{aligned}$$

By the dominated convergence theorem,

$$\lim_{t\to 0}\frac{J(u+tv)-J(u)}{t}=\int_G f(x,u(x))v(x)dx,$$

so that J is Gâteaux differentiable and

$$\langle J'(u),v\rangle=\int_G f(x,u(x))v(x)dx.$$

Next we show that J' is continuous in $(W_0^{1,p(\cdot)}(G))'$. Let $\{u_n\}\subset W_0^{1,p(\cdot)}(G)$ converge to $u\in W_0^{1,p(\cdot)}(G)$ in $W_0^{1,p(\cdot)}(G)$. Since $\{u_n\}$ is bounded in $W_0^{1,p(\cdot)}(G)$, $\{u_n\}$ converges to u in $L^{\Phi_A(\cdot,\cdot)}(G)$ by Theorem 3.1. Therefore, there exists $\tilde{u}\in L^{\Phi_A(\cdot,\cdot)}(G)$ such that $|u(x)|\leq\tilde{u}(x)$ for almost every $x\in G$. In view of (f1), (4.1) and (4.2), we see that

$$\begin{aligned}
&|f(x,u_n(x))-f(x,u(x))||v(x)| \\
\leq\ & C\left(\widetilde{\Phi}_A(x,|f(x,u_n(x))|)+\widetilde{\Phi}_A(x,|f(x,u(x))|)+\Phi_A(x,|v(x)|)\right) \\
\leq\ & C\left(\widetilde{\Phi}_A(x,1)+\Phi_A(x,|u_n(x)|)+\Phi_A(x,|u(x)|)+\Phi_A(x,|v(x)|)\right) \\
\leq\ & C\left(\widetilde{\Phi}_A(x,1)+\Phi_A(x,\tilde{u}(x))+\Phi_A(x,|u(x)|)+\Phi_A(x,|v(x)|)\right)
\end{aligned}$$

for each $v\in W_0^{1,p(\cdot)}(G)$, so that the dominated convergence theorem gives the required result. □

Lemma 4.3. *Suppose* $\{u_n\}\subset W_0^{1,p(\cdot)}(G)$ *converges weakly to* $u\in W_0^{1,p(\cdot)}(G)$ *and a Carathéodory function* f *satisfies* (f1). *Then* $J'(u_n)$ *converges to* $J'(u)$ *in* $(W_0^{1,p(\cdot)}(G))'$.

Proof. Since $\{u_n\} \subset W_0^{1,p(\cdot)}(G)$ converges weakly to $u \in W_0^{1,p(\cdot)}(G)$, $\{u_n\}$ is bounded in $W_0^{1,p(\cdot)}(G)$, so that $\{u_n\}$ converges to u in $L^{\Phi_A(\cdot,\cdot)}(G)$ by Theorem 3.1. Hence, we can apply the considerations in Lemma 4.2 to obtain the required result. □

Let X be a Banach space. We say that $u \in X$ is a critical point of $I \in C^1(X; \mathbf{R})$ if the Fréchet derivative $I'(u)$ of I at u is zero. We say that $\{u_n\} \subset X$ is a Palais-Smale sequence for I if $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$ in the dual space of X. We further say that I satisfies the Palais-Smale condition if every Palais-Smale sequence for I has a convergent subsequence.

We define a functional $I : W_0^{1,p(\cdot)}(G) \to \mathbf{R}$ by

$$I(u) = L(u) - J(u) \quad \text{for } u \in W_0^{1,p(\cdot)}(G).$$

Note from Lemma 4.2 that if a Carathéodory function f satisfies (f1), then the Fréchet derivative $I'(u)$ of I at $u \in W_0^{1,p(\cdot)}(G)$ exists and

$$\langle I'(u), v\rangle = \langle L'(u) - J'(u), v\rangle = \int_G \left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x) - f(x,u(x))v(x)\right) dx$$

for each $v \in W_0^{1,p(\cdot)}(G)$.

LEMMA 4.4. *Let $f : G \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function satisfying* (f1) *and*

(f2) *there exist constants $M > 0$ and $\theta > p_+$ such that*

$$0 < \theta \int_M^t f(x,s)ds \le tf(x,t) \text{ for all } x \in G \text{ and } t > M, \text{ and}$$
$$0 < \theta \int_{-M}^t f(x,s)ds \le tf(x,t) \text{ for all } x \in G \text{ and } t < -M.$$

Then I satisfies the Palais-Smale condition.

Proof. Let $\{u_n\} \subset W_0^{1,p(\cdot)}(G)$ be a Palais-Smale sequence for I. By (f1), (f2) and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
C &\ge I(u_n) \\
&\ge \int_G \frac{1}{p(x)}|\nabla u_n(x)|^{p(x)}dx - \int_G \frac{u_n(x)}{\theta} f(x,u_n(x))dx - C \\
&= \int_G \left(\frac{1}{p(x)} - \frac{1}{\theta}\right)|\nabla u_n(x)|^{p(x)}dx + \frac{1}{\theta}\langle I'(u_n), u_n\rangle - C \\
&\ge \left(\frac{1}{p_+} - \frac{1}{\theta}\right) \min\left\{\|\nabla u_n\|^{p_-}_{L^{p(\cdot)}(G)}, \|\nabla u_n\|^{p_+}_{L^{p(\cdot)}(G)}\right\} \\
&\qquad\qquad - \frac{1}{\theta}\|I'(u_n)\|_{(W_0^{1,p(\cdot)}(G))'}\|u_n\|_{W^{1,p(\cdot)}(G)} - C \\
&\ge \left(\frac{1}{p_+} - \frac{1}{\theta}\right) \min\left\{\|\nabla u_n\|^{p_-}_{L^{p(\cdot)}(G)}, \|\nabla u_n\|^{p_+}_{L^{p(\cdot)}(G)}\right\} \\
&\qquad\qquad - C\left(\|I'(u_n)\|_{(W_0^{1,p(\cdot)}(G))'}\|\nabla u_n\|_{L^{p(\cdot)}(G)} + 1\right),
\end{aligned}$$

so that $\{u_n\}$ is bounded in $W_0^{1,p(\cdot)}(G)$ since $\theta > p_+$. Hence, passing to a subsequence, we may assume that $\{u_n\}$ converges weakly to a function u in $W_0^{1,p(\cdot)}(G)$. Since $I'(u_n)$ converges to 0 in $(W_0^{1,p(\cdot)}(G))'$ and $J'(u_n)$ converges to $J'(u)$ in $(W_0^{1,p(\cdot)}(G))'$ by Lemma 4.3, we have $L'(u_n)$ converges to $J'(u)$ in $(W_0^{1,p(\cdot)}(G))'$. It follows from Lemma 4.1 that I satisfies the Palais-Smale condition. □

We recall the following variant of the mountain pass theorem; see e.g., [12].

LEMMA 4.5. *Let X be a Banach space and let I be a C^1 functional on X with $I(0) = 0$, for which there exist positive constants $\kappa, r > 0$ such that*

(1) *$I(u) \geq \kappa$ for all $u \in X$ with $\|u\|_X = r$, and*

(2) *there exists an element $v \in X$ satisfying $I(v) < 0$ and $\|v\|_X > r$.*

Define

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0, 1]; X) : \gamma(0) = 0,\ I(\gamma(1)) < 0,\ \|\gamma(1)\| > r\}.$$

Then $c > 0$ and for each $\varepsilon > 0$, there exists $u \in X$ such that $|I(u) - c| \leq \varepsilon$ and $\|I'(u)\|_{X'} \leq \varepsilon$.

THEOREM 4.6. *Suppose $p_+ < p_-^*$. Let $f : G \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function satisfying* (f1), (f2) *and*

(f3) $\lim_{t\to 0} \sup_{x\in G} f(x, t)/|t|^{p_+-1} = 0.$

Then there exists a nontrivial weak solution of (1.1).

Proof. First we show that

$$\inf \left\{ I(u) : u \in W_0^{1,p(\cdot)}(G),\ \|u\|_{W^{1,p(\cdot)}(G)} = r \right\} > 0 \tag{4.3}$$

if $r > 0$ is sufficiently small. Note from (f1) and (f3) that

$$F(x, t) \leq \varepsilon|t|^{p_+} + C(\varepsilon)\Phi_A(x, t)$$

for all $t \in \mathbf{R}$ and $\varepsilon > 0$. Taking $r > 0$ so small, by Lemma 2.6, we have $\|\nabla u\|_{L^{p(\cdot)}(G)} \leq 1$ and $\|u\|_{L^{\Phi_A(\cdot,\cdot)}(G)} \leq 1$ for all $u \in W_0^{1,p(\cdot)}(G)$ with $\|u\|_{W^{1,p(\cdot)}(G)} = r$. Then for each $u \in W_0^{1,p(\cdot)}(G)$ with $\|u\|_{W^{1,p(\cdot)}(G)} = r$, we see from Lemmas 2.1, 2.2 and 2.7 that

$$\begin{aligned} I(u) &\geq \frac{1}{p_+}\int_G |\nabla u(x)|^{p(x)}dx - \varepsilon\int_G |u(x)|^{p_+}dx - C(\varepsilon)\int_G \Phi_A(x, |u(x)|)dx \\ &\geq \left(\frac{1}{p_+} - C\varepsilon\right)\|\nabla u\|^{p_+}_{L^{p(\cdot)}(G)} - C(\varepsilon)\|\nabla u\|^{p_0}_{L^{p(\cdot)}(G)}, \end{aligned}$$

where $p_+ < p_0 < p^*_-$. Choosing $\varepsilon > 0$ as $1/p_+ - C\varepsilon > 0$ in the last expression, we have (4.3) if $r > 0$ is small.

Next we prove $I(tu) \to -\infty$ as $t \to \infty$ for $u \in W_0^{1,p(\cdot)}(G)$ with $u \neq 0$. Note from (f2) that $F(x,t) \geq C|t|^\theta$ for all $x \in G$ and $|t| > M$. If $u \in W_0^{1,p(\cdot)}(G)$ such that $u \neq 0$, then we see that

$$I(tu) \leq t^{p_+} \int_G \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx - t^\theta \int_G |u(x)|^\theta dx \to -\infty$$

as $t \to \infty$, since $p_+ < \theta$.

Now the required result follows from Lemmas 4.4 and 4.5. □

Remark 4.7. Let $p_+ < p^*_-$ and $\varphi(r) = (\log(e+r))^a$ for $a > 0$. Then Theorem 4.6 implies that there exists a nontrivial weak solution to the nonlinear elliptic problem

$$\begin{cases} -\mathrm{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right) = |u(x)|^{p^*(x)-2}u(x)(\log(e+|u(x)|))^{-a'Np^*(x)/p(x)^2} & \text{in } G, \\ u(x) = 0 & \text{on } \partial G, \end{cases}$$

where $a' > a$.

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