BOUNDEDNESS OF FRACTIONAL INTEGRAL OPERATORS ON MORREY SPACES AND SOBOLEV EMBEDDINGS FOR GENERALIZED RIESZ POTENTIALS

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Abstract. Our aim in this paper is to deal with boundedness of fractional integral operators on Morrey spaces $L^{(1,\varphi)}(G)$ and the Sobolev embeddings for generalized Riesz potentials. Target spaces are Orlicz-Morrey, Orlicz-Campanato, and generalized Hölder spaces.

1. Introduction

The space introduced by Morrey [12] in 1938 has become a useful tool of the study for the existence and regularity of solutions of partial differential equations. In the present paper, we aim to show boundedness of fractional integral operators from Morrey spaces $L^{(1,\varphi)}$ to Orlicz-Morrey spaces, to Orlicz-Campanato spaces, or, to generalized Hölder spaces, and consequently establish Sobolev embeddings for generalized Riesz potentials, as an extension of Trudinger [26], Serrin [23] and the authors [16, 10].

Let G be a bounded open subset of $\mathbb{R}^n$ whose diameter is denoted by $d_G = \sup\{|x-y| : x, y \in G\}$. For an integrable function f on G, the Riesz potential of order α $(0 < \alpha < n)$ is defined by

$$I_\alpha f(x) = \int_G \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

The operator I_α is also called the fractional integral operator.

We denote by $B(z,r)$ the ball $\{x \in \mathbb{R}^n : |x-z| < r\}$ with center z and of radius $r > 0$, and by $|B(z,r)|$ its Lebesgue measure, i.e. $|B(z,r)| = \omega_n r^n$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. For $u \in L^1(G)$, we define the

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integral mean over $B(z,r)$ by

$$u_{B(z,r)} = \fint_{B(z,r)} u(x)\,dx = \frac{1}{|B(z,r)|}\int_{G\cap B(z,r)} u(x)\,dx.$$

Let $1 \le p < \infty$. If φ is a positive function on the interval $(0,\infty)$ satisfying the doubling condition, then we define the Morrey space $L^{(p,\varphi)}(G)$ to be the family of all $f \in L^p_{loc}(G)$ for which there is a positive constant C such that

$$\fint_{B(z,r)} |f(x)|^p\,dx \le C^p\varphi(r) \quad \text{whenever } z \in G \text{ and } 0 < r \le d_G.$$

The norm of $f \in L^{(p,\varphi)}(G)$ is defined by the infimum of the constants C satisfying the inequality. When $\varphi(r) = r^{-\lambda}$, $L^{(p,\varphi)}(G)$ is denoted by $L_{p,\lambda}(G)$.

There are many results for the case $p > 1$. Adams [1, Theorem 3.1] showed the boundedness of I_α from $L_{p,\lambda}(G)$ to $L_{q,\lambda}(G)$ with $0 < \alpha < n$, $1 < p < q < \infty$, $0 < \lambda \le n$ and $1/q = 1/p - \alpha/\lambda$. See also [22, 2, 13, 20, 9, 19, 5, 7, 25, 6].

On the other hand, a few results are known for the case $p = 1$. Trudinger [26, Theorem 1] proved that, if $f \in L_{1,1}(G)$ then $\exp(a|I_1 f(x)|) \in L^1(G)$ for some constant $a > 0$; this implies that the operator I_1 is bounded from $L_{1,1}(G)$ to $\exp(L^1)(G)$. See also Serrin [23] for another proof. In [16] the boundedness from $L^{(1,\varphi)}$ to another Morrey space $L^{(1,\psi)}$ was shown. Recently, the authors [10] gave a result on Sobolev embeddings for Riesz potentials of functions in $L^{(1,\varphi)}(G)$ with $\varphi(r) = r^{-\beta}(\log(2 + r^{-1}))^{-\beta_1}$.

Our aim in this paper is to show that, for the case $p = 1$, the operator I_α and its generalization I_ρ are bounded from Morrey spaces $L^{(1,\varphi)}$ to Orlicz-Morrey spaces, to Orlicz-Campanato spaces, or, to generalized Hölder spaces, whose definitions will be given in the next section. Our result is an extension of the results in [26, 16, 23, 10]. The definition of I_ρ is the following: Let ρ be a function from $(0,\infty)$ to itself with the doubling condition and $\displaystyle\int_0^1 \frac{\rho(t)}{t}\,dt < +\infty$. We define

$$I_\rho f(x) = \int_G \frac{\rho(|x-y|)}{|x-y|^n} f(y)\,dy,$$

where $f \in L^1(G)$. If $\rho(r) = r^\alpha$ for $0 < \alpha < n$, then $I_\rho f$ coincides with the usual Riesz potential of order α. Using the operator I_ρ, we can give a systematic proof and several new results as corollaries. For the boundedness of $I_\rho f$, we also refer the reader to [14, 15, 7, 6, 17].

In view of [16, Theorem 3.4], we know that the Orlicz space $L^{\Phi}(G)$ is included in the Morrey space $L^{(1,\varphi)}(G)$ when $\varphi(r) = \Phi^{-1}(r^{-n})$. In the final section we give a sufficient condition for the boundedness of I_ρ from $L^{\Phi}(G)$ to $L^{\Psi}(G)$ by applying the same discussions as in Morrey spaces $L^{(1,\varphi)}(G)$. O'Neil [21, Theorem 5.2] gave a sufficient condition for the boundedness of convolution operators in Orlicz spaces near L^1. We give another sufficient condition. Our statement and proof are simpler than O'Neil's and we can check easily whether the pair (ρ, Φ, Ψ) satisfies the assumption.

The next section is for the definitions of function spaces. Our main results and their corollaries are in Section 3 and Section 4, respectively. Section 5 is for lemmas to prove the main results in Section 6. In Section 7 we give results for Orlicz spaces.

2. Notation and terminologies

Let $\mathcal{G}$ be the set of all continuous functions from $(0,\infty)$ to itself with the doubling condition; that is, there exists a constant $c_\varphi \geq 1$ such that

$$\frac{1}{c_\varphi} \leq \frac{\varphi(r)}{\varphi(s)} \leq c_\varphi \quad \text{for} \quad \frac{1}{2} \leq \frac{r}{s} \leq 2. \tag{2.1}$$

We call c_φ the doubling constant of φ. For $\varphi \in \mathcal{G}$, we define the Morrey space $L^{(1,\varphi)}(G)$ as follows:

$$L^{(1,\varphi)}(G) = \left\{ f \in L^1_{\mathrm{loc}}(G) : \|f\|_{L^{(1,\varphi)}(G)} < \infty \right\}$$

with the norm

$$\|f\|_{L^{(1,\varphi)}(G)} = \sup_{z \in G,\, 0<r\leq d_G} \frac{1}{\varphi(r)} \fint_{B(z,r)} |f(x)|\, dx.$$

Then $L^{(1,\varphi)}(G)$ is a Banach space. Note here that if $\varphi_1,\ \varphi_2 \in \mathcal{G}$ and $\varphi_1(r)/\varphi_2(r)$ is bounded above, then

$$L^{(1,\varphi_1)}(G) \subset L^{(1,\varphi_2)}(G);$$

in particular, if there exists a constant $C \geq 1$ such that $C^{-1}\varphi_1(r) \leq \varphi_2(r) \leq C\varphi_1(r)$ for all $r > 0$, then

$$L^{(1,\varphi_1)}(G) = L^{(1,\varphi_2)}(G)$$

with equivalent norms.

When $\varphi(r) = r^{-\lambda}$, $L^{(1,\varphi)}(G)$ is denoted by $L_{1,\lambda}(G)$. If $\varphi(r) = r^{-n/p}$ with $1 \le p < \infty$, then Jensen's inequality yields

$$L^p(G) \subset L^{(1,\varphi)}(G) = L_{1,n/p}(G);$$

in particular, if $\varphi(r) = r^{-n}$, then

$$L^1(G) = L^{(1,\varphi)}(G) = L_{1,n}(G).$$

Note that $L^{(1,\varphi)}(G) = \{0\}$ when $\varphi(r) \to 0$ as $r \to 0$ by Lebesgue's differentiation theorem.

Let

$$G^* = \{x \in \mathbb{R}^n : \mathrm{dist}(x, G) < d_G\} = \bigcup_{z \in G,\, 0 < r \le d_G} B(z, r).$$

For $\varphi \in \mathcal{G}$, we define the generalized Campanato space $\mathcal{L}^{(1,\varphi)}(G)$ as follows:

$$\mathcal{L}^{(1,\varphi)}(G) = \left\{ f \in L^1_{\mathrm{loc}}(G^*) : \|f\|_{\mathcal{L}^{(1,\varphi)}(G)} < \infty \right\}$$

and

$$\|f\|_{\mathcal{L}^{(1,\varphi)}(G)} = \sup_{z \in G,\, 0 < r \le d_G} \frac{1}{\varphi(r)|B(z,r)|} \int_{B(z,r)} |f(x) - f_{B(z,r)}|\, dx.$$

Then $\|f\|_{\mathcal{L}^{(1,\varphi)}(G)}$ is a norm modulo constants and thereby $\mathcal{L}^{(1,\varphi)}(G)$ is a Banach space.

Let us consider the family $\mathcal{Y}$ of all continuous, increasing, convex and bijective functions from $[0, \infty)$ to itself. For $\Phi \in \mathcal{Y}$, the Orlicz space $L^\Phi(G)$ is defined by

$$L^\Phi(G) = \left\{ f \in L^1_{\mathrm{loc}}(G) : \|f\|_{L^\Phi(G)} < \infty \right\},$$

where

$$\|f\|_{L^\Phi(G)} = \inf \left\{ \lambda > 0 : \int_G \Phi\left(\frac{|f(x)|}{\lambda} \right) dx \le 1 \right\}.$$

If Φ_1, $\Phi_2 \in \mathcal{Y}$ and there exists a constant $C \ge 1$ such that $\Phi_1(C^{-1}r) \le \Phi_2(r) \le \Phi_1(Cr)$ for all $r > 0$, then we see easily that

$$L^{\Phi_1}(G) = L^{\Phi_2}(G)$$

with equivalent norms. If $\Phi \in \mathcal{Y}$ and

$$\Phi(r) = \exp(r^p),\ \exp(\exp(r^p)),\ r^p(\log r)^\lambda \text{ or } r^p(\log r)^q(\log(\log r))^\lambda$$

for large $r > 0$, then $L^\Phi(G)$ will be denoted by

$$\exp(L^p)(G),\ \exp\exp(L^p)(G),\ L^p(\log L)^\lambda(G) \text{ or } L^p(\log L)^q(\log\log L)^\lambda(G),$$

respectively.

We know that Orlicz spaces are included in Morrey spaces. For example, if $\varphi(r) = \Phi^{-1}(r^{-n})$, then we can show that

$$L^{\Phi}(G) \subset L^{(1,\varphi)}(G)$$

(see e.g. [16, Theorem 3.4]); if in addition $\Phi(r)/r \to \infty$ as $r \to \infty$, then $L^{\Phi}(G)$ is a proper subset of $L^{(1,\varphi)}(G)$ on account of [18, Theorem 4.9].

For $\Phi \in \mathcal{Y}$ and $\varphi \in \mathcal{G}$, the Orlicz-Morrey space $L^{(\Phi,\varphi)}(G)$ is defined by

$$L^{(\Phi,\varphi)}(G) = \left\{ f \in L^1_{\mathrm{loc}}(G) : \|f\|_{L^{(\Phi,\varphi)}(G)} < \infty \right\},$$

where

$$\|f\|_{L^{(\Phi,\varphi)}(G)} = \sup_{z \in G,\, 0<r\le d_G} \inf \left\{ \lambda > 0 : \frac{1}{\varphi(r)} \fint_{B(z,r)} \Phi\left(\frac{|f(x)|}{\lambda} \right) dx \le 1 \right\}$$

(see [17, 18]). Then $\|f\|_{L^{(\Phi,\varphi)}(G)}$ is a norm and $L^{(\Phi,\varphi)}(G)$ is a Banach space.

For $\Phi \in \mathcal{Y}$ and $\varphi \in \mathcal{G}$, we also define the Orlicz-Campanato space $\mathcal{L}^{(\Phi,\varphi)}(G)$ as follows:

$$\mathcal{L}^{(\Phi,\varphi)}(G) = \left\{ f \in L^1_{\mathrm{loc}}(G^*) : \|f\|_{\mathcal{L}^{(\Phi,\varphi)}(G)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{L}^{(\Phi,\varphi)}(G)} = \sup_{z \in G,\, 0<r\le d_G} \inf \Bigg\{ \lambda > 0 : \frac{1}{\varphi(r)|B(z,r)|} \int_{B(z,r)} \Phi\left(\frac{|f(x) - f_{B(z,r)}|}{\lambda} \right) dx \le 1 \Bigg\}.$$

Then $\|f\|_{\mathcal{L}^{(\Phi,\varphi)}(G)}$ is a norm modulo constants and thereby $\mathcal{L}^{(\Phi,\varphi)}(G)$ is a Banach space.

For $\varphi \in \mathcal{G}$ such that φ is bounded, the generalized Hölder space is defined by

$$\Lambda_{\varphi}(G) = \left\{ f : \|f\|_{\Lambda_{\varphi}(G)} < \infty \right\},$$

where

$$\|f\|_{\Lambda_{\varphi}(G)} = \sup_{x,y \in G, x \ne y} \frac{|f(x) - f(y)|}{\varphi(|x-y|)}.$$

Then $\|f\|_{\Lambda_{\varphi}(G)}$ is a norm modulo constants and thereby $\Lambda_{\varphi}(G)$ is a Banach space. Since φ is bounded, every $f \in \Lambda_{\varphi}(G)$ is bounded. If $\varphi(r) \to 0$ as $r \to 0$, then every $f \in \Lambda_{\varphi}(G)$ is continuous.

3. Main results

In this section, we state our main theorems, whose proofs are given in Section 6.

Throughout this paper, let G be a bounded open set in $\mathbb{R}^n$ and denote by $c_\rho, c_\varphi, c_{\tilde\rho}$, the doubling constants of $\rho, \varphi, \tilde\rho \in \mathcal{G}$, respectively.

Let us begin with the following result.

Theorem 3.1. *Let $\rho, \varphi \in \mathcal{G}$, and define*

$$\psi(r) = \left(\int_0^r \frac{\rho(t)}{t}\,dt\right)\varphi(r) + \int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t}\,dt \tag{3.1}$$

for $0 < r \le d_G$. Then I_ρ is bounded from $L^{(1,\varphi)}(G)$ to $L^{(1,\psi)}(G)$. More precisely,

$$\|I_\rho f\|_{L^{(1,\psi)}(G)} \le C\|f\|_{L^{(1,\varphi)}(G)},$$

where $C > 0$ is a constant depending only on n, c_ρ and c_φ.

Remark 3.1. Theorem 3.1 is proved in [16] when $G = \mathbb{R}^n$ and $d_G = \infty$, under the assumption that there exists $C > 0$ such that $\rho(r)/r^n \le C\rho(s)/s^n$ for all $s < r$. However, we don't need this assumption. For example, the theorem is valid in the case $\rho(r) = r^n(\log r^{-1})^{-1/2}$ and $\varphi(r) = r^{-n}(\log r^{-1})^{-1/2}$ for small $r > 0$. We give a proof in Section 6 for convenience, though it is almost same as [16].

Theorem 3.2. *Let $\rho, \varphi \in \mathcal{G}$ such that $\displaystyle\int_0^1 \frac{\rho(t)\varphi(t)}{t}\,dt = \infty$. Define*

$$\psi_1(r) = \int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t}\,dt$$

for $0 < r \le d_G$. For $\tilde\rho \in \mathcal{G}$ such that $\tilde\rho/\rho$ is continuous and decreasing, consider

$$\kappa(r) = \psi_1(r)\tilde\rho(r)/\rho(r)$$

and

$$\psi(r) = \left(\int_0^r \frac{\tilde\rho(t)}{t}\,dt\right)\varphi(r) + \int_r^{2d_G} \frac{\tilde\rho(t)\varphi(t)}{t}\,dt$$

for $0 < r \le d_G$. If $\Phi \in \mathcal{Y}$ satisfies

$$C_G = \sup\left\{\frac{(\psi_1 \circ \kappa^{-1})(s)}{\Phi^{-1}(s)} : \kappa(d_G) \le s < \infty\right\} < \infty,$$

then there exists a constant $A > 0$ such that

$$\fint_{B(z,r)} \Phi\left(\frac{|I_\rho f(x)|}{A\|f\|_{L^{(1,\varphi)}(G)}}\right) dx \le \psi(r)$$

for $z \in G$, $0 < r \le d_G$ *and* $f \in L^{(1,\varphi)}(G)$, *that is,*

$$\|I_\rho f\|_{L^{(\Phi,\psi)}(G)} \le A\|f\|_{L^{(1,\varphi)}(G)},$$

where $A > 0$ *is a constant depending only on* $n, c_\rho, c_{\tilde{\rho}}, c_\varphi$ *and* C_G.

Remark 3.2. Note that κ is bijective from $(0, d_G]$ to $[\kappa(d_G), \infty)$ by the assumptions in the theorem.

In Theorems 3.3 and 3.4 below, we consider the following condition on ρ:

$$\left|\frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n}\right| \le c'_\rho |r - s| \frac{\rho(s)}{s^{n+1}} \quad \text{for} \quad \frac{1}{2} \le \frac{r}{s} \le 2. \tag{3.2}$$

For $f \in L^{(1,\varphi)}(G)$, letting $f = 0$ outside G, we can regard $I_\rho f$ as a function on G^*.

Theorem 3.3. *Let* $\rho, \varphi \in \mathcal{G}$ *and* (3.2) *hold. If*

$$\psi(r) = \left(\int_0^r \frac{\rho(t)}{t}\,dt\right)\varphi(r) + r\int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t^2}\,dt \quad \textit{for} \quad 0 < r \le d_G,$$

then I_ρ *is bounded from* $L^{(1,\varphi)}(G)$ *to* $\mathcal{L}^{(1,\psi)}(G)$. *More precisely,*

$$\|I_\rho f\|_{\mathcal{L}^{(1,\psi)}(G)} \le C\|f\|_{L^{(1,\varphi)}(G)},$$

where $C > 0$ *is a constant depending only on* n, c_ρ, c'_ρ *and* c_φ.

Moreover, if there exists a constant $A' \ge 1$ *such that* $\psi(t) \le A'\psi(r)$ *for* $0 < t \le r \le d_G$, *then*

$$\|I_\rho f\|_{\mathcal{L}^{(\Phi,\psi)}(G)} \le AC\|f\|_{L^{(1,\varphi)}(G)},$$

where $\Phi(r) = \exp(r) - 1$ *and* $A > 0$ *is a constant depending only on* n, A' *and* $\psi(d_G)$.

Note that, if $\displaystyle\int_0^1 \frac{\rho(t)\varphi(t)}{t}\,dt < \infty$, then $\displaystyle r\int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t^2}\,dt$ is bounded.

Theorem 3.4. *Let* $\rho, \varphi \in \mathcal{G}$, (3.2) *hold and* $\displaystyle\int_0^1 \frac{\rho(t)\varphi(t)}{t}\,dt < \infty$. *If*

$$\psi(r) = \int_0^r \frac{\rho(t)\varphi(t)}{t}\,dt + r\int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t^2}\,dt \quad \textit{for} \quad 0 < r \le d_G, \tag{3.3}$$

then I_ρ *is bounded from* $L^{(1,\varphi)}(G)$ *to* $\Lambda_\psi(G)$. *More precisely,*

$$\|I_\rho f\|_{\Lambda_\psi(G)} \le C\|f\|_{L^{(1,\varphi)}(G)},$$

where $C > 0$ *is a constant depending only on* n, c_ρ, c_φ *and* c'_ρ.

Remark 3.3. In Theorem 3.4, if there exist $c_{\rho,\varphi} > 0$ and $0 < \varepsilon < 1$ such that

$$\frac{\rho(r)\varphi(r)}{r^\varepsilon} \le c_{\rho,\varphi} \frac{\rho(s)\varphi(s)}{s^\varepsilon} \quad \text{for} \quad 0 < r \le s < \infty, \tag{3.4}$$

or

$$\frac{\rho(s)\varphi(s)}{s^\varepsilon} \le c_{\rho,\varphi} \frac{\rho(r)\varphi(r)}{r^\varepsilon} \quad \text{for} \quad 0 < r \le s < \infty, \tag{3.5}$$

then $\psi(r) \to 0$ as $r \to 0$, that is, $I_\rho f$ is continuous. Actually, if (3.4) holds, then

$$\begin{aligned} r \int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t^2}\, dt &\le c_{\rho,\varphi} \frac{\rho(2d_G)\varphi(2d_G)}{(2d_G)^\varepsilon}\, r \int_r^{2d_G} \frac{1}{t^{2-\varepsilon}}\, dt \\ &\le c_{\rho,\varphi} \frac{\rho(2d_G)\varphi(2d_G)}{(2d_G)^\varepsilon (1-\varepsilon)} r^\varepsilon, \end{aligned}$$

and if (3.5) holds, then

$$\begin{aligned} r \int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t^2}\, dt &\le c_{\rho,\varphi} \frac{\rho(r)\varphi(r)}{r^\varepsilon}\, r \int_r^{2d_G} \frac{1}{t^{2-\varepsilon}}\, dt \le \frac{c_{\rho,\varphi}}{1-\varepsilon} \rho(r)\varphi(r) \\ &\le \frac{c_{\rho,\varphi}}{1-\varepsilon} \frac{c_\rho c_\varphi}{\log 2} \int_{r/2}^r \frac{\rho(t)\varphi(t)}{t}\, dt \le \frac{c_{\rho,\varphi}}{1-\varepsilon} \frac{c_\rho c_\varphi}{\log 2} \int_0^r \frac{\rho(t)\varphi(t)}{t}\, dt. \end{aligned}$$

Therefore, if (3.5) holds, then we can take $\psi(r) = \int_0^r \frac{\rho(t)\varphi(t)}{t}\, dt$ instead of (3.3).

4. Corollaries

In this section we collect several results, which are special cases of our theorems. Recall that we always assume that

$$\int_0^1 \frac{\rho(t)}{t}\, dt < \infty$$

in the definition of I_ρ. Note that, if $\varphi \in \mathcal{G}$, $\theta > 0$ and $\varphi(r) = (\log r^{-1})^{-\theta}$ for small $r > 0$, then

$$L^{(1,\varphi)}(G) = \{0\}.$$

If $\varphi_0(r) = r^{-n}$, $\varphi \in \mathcal{G}$, $\theta < 0$ and $\varphi(r) = r^{-n}(\log r^{-1})^{-\theta}$ for small $r > 0$, then

$$L^{(1,\varphi)}(G) = L^{(1,\varphi_0)}(G) = L^1(G).$$

Therefore we need the condition "$\theta \le 0$ if $\beta = 0$" and "$\theta \ge 0$ if $\beta = n$" on $\varphi(r) = r^{-\beta}(\log r^{-1})^{-\theta}$.

Corollary 4.1. *For $0 \le \alpha < \beta \le n$, $\gamma > 1$ and $\alpha_1, \beta_1 \in \mathbb{R}$ ($\alpha_1 \ge \gamma$ if $\alpha = 0$: $\beta_1 \ge 0$ if $\beta = n$), let $p = \beta/(\beta - \alpha)$, $\rho, \varphi, \psi \in \mathcal{G}$, $\Phi \in \mathcal{Y}$ and*

$$\rho(r) = r^{\alpha}(\log r^{-1})^{-\alpha_1}, \quad \varphi(r) = r^{-\beta}(\log r^{-1})^{-\beta_1} \quad \textit{for small } r > 0,$$
$$\psi(r) = r^{-\beta}(\log r^{-1})^{-\beta_1-\gamma+1} \quad \textit{for small } r > 0,$$
$$\Phi(r) = r^{p}(\log r)^{-\gamma+(\alpha_1\beta+\alpha\beta_1)/(\beta-\alpha)} \quad \textit{for large } r > 0.$$

Then

$$\|I_\rho f\|_{L^{(\Phi,\psi)}(G)} \le A\|f\|_{L^{(1,\varphi)}(G)}.$$

That is

$$⨍_{B(z,r)} \Phi\left(\frac{|I_\rho f(x)|}{A\|f\|_{L^{(1,\varphi)}(G)}}\right) dx \le \psi(r)$$

for $z \in G$, $0 < r \le d_G$ and $f \in L^{(1,\varphi)}(G)$.

Proof. We use Theorem 3.2. Let

$$\tilde{\rho}(r) = (\log r^{-1})^{-\gamma} \quad \text{for small } r > 0.$$

Then we have

$$\begin{aligned}
\psi(r) &= \left(\int_0^r \frac{\tilde{\rho}(t)}{t}\,dt\right)\varphi(r) + \int_r^{2d_G} \frac{\tilde{\rho}(t)\varphi(t)}{t}\,dt \\
&\sim (\log r^{-1})^{-\gamma+1} r^{-\beta}(\log r^{-1})^{-\beta_1} + r^{-\beta}(\log r^{-1})^{-(\beta_1+\gamma)} \\
&\sim r^{-\beta}(\log r^{-1})^{-\beta_1-\gamma+1}, \\
\psi_1(r) &= \int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t}\,dt \sim r^{-(\beta-\alpha)}(\log r^{-1})^{-(\alpha_1+\beta_1)}, \\
\kappa(r) &= \psi_1(r)\tilde{\rho}(r)/\rho(r) \sim r^{-\beta}(\log r^{-1})^{-(\beta_1+\gamma)}
\end{aligned}$$

for small $r > 0$, and

$$\kappa^{-1}(s) \sim s^{-1/\beta}(\log s)^{-(\beta_1+\gamma)/\beta},$$
$$\psi_1 \circ \kappa^{-1}(s) \sim s^{(\beta-\alpha)/\beta}(\log s)^{(\gamma(\beta-\alpha)-\alpha\beta_1-\alpha_1\beta)/\beta} \sim \Phi^{-1}(s)$$

for large $s > 0$. □

Let $\alpha_1 = 0$ in the above. Then we have the following.

Corollary 4.2 ([10, Theorem 1.2])**.** *For $0 < \alpha < \beta \le n$, $\gamma > 1$ and $\beta_1 \in \mathbb{R}$ ($\beta_1 \ge 0$ if $\beta = n$), let $p = \beta/(\beta - \alpha)$, $\varphi, \psi \in \mathcal{G}$, $\Phi \in \mathcal{Y}$ and*

$$\varphi(r) = r^{-\beta}(\log r^{-1})^{-\beta_1} \quad \textit{for small } r > 0,$$
$$\psi(r) = r^{-\beta}(\log r^{-1})^{-\beta_1-\gamma+1} \quad \textit{for small } r > 0,$$
$$\Phi(r) = r^{p}(\log r)^{-\gamma+\alpha\beta_1 p/\beta} \quad \textit{for large } r > 0.$$

Then

$$\|I_\alpha f\|_{L^{(\Phi,\psi)}(G)} \leq A\|f\|_{L^{(1,\varphi)}(G)}.$$

In Corollary 4.2, we have

$$L^{n/\beta}(\log L)^{\beta_1 n/\beta}(G) \subsetneqq L^{(1,\varphi)}(G) \xrightarrow{I_\alpha} L^{(\Phi,\psi)}(G) \subsetneqq L^{\Phi}(G).$$

Remark 4.1. O'Neil [21] proved the boundedness of I_α on Orlicz spaces. Our results are independent of them. Let $\beta = n$ and $\beta_1 = 1 - \alpha/n > 0$ in Corollary 4.2, then we have the following:

$$\begin{array}{llcll} \text{Corollary 4.2:} & L^{(1,\varphi)}(G) & \xrightarrow{I_\alpha} & L^{(\Phi,\psi)}(G) & \subsetneqq \\ & \cup\!\!\!\!\!\not\,\,\,\, & & \cup\!\!\!\!\!\not\,\,\,\, & \qquad L^{\Phi}(G) \\ \text{O'Neil:} & L^1(\log L)^{1-\alpha/n}(G) & \xrightarrow{I_\alpha} & L^{n/(n-\alpha)}(G) & \subsetneqq \end{array}$$

For the inclusion above, see [18].

Corollary 4.3. *For* $0 \leq \alpha \leq n$, $\gamma \geq 0$ *and* $-\infty < \alpha_1 + \beta_1 < 1$ $(\alpha_1 > \gamma + 1$ *if* $\alpha = 0:$ $\beta_1 \geq 0$ *if* $\alpha = n)$, *let* $p = (1 - \alpha_1 - \beta_1 + \gamma)/(1 - \alpha_1 - \beta_1)$, $\rho, \varphi, \psi \in \mathcal{G}$ *and*

$$\rho(r) = r^{\alpha}(\log r^{-1})^{-\alpha_1}, \quad \varphi(r) = r^{-\alpha}(\log r^{-1})^{-\beta_1} \quad \textit{for small } r > 0,$$
$$\psi(r) = (\log r^{-1})^{1-\alpha_1-\beta_1+\gamma} \quad \textit{for small } r > 0,$$
$$\Phi(r) = r^p \quad \textit{for all } r \geq 0.$$

Then

$$\|I_\rho f\|_{L^{(\Phi,\psi)}(G)} \leq A\|f\|_{L^{(1,\varphi)}(G)}.$$

That is

$$⨍_{B(z,r)} \left(\frac{|I_\rho f(x)|}{A\|f\|_{L^{(1,\varphi)}(G)}} \right)^p dx \leq \psi(r)$$

for $z \in G$, $0 < r \leq d_G$ *and* $f \in L^{(1,\varphi)}(G)$.

Proof. We use Theorem 3.2. Let

$$\tilde{\rho}(r) = r^{\alpha}(\log r^{-1})^{-\alpha_1+\gamma} \quad \text{for small } r > 0.$$

Then we have

$$\begin{aligned}
\psi(r) &= \left(\int_0^r \frac{\tilde\rho(t)}{t}\,dt\right)\varphi(r) + \int_r^{2d_G} \frac{\tilde\rho(t)\varphi(t)}{t}\,dt \\
&\sim \begin{cases} r^{\alpha}(\log r^{-1})^{-\alpha_1+\gamma} r^{-\alpha}(\log r^{-1})^{-\beta_1} + (\log r^{-1})^{1-\alpha_1-\beta_1+\gamma} & (\alpha>0) \\ (\log r^{-1})^{1-\alpha_1+\gamma}(\log r^{-1})^{-\beta_1} + (\log r^{-1})^{1-\alpha_1-\beta_1+\gamma} & (\alpha=0) \end{cases} \\
&\sim (\log r^{-1})^{1-\alpha_1-\beta_1+\gamma}, \\
\psi_1(r) &= \int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t}\,dt \sim (\log r^{-1})^{1-\alpha_1-\beta_1}, \\
\kappa(r) &= \psi_1(r)\tilde\rho(r)/\rho(r) \sim (\log r^{-1})^{1-\alpha_1-\beta_1+\gamma}
\end{aligned}$$

for small $r>0$, and

$$\kappa^{-1}(s) \sim \exp\left(-s^{1/(1-\alpha_1-\beta_1+\gamma)}\right),$$
$$\psi_1\circ\kappa^{-1}(s) \sim s^{(1-\alpha_1-\beta_1)/(1-\alpha_1-\beta_1+\gamma)} = s^{1/p} = \Phi^{-1}(s)$$

for large $s>0$. □

Let $\alpha_1 = 0$ in the above. Then we have the following.

Corollary 4.4. *For $0<\alpha<n$, $-\infty<\beta_1<1$ and $\gamma\ge 0$, let $p = (1-\beta_1+\gamma)/(1-\beta_1)$, $\varphi,\psi\in\mathcal{G}$ and*

$$\varphi(r) = r^{-\alpha}(\log r^{-1})^{-\beta_1} \quad \textit{for small } r>0,$$
$$\psi(r) = (\log r^{-1})^{1-\beta_1+\gamma} \quad \textit{for small } r>0,$$
$$\Phi(r) = r^p \quad \textit{for all } r\ge 0.$$

Then

$$\|I_\alpha f\|_{L^{(\Phi,\psi)}(G)} \le A\|f\|_{L^{(1,\varphi)}(G)}.$$

Corollary 4.5. *For $0<\varepsilon<\alpha\le n$ and $-\infty<\alpha_1+\beta_1<1$, ($\beta_1\ge 0$ if $\alpha = n$), let $p = 1/(1-\alpha_1-\beta_1)$, $\rho,\varphi\in\mathcal{G}$, $\Phi\in\mathcal{Y}$ and*

$$\rho(r) = r^{\alpha}(\log r^{-1})^{-\alpha_1}, \quad \varphi(r) = r^{-\alpha}(\log r^{-1})^{-\beta_1} \quad \textit{for small } r>0,$$
$$\psi(r) = r^{-\alpha+\varepsilon} \quad \textit{for all } r>0,$$
$$\Phi(r) = \exp(r^p) \quad \textit{for large } r>0.$$

Then

$$\|I_\rho f\|_{L^{(\Phi,\psi)}(G)} \le A\|f\|_{L^{(1,\varphi)}(G)}.$$

That is

$$\fint_{B(z,r)} \Phi\left(\frac{|I_\rho f(x)|}{A\|f\|_{L^{(1,\varphi)}(G)}}\right) dx \le r^{-\alpha+\varepsilon}$$

for $z \in G$, $0 < r \le d_G$ *and* $f \in L^{(1,\varphi)}(G)$.

Proof. We use Theorem 3.2. Let

$$\tilde\rho(r) = r^{\varepsilon}(\log r^{-1})^{\beta_1} \quad \text{for small } r > 0.$$

Then we have

$$\begin{aligned}
\psi(r) &= \left(\int_0^r \frac{\tilde\rho(t)}{t}\,dt\right)\varphi(r) + \int_r^{2d_G} \frac{\tilde\rho(t)\varphi(t)}{t}\,dt \\
&\sim r^{\varepsilon}(\log r^{-1})^{\beta_1} r^{-\alpha}(\log r^{-1})^{-\beta_1} + r^{-(\alpha-\varepsilon)} \\
&\sim r^{-(\alpha-\varepsilon)}, \\
\psi_1(r) &= \int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t}\,dt \sim (\log r^{-1})^{1-\alpha_1-\beta_1}, \\
\kappa(r) &= \psi_1(r)\tilde\rho(r)/\rho(r) \sim r^{-(\alpha-\varepsilon)}\log r^{-1}
\end{aligned}$$

for small $r > 0$. Then

$$\begin{aligned}
\kappa^{-1}(s) &\sim s^{-1/(\alpha-\varepsilon)}(\log s)^{1/(\alpha-\varepsilon)}, \\
\psi_1 \circ \kappa^{-1}(s) &\sim (\log s)^{1-\alpha_1-\beta_1} = \Phi^{-1}(s)
\end{aligned}$$

for large $s > 0$. □

Let $\alpha_1 = 0$ in the above. Then we have the following.

Corollary 4.6 (cf. [10, Theorem 1.1 (1)]). *For* $0 < \varepsilon < \alpha < n$ *and* $-\infty < \beta_1 < 1$, *let* $p = 1/(1-\beta_1)$, $\varphi \in \mathcal{G}$, $\Phi \in \mathcal{Y}$ *and*

$$\begin{aligned}
\varphi(r) &= r^{-\alpha}(\log r^{-1})^{-\beta_1} \quad \textit{for small } r > 0, \\
\psi(r) &= r^{-\alpha+\varepsilon} \quad \textit{for all } r > 0, \\
\Phi(r) &= \exp(r^p) \quad \textit{for large } r > 0.
\end{aligned}$$

Then

$$\|I_\alpha f\|_{L^{(\Phi,\psi)}(G)} \le A\|f\|_{L^{(1,\varphi)}(G)}.$$

In Corollary 4.6, we have

$$L^{n/\alpha}(\log L)^{\beta_1 n/\alpha}(G) \subsetneqq L^{(1,\varphi)}(G) \xrightarrow{I_\alpha} L^{(\Phi,\psi)}(G) \subsetneqq L^{\Phi}(G)$$

for $0 < \varepsilon < \alpha$.

Remark 4.2. For the boundedness of I_α on Orlicz spaces, Edmunds, Gurka and Opic [4] and Mizuta and Shimomura [11] proved

$$I_\alpha : L^{n/\alpha}(\log L)^{\lambda}(G) \to \exp\left(L^{n/(n-\alpha-\lambda\alpha)}\right)(G) \qquad \text{when } \lambda < n/\alpha - 1$$

and

$$I_\alpha : L^{n/\alpha}(\log L)^\lambda(G) \to \exp\exp\left(L^{n/(n-\alpha)}\right)(G) \qquad \text{when } \lambda \geq n/\alpha - 1.$$

Our results are independent of them.

Corollary 4.7. *For $0 < \varepsilon < \alpha \leq n$, $\alpha_1 + \beta_1 = 1$ and $-\infty < \alpha_2 + \beta_2 < 1$, ($\beta_1 \geq 0$ if $\alpha = n$: $\beta_2 \geq 0$ if $\alpha = n$ and $\beta_1 = 0$), let $p = 1/(1 - \alpha_2 - \beta_2)$, $\rho, \varphi \in \mathcal{G}$, $\Phi \in \mathcal{Y}$ and*

$$\begin{aligned}
\rho(r) &= r^\alpha(\log r^{-1})^{-\alpha_1}(\log\log r^{-1})^{-\alpha_2}, \quad \textit{for small } r > 0,\\
\varphi(r) &= r^{-\alpha}(\log r^{-1})^{-\beta_1}(\log\log r^{-1})^{-\beta_2} \quad \textit{for small } r > 0,\\
\psi(r) &= r^{-\alpha+\varepsilon} \quad \textit{for all } r > 0,\\
\Phi(r) &= \exp\exp(r^p) \quad \textit{for large } r > 0.
\end{aligned}$$

Then

$$\|I_\rho f\|_{L^{(\Phi,\psi)}(G)} \leq A\|f\|_{L^{(1,\varphi)}(G)}.$$

That is

$$⨍_{B(z,r)} \Phi\left(\frac{|I_\rho f(x)|}{A\|f\|_{L^{(1,\varphi)}(G)}}\right) dx \leq r^{-\alpha+\varepsilon}$$

for $z \in G$, $0 < r \leq d_G$ and $f \in L^{(1,\varphi)}(G)$.

Proof. We use Theorem 3.2. Let

$$\tilde\rho(r) = r^\varepsilon(\log r^{-1})^{\beta_1}(\log\log r^{-1})^{\beta_2} \quad \text{for small } r > 0.$$

Then we have

$$\begin{aligned}
\psi(r) &= \left(\int_0^r \frac{\tilde\rho(t)}{t}\,dt\right)\varphi(r) + \int_r^{2d_G} \frac{\tilde\rho(t)\varphi(t)}{t}\,dt \sim r^{-(\alpha-\varepsilon)},\\
\psi_1(r) &= \int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t}\,dt \sim (\log\log r^{-1})^{1-\alpha_2-\beta_2},\\
\kappa(r) &= \psi_1(r)\tilde\rho(r)/\rho(r) \sim r^{-(\alpha-\varepsilon)}(\log r^{-1})(\log\log r^{-1})
\end{aligned}$$

for small $r > 0$. Then

$$\begin{aligned}
\kappa^{-1}(s) &\sim s^{-1/(\alpha-\varepsilon)}(\log s)^{1/(\alpha-\varepsilon)}(\log\log s)^{1/(\alpha-\varepsilon)},\\
\psi_1 \circ \kappa^{-1}(s) &\sim (\log\log s)^{1-\alpha_2-\beta_2} = \Phi^{-1}(s)
\end{aligned}$$

for large $s > 0$. □

Let $\alpha_1 = \alpha_2 = 0$ in the above. Then we have the following.

Corollary 4.8. *For $0 < \varepsilon < \alpha < n$ and $-\infty < \beta_2 < 1$, let $p = 1/(1-\beta_2)$, $\varphi \in \mathcal{G}$, $\Phi \in \mathcal{Y}$ and*

$$\varphi(r) = r^{-\alpha}(\log r^{-1})^{-1}(\log\log r^{-1})^{-\beta_2} \quad \text{for small } r > 0,$$
$$\psi(r) = r^{-\alpha+\varepsilon} \quad \text{for all } r > 0,$$
$$\Phi(r) = \exp\exp(r^p) \quad \text{for large } r > 0.$$

Then

$$\|I_\alpha f\|_{L^{(\Phi,\psi)}(G)} \le A\|f\|_{L^{(1,\varphi)}(G)}.$$

In Corollary 4.8, we have

$$L^{n/\alpha}(\log L)^{n/\alpha}(\log\log L)^{\beta_2 n/\alpha}(G) \subsetneqq L^{(1,\varphi)}(G) \xrightarrow{I_\alpha} L^{(\Phi,\psi)}(G) \subsetneqq L^{\Phi}(G)$$

for $0 < \varepsilon < \alpha$.

Let $\beta_2 = 0$ in the above. Then we have the following.

Corollary 4.9 (cf. [10, Theorem 1.1 (2)])**.** *For $0 < \varepsilon < \alpha < n$, let $\varphi \in \mathcal{G}$ and*

$$\varphi(r) = r^{-\alpha}(\log r^{-1})^{-1} \quad \text{for small } r > 0,$$
$$\psi(r) = r^{-\alpha+\varepsilon} \quad \text{for all } r > 0,$$
$$\Phi(r) = \exp\exp r - e \quad \text{for all } r \ge 0.$$

Then

$$\|I_\alpha f\|_{L^{(\Phi,\psi)}(G)} \le A\|f\|_{L^{(1,\varphi)}(G)}.$$

Corollary 4.10. *For $0 \le \alpha \le n$ and $\alpha_1, \beta_1 \in \mathbb{R}$ ($\alpha_1 > 1$, $\beta_1 \le 0$ if $\alpha = 0$: $\beta_1 \ge 0$ if $\alpha = n$), let $\rho, \varphi, \psi \in \mathcal{G}$ and*

$$\rho(r) = r^{\alpha}(\log r^{-1})^{-\alpha_1}, \quad \varphi(r) = r^{-\alpha}(\log r^{-1})^{-\beta_1} \quad \text{for small } r > 0,$$
$$\psi(r) = \begin{cases} (\log r^{-1})^{-\alpha_1-\beta_1} & (\alpha > 0) \\ (\log r^{-1})^{1-\alpha_1-\beta_1} & (\alpha = 0) \end{cases} \quad \text{for small } r > 0,$$
$$\Phi(r) = \exp r - 1 \quad \text{for all } r \ge 0.$$

Then

$$\|I_\rho f\|_{\mathcal{L}^{(1,\psi)}(G)} \le A\|f\|_{L^{(1,\varphi)}(G)}.$$

Moreover, if $\alpha_1 + \beta_1 \ge 0$ ($\alpha_1 + \beta_1 \ge 1$ if $\alpha = 0$), then

$$\|I_\rho f\|_{\mathcal{L}^{(\Phi,\psi)}(G)} \le A\|f\|_{L^{(1,\varphi)}(G)}.$$

That is

$$\fint_{B(z,r)} \left(\exp\left(\frac{|I_\rho f(x) - (I_\rho f)_{B(z,r)}|}{A\|f\|_{L^{(1,\varphi)}(G)}} \right) - 1 \right) dx \le \psi(r)$$

for $z \in G$, $0 < r \le d_G$ *and* $f \in L^{(1,\varphi)}(G)$.

Proof. We use Theorem 3.3. Then we have

$$\begin{aligned} \psi(r) &= \left(\int_0^r \frac{\rho(t)}{t}\, dt \right) \varphi(r) + r \int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t^2}\, dt \\ &\sim \begin{cases} (\log r^{-1})^{-\alpha_1-\beta_1} & (\alpha > 0) \\ (\log r^{-1})^{1-\alpha_1-\beta_1} & (\alpha = 0) \end{cases} \end{aligned}$$

for small $r > 0$. □

Remark 4.3. In Corollary 4.10, if $\alpha_1 + \beta_1 = 0$ ($\alpha_1 + \beta_1 = 1$ if $\alpha = 0$), then $\mathcal{L}^{(1,\psi)}(G) = \mathrm{BMO}(G)$. In the case $0 < \alpha < n$ and $\alpha_1 = \beta_1 = 0$, the result is proved by Peetre [22]. See also [14].

Remark 4.4. In Corollary 4.10, if $\alpha_1 + \beta_1 > 1$ ($\alpha_1 + \beta_1 > 2$ if $\alpha = 0$), then $\mathcal{L}^{(1,\psi)}(G) \subset \Lambda_{\tilde\psi}(G)$ for $\tilde\psi(r) = (\log r^{-1})^{-\theta}$ for small $r > 0$ with $\theta = \alpha_1+\beta_1-1$ ($\theta = \alpha_1+\beta_1-2$ if $\alpha = 0$) (Spanne [24, p. 601]). Therefore $I_\rho f$ is continuous for all $f \in L^{(1,\varphi)}(G)$. In this case, the assumption of Theorem 3.4 holds and thereby we also get $I_\rho f \in \Lambda_{\tilde\psi}(G)$.

Let $\alpha_1 = 0$ in the above. Then we have the following.

Corollary 4.11 ([10, Theorem 1.1 (3)])**.** *For* $0 < \alpha < n$ *and* $\beta_1 > 1$, *let* $\varphi, \psi \in \mathcal{G}$ *and*

$$\varphi(r) = r^{-\alpha}(\log r^{-1})^{-\beta_1}, \quad \psi(r) = (\log r^{-1})^{-\beta_1+1} \quad \textit{for small } r > 0.$$

Then

$$\|I_\alpha f\|_{\Lambda_\psi(G)} \le A\|f\|_{L^{(1,\varphi)}(G)}.$$

5. Preliminary lemmas

For proofs of our theorems, we prepare several lemmas.

Lemma 5.1. *Let* $k \ge 0$. *If* $\rho, \varphi \in \mathcal{G}$, *then*

$$(5.1)\quad \int_{B(x,r)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|\, dy \le \frac{2^n \omega_n c_\rho^2 c_\varphi}{\log 2} \left(\int_0^r \frac{\rho(t)\varphi(t)}{t}\, dt \right) \|f\|_{L^{(1,\varphi)}(G)}$$

and

(5.2)

$$\int_{B(x,d_G)\setminus B(x,r)} \frac{\rho(|x-y|)}{|x-y|^{n+k}}|f(y)|\,dy \le \frac{2^{n+k}\omega_n c_\rho^2 c_\varphi}{\log 2}\left(\int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t^{1+k}}\,dt\right)\|f\|_{L^{(1,\varphi)}(G)}.$$

Proof. If $y \in B(x,2^j r)\setminus B(x,2^{j-1}r)$, $j\in\mathbb{Z}$, then

$$\frac{\rho(|x-y|)}{|x-y|^{n+k}} \le \frac{c_\rho \rho(2^j r)}{(2^{j-1}r)^{n+k}}.$$

Hence

$$\begin{aligned}
&\int_{B(x,2^j r)\setminus B(x,2^{j-1}r)} \frac{\rho(|x-y|)}{|x-y|^{n+k}}|f(y)|\,dy \\
\le\ & \frac{c_\rho\rho(2^j r)}{(2^{j-1}r)^{n+k}}\int_{B(x,2^j r)}|f(y)|\,dy \\
\le\ & 2^{n+k}\omega_n c_\rho \frac{\rho(2^j r)\varphi(2^j r)}{(2^j r)^k}\|f\|_{L^{(1,\varphi)}} \\
\le\ & \frac{2^{n+k}\omega_n c_\rho^2 c_\varphi}{\log 2}\left(\int_{2^{j-1}r}^{2^j r}\frac{\rho(t)\varphi(t)}{t^{1+k}}\,dt\right)\|f\|_{L^{(1,\varphi)}}.
\end{aligned}$$

Noting that

$$\int_{B(x,r)}\frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy = \sum_{j=0}^{\infty}\int_{B(x,2^{-j}r)\setminus B(x,2^{-j-1}r)}\frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy$$

and

$$\int_{B(x,d_G)\setminus B(x,r)}\frac{\rho(|x-y|)}{|x-y|^{n+k}}|f(y)|\,dy \le \sum_{j=1}^{j_0}\int_{B(x,2^j r)\setminus B(x,2^{j-1}r)}\frac{\rho(|x-y|)}{|x-y|^{n+k}}|f(y)|\,dy,$$

where $d_G \le 2^{j_0} < 2d_G$, we have the conclusion. □

Lemma 5.2. *If $\rho\in\mathcal{G}$, then*

$$⨍_{B(z,r)}\left(\int_{B(z,r)}\frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy\right)dx \le n\omega_n c_\rho\varphi(r)\left(\int_0^r\frac{\rho(t)}{t}\,dt\right)\|f\|_{L^{(1,\varphi)}(G)}.$$

Proof. By Fubini's theorem we have

$$\begin{aligned}
& \int_{B(z,r)} \left(\int_{B(z,r)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|\, dy \right) dx \\
= & \int_{B(z,r)} |f(y)| \left(\int_{B(z,r)} \frac{\rho(|x-y|)}{|x-y|^n}\, dx \right) dy \\
\le & \int_{B(z,r)} |f(y)| \left(\int_{B(y,2r)} \frac{\rho(|x-y|)}{|x-y|^n}\, dx \right) dy \\
\le & \; n\omega_n c_\rho \left(\int_0^r \frac{\rho(t)}{t}\, dt \right) \int_{B(z,r)} |f(y)|\, dy \\
\le & \; n\omega_n c_\rho \varphi(r) |B(z,r)| \left(\int_0^r \frac{\rho(t)}{t}\, dt \right) \|f\|_{L^{(1,\varphi)}(G)},
\end{aligned}$$

as required. □

6. Proofs of the theorems

We are now ready to prove our theorems.

Proof of Theorem 3.1. We write

$$\begin{aligned}
⨍_{B(z,r)} |I_\rho f(x)|\, dx \;\le\; & ⨍_{B(z,r)} \left| \int_{B(x,r)} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy \right| dx \\
& + ⨍_{B(z,r)} \left| \int_{B(x,d_G)\setminus B(x,r)} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy \right| dx \\
\le\; & ⨍_{B(z,2r)} \left| \int_{B(z,2r)} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy \right| dx \\
& + ⨍_{B(z,r)} \left| \int_{B(x,d_G)\setminus B(x,r)} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy \right| dx \\
=\; & I_1 + I_2
\end{aligned}$$

for $z \in G$ and $0 < r \le d_G$. By Lemma 5.2 we have

$$\begin{aligned}
I_1 \;\le\; & C_1 \varphi(r) \left(\int_0^{2r} \frac{\rho(t)}{t}\, dt \right) \|f\|_{L^{(1,\varphi)}(G)} \\
\le\; & C_1 \psi(r) \|f\|_{L^{(1,\varphi)}(G)}.
\end{aligned}$$

By Lemma 5.1 we have

$$\begin{aligned}
I_2 \;\le\; & C_2 \left(\int_r^{2d_G} \frac{\rho(t)\varphi(t)}{t}\, dt \right) \|f\|_{L^{(1,\varphi)}(G)} \\
\le\; & C_2 \psi(r) \|f\|_{L^{(1,\varphi)}(G)}.
\end{aligned}$$

Hence it follows that

$$\fint_{B(z,r)} |I_\rho f(x)|\, dx \le C\psi(r)\|f\|_{L^{(1,\varphi)}(G)},$$

where $C > 0$ depends only on n, c_ρ, c_φ. □

Proof of Theorem 3.2. By Theorem 3.1, we have

$$\fint_{B(z,r)} |I_{\tilde\rho} f(x)|\, dx \le C_1\psi(r)\|f\|_{L^{(1,\varphi)}(G)} \tag{6.1}$$

for $z \in G$ and $0 < r \le d_G$.

Let $g = |f|/\|f\|_{L^{(1,\varphi)}(G)}$. For $x \in G$ and $0 < \delta \le d_G$, we have by Lemma 5.1

$$\begin{aligned} I_\rho g(x) &= \int_{B(x,\delta)} \frac{\rho(|x-y|)}{|x-y|^n} g(y)\, dy + \int_{B(x,d_G)\setminus B(x,\delta)} \frac{\rho(|x-y|)}{|x-y|^n} g(y)\, dy \\ &\le \frac{\rho(\delta)}{\tilde\rho(\delta)} \int_{B(x,\delta)} \frac{\tilde\rho(|x-y|)}{|x-y|^n} g(y)\, dy + C_2 \int_\delta^{2d_G} \frac{\rho(t)\varphi(t)}{t}\, dt \\ &\le \frac{\rho(\delta)}{\tilde\rho(\delta)} I_{\tilde\rho} g(x) + C_2\psi_1(\delta). \end{aligned}$$

Now let

$$\delta = \begin{cases} \kappa^{-1}(I_{\tilde\rho} g(x)) & \text{when } I_{\tilde\rho} g(x) \ge \kappa(d_G), \\ d_G & \text{when } I_{\tilde\rho} g(x) < \kappa(d_G). \end{cases}$$

Then it follows that

$$I_\rho g(x) \le (1 + C_2) \max\left\{\psi_1(\kappa^{-1}(I_{\tilde\rho} g(x))), \psi_1(d_G)\right\}.$$

Note that

$$(\psi_1 \circ \kappa^{-1})(s) \le C_G \Phi^{-1}(s) \quad \text{for} \quad \kappa(d_G) \le s < \infty.$$

Hence, taking $A = (1 + C_2)C_G(C_1 + c_\rho c_{\tilde\rho})$, we establish

$$\frac{|I_\rho f(x)|}{A\|f\|_{L^{(1,\varphi)}(G)}} \le \frac{I_\rho g(x)}{A} \le \frac{\max\{\Phi^{-1}(I_{\tilde\rho} g(x)), \Phi^{-1}(\kappa(d_G))\}}{C_1 + c_\rho c_{\tilde\rho}}.$$

On the other hand, we see that

$$\begin{aligned} \kappa(d_G) &= \left(\int_{d_G}^{2d_G} \frac{\rho(t)\varphi(t)}{t}\, dt\right) \frac{\tilde\rho(d_G)}{\rho(d_G)} \le c_\rho c_{\tilde\rho} \int_{d_G}^{2d_G} \frac{\tilde\rho(t)\varphi(t)}{t}\, dt \\ &\le c_\rho c_{\tilde\rho}\psi(r). \end{aligned}$$

Hence, with the aid of (6.1), we have

$$
\begin{aligned}
&\fint_{B(z,r)} \Phi\left(\frac{|I_\rho f(x)|}{A\|f\|_{L^{(1,\varphi)}(G)}}\right) dx \\
\le\ & \frac{1}{C_1 + c_\rho c_{\tilde\rho}} \fint_{B(z,r)} \max\left\{I_{\tilde\rho} g(x), \kappa(d_G)\right\} dx \\
\le\ & \frac{1}{C_1 + c_\rho c_{\tilde\rho}} \left(\fint_{B(z,r)} I_{\tilde\rho} g(x)\, dx + \fint_{B(z,r)} \kappa(d_G)\, dx\right) \\
\le\ & \frac{1}{C_1 + c_\rho c_{\tilde\rho}} \left(C_1 \psi(r) + c_\rho c_{\tilde\rho} \psi(r)\right) = \psi(r),
\end{aligned}
$$

which proves the conclusion. □

Proof of Theorem 3.3. For $f \in L^{(1,\varphi)}(G)$, letting $f = 0$ outside G, we can regard $I_\rho f$ as a function on $\mathbb{R}^n$. The first part

$$\|I_\rho f\|_{\mathcal{L}^{(1,\psi)}(G)} \le C\|f\|_{L^{(1,\varphi)}(G)}$$

is proved in [16] when $G = \mathbb{R}^n$ and $d_G = \infty$. Actually, we have

$$\sup_{B(z,r)\subset\mathbb{R}^n,\, 0<r\le d_G} \frac{1}{|B(z,r)|} \int_{B(z,r)} |I_\rho f(x) - (I_\rho f)_{B(z,r)}|\, dx \le C\psi(r)\|f\|_{L^{(1,\varphi)}(G)}.$$

The second part is shown by John-Nirenberg's inequality [8]. Let

$$g(x) = \frac{I_\rho f(x)}{C\|f\|_{L^{(1,\varphi)}(G)}}.$$

Then, for $0 < t \le r \le d_G$, we have

$$\frac{1}{|B(z,t)|} \int_{B(z,t)} |g(x) - g_{B(z,t)}|\, dx \le \psi(t) \le A'\psi(r).$$

By John-Nirenberg's inequality, there exist constants $c_1, c_2 > 0$, depending only on n, such that

$$\frac{1}{|B(z,r)|} \int_{B(z,r)} \left\{\exp\left(c|g(x) - g_{B(z,r)}|\right) - 1\right\} dx \le \frac{c_1 c}{(A'\psi(r))^{-1} c_2 - c}$$

for $0 < c < (A'\psi(r))^{-1}c_2$. Let $A = c^{-1}\max\{1, c_1/(A'\psi(d_G))\}$ and $c = c_2/(2A'^2\psi(d_G))$. Then $0 < c \le (A'\psi(r))^{-1}c_2/2$, since $\psi(r) \le A'\psi(d_G)$. Hence

$$\frac{c_1 c}{(A'\psi(r))^{-1}c_2 - c} \le \frac{c_1 c_2/(2A'^2\psi(d_G))}{(A'\psi(r))^{-1}c_2/2} = \frac{c_1\psi(r)}{A'\psi(d_G)}.$$

Using the inequality $\exp(r/a) - 1 \leq (\exp(r) - 1)/a$ for $r \geq 0$ and $a \geq 1$, we have

$$\begin{aligned}
&\frac{1}{|B(z,r)|} \int_{B(z,r)} \left\{ \exp\left(\frac{|g(x) - g_{B(z,r)}|}{A} \right) - 1 \right\} dx \\
\leq\ & \frac{1}{\max\{1, c_1/(A'\psi(d_G))\}} \fint_{B(z,r)} \left\{ \exp\left(c|g(x) - g_{B(z,r)}| \right) - 1 \right\} dx \\
\leq\ & \frac{1}{\max\{1, c_1/(A'\psi(d_G))\}} \frac{c_1 \psi(r)}{A'\psi(d_G)} \leq \psi(r).
\end{aligned}$$

This shows the conclusion. □

Proof of Theorem 3.4. Write

$$\begin{aligned}
&I_\rho f(x) - I_\rho f(z) \\
=\ & \int_{B(x,2|x-z|)} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy - \int_{B(x,2|x-z|)} \frac{\rho(|z-y|)}{|z-y|^n} f(y)\, dy \\
&+ \int_{G \setminus B(x,2|x-z|)} \left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|z-y|)}{|z-y|^n} \right) f(y)\, dy.
\end{aligned}$$

By (5.1), we have

$$\int_{B(x,2|x-z|)} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| dy \leq C_1 \psi(|x-z|) \|f\|_{L^{(1,\varphi)}(G)}$$

and

$$\begin{aligned}
\int_{B(x,2|x-z|)} \frac{\rho(|z-y|)}{|z-y|^n} |f(y)|\, dy &\leq \int_{B(z,3|x-z|)} \frac{\rho(|z-y|)}{|z-y|^n} |f(y)|\, dy \\
&\leq C_1' \psi(|x-z|) \|f\|_{L^{(1,\varphi)}(G)}
\end{aligned}$$

for $x, z \in G$. On the other hand, we have by (3.2) and (5.2)

$$\begin{aligned}
&\int_{G \setminus B(x,2|x-z|)} \left| \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|z-y|)}{|z-y|^n} \right| |f(y)|\, dy \\
\leq\ & c_\rho' |x-z| \int_{G \setminus B(x,2|x-z|)} \frac{\rho(|x-y|)}{|x-y|^{n+1}} |f(y)|\, dy \\
\leq\ & C_2 |x-z| \left(\int_{2|x-z|}^{2d_G} \frac{\rho(t)\varphi(t)}{t^2}\, dt \right) \|f\|_{L^{(1,\varphi)}(G)} \\
\leq\ & C_2 \psi(|x-z|) \|f\|_{L^{(1,\varphi)}(G)}.
\end{aligned}$$

Now we establish

$$|I_\rho f(x) - I_\rho f(z)| \leq C \psi(|x-z|) \|f\|_{L^{(1,\varphi)}(G)}$$

for $x, z \in G$, as required. □

7. Orlicz spaces

Our discussions can be applicable to the study of the boundedness of I_ρ from $L^\Phi(G)$ to $L^\Psi(G)$ and Sobolev embeddings of Riesz potentials for Orlicz spaces.

O'Neil [21, Theorem 5.2] gave a sufficient condition for the boundedness of the convolution operators in Orlicz spaces near L^1. He used other function spaces M^Φ in which L^Φ is a proper subspace. In this section we give another sufficient condition. Our statement and proof are simpler than O'Neil's and we can easily check whether the pair (ρ, Φ, Ψ) satisfies the assumption.

Let $\mathcal{L}$ be the set of all positive continuous functions ℓ on $[0,\infty)$ for which there exists a constant $c \ge 1$ such that

$$c^{-1} \le \frac{\ell(s)}{\ell(r)} \le c \quad \text{whenever} \quad \frac{1}{2} \le \frac{\log s}{\log r} \le 2. \tag{7.1}$$

Here we collect the fundamental properties on functions $\ell \in \mathcal{L}$.

(i) $\ell \in \mathcal{G}$ and $1/\ell \in \mathcal{L}$.

(ii) $\ell(e^t)$ satisfies the doubling condition, so that there exist positve constants c and β such that

$$\ell(r) \le c(\log(2+r))^\beta \quad \text{for} \quad 0 < r < \infty. \tag{7.2}$$

(iii) For all $\alpha > 0$, there exists a constant $c_\alpha \ge 1$ such that

$$c_\alpha^{-1}\ell(r) \le \ell(r^\alpha) \le c_\alpha \ell(r) \quad \text{for} \quad 0 < r < \infty. \tag{7.3}$$

(iv) For each $\varepsilon > 0$, $r^\varepsilon \ell(r)$ is almost increasing, that is, there exists a constant $c_\varepsilon \ge 1$ such that

$$r^\varepsilon \ell(r) \le c_\varepsilon s^\varepsilon \ell(s) \quad \text{for} \quad 0 < r < s < \infty. \tag{7.4}$$

(v) If $\ell, \ell_1 \in \mathcal{L}$ and $\alpha > 0$, then there exists a constant $c_\alpha \ge 1$ such that

$$c_\alpha{}^{-1}\ell(r) \le \ell(r^\alpha \ell_1(r)) \le c_\alpha \ell(r) \quad \text{for} \quad 0 < r < \infty. \tag{7.5}$$

(vi) If $p \ge 1$, $\ell \in \mathcal{L}$, $\Phi \in \mathcal{Y}$ and $\Phi(r) \le r^p \ell(r)$, then there exists a constant $c > 0$ such that

$$r^{1/p}\ell(r)^{-1/p} \le c\Phi^{-1}(r) \quad \text{for} \quad 0 < r < \infty. \tag{7.6}$$

7.1. **Riesz potentials.**

Theorem 7.1. *Let $0 < \alpha < n$ and $p = n/(n-\alpha)$. Let $\rho \in \mathcal{G}$ and $\Phi \in \mathcal{Y}$ be of the form*

$$\rho(r) = r^\alpha \ell(r^{-1})^{-1}$$

and

$$\Phi(r) = r\ell_1(r),$$

where $\ell, \ell_1 \in \mathcal{L}$. *Take functions* $\ell_2 \in \mathcal{L}$ *and* $\Psi \in \mathcal{Y}$ *satisfying*

$$\int_{d_G^{-1}}^{r} \frac{\ell_2(t)}{t}\, dt \le \ell_1(r) \quad \textit{for} \quad d_G^{-1} \le r < \infty, \tag{7.7}$$

$$\Psi(r) \le r^p \ell(r)^p \ell_1(r)^{p-1}\ell_2(r) \quad \textit{for} \quad 0 \le r < \infty. \tag{7.8}$$

Then there exists a constant $A > 0$ *such that*

$$\|I_\rho f\|_{L^\Psi(G)} \le A\|f\|_{L^\Phi(G)}.$$

Proof. We may assume that $\|f\|_{L^\Phi(G)} = 1$. Then

$$\int_G \Phi(|f(y)|)\, dy \le 1.$$

Note that ℓ_1 is nondecreasing, since Φ is convex and $\Phi(0) = 0$ by our assumption.

For $0 < \gamma < \alpha$, let

$$J(x) = \int_{\{y\in G: |x-y|^{-\gamma} < |f(y)|\}} \frac{\ell_2(|x-y|^{-1})}{|x-y|^n}|f(y)|\, dy.$$

Then, for $0 < \delta \le d_G$, which is determined later, we have by (7.3) and (7.4)

$$\begin{aligned}
&\int_{G\cap B(x,\delta)} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\, dy \\
\le\ & \int_{\{y\in G\cap B(x,\delta): |x-y|^{-\gamma} < |f(y)|\}} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\, dy + \int_{G\cap B(x,\delta)} \frac{\ell(|x-y|^{-1})^{-1}}{|x-y|^{n-\alpha+\gamma}}\, dy \\
\le\ & C\delta^\alpha \ell(\delta^{-1})^{-1}\ell_2(\delta^{-1})^{-1}J(x) + C\delta^{\alpha-\gamma}\ell(\delta^{-1})^{-1}.
\end{aligned}$$

Similarly, for $\alpha < \gamma' < n$, we obtain

$$\begin{aligned}
&\int_{G\setminus B(x,\delta)} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\, dy \\
\le\ & C\int_{G\setminus B(x,\delta)} \frac{\rho(|x-y|)}{|x-y|^n}\left\{\Phi(|f(y)|)\ell_1(|x-y|^{-1})^{-1} + |x-y|^{-\gamma'}\right\} dy \\
\le\ & C\delta^{\alpha-n}\ell(\delta^{-1})^{-1}\ell_1(\delta^{-1})^{-1}\int_G \Phi(|f(y)|)\, dy + C\delta^{\alpha-\gamma'}\ell(\delta^{-1})^{-1} \\
\le\ & C\delta^{\alpha-n}\ell(\delta^{-1})^{-1}\ell_1(\delta^{-1})^{-1} + C\delta^{\alpha-\gamma'}\ell(\delta^{-1})^{-1}.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
|I_\rho f(x)| &\le \int_{G\cap B(x,\delta)} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy + \int_{G\setminus B(x,\delta)} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \\
&\le C\delta^\alpha \ell(\delta^{-1})^{-1}\ell_2(\delta^{-1})^{-1}J(x) + C\delta^{\alpha-\gamma}\ell(\delta^{-1})^{-1} \\
&\quad + C\delta^{\alpha-n}\ell(\delta^{-1})^{-1}\ell_1(\delta^{-1})^{-1} + C\delta^{\alpha-\gamma'}\ell(\delta^{-1})^{-1} \\
&= C\delta^\alpha \ell(\delta^{-1})^{-1}\ell_2(\delta^{-1})^{-1}J(x) \\
&\quad + C\delta^{\alpha-n}\ell(\delta^{-1})^{-1}\ell_1(\delta^{-1})^{-1}\left(1+\delta^{n-\gamma}\ell_1(\delta^{-1})+\delta^{n-\gamma'}\ell_1(\delta^{-1})\right) \\
&\le C\delta^\alpha \ell(\delta^{-1})^{-1}\ell_2(\delta^{-1})^{-1}J(x) + C\delta^{\alpha-n}\ell(\delta^{-1})^{-1}\ell_1(\delta^{-1})^{-1}.
\end{aligned}$$

Now, let

$$\delta = \min\left\{J(x)^{-1/n}\ell_1(J(x))^{-1/n}\ell_2(J(x))^{1/n}, d_G\right\}.$$

If $\delta = J(x)^{-1/n}\ell_1(J(x))^{-1/n}\ell_2(J(x))^{1/n}$, then it follows from (7.5) that

$$\ell(\delta^{-1}) \sim \ell(J(x)), \quad \ell_1(\delta^{-1}) \sim \ell_1(J(x)), \quad \ell_2(\delta^{-1}) \sim \ell_2(J(x)),$$

so that we have by (7.8) and (7.6)

$$\begin{aligned}
|I_\rho f(x)| &\le CJ(x)^{(n-\alpha)/n}\ell(J(x))^{-1}\ell_1(J(x))^{-\alpha/n}\ell_2(J(x))^{-(n-\alpha)/n} \\
&= CJ(x)^{1/p}\ell(J(x))^{-1}\ell_1(J(x))^{-(p-1)/p}\ell_2(J(x))^{-1/p} \\
&\le C\Psi^{-1}(J(x)).
\end{aligned}$$

If $\delta = d_G$, then

$$|I_\rho f(x)| \le C.$$

Therefore

$$\Psi\left(\frac{|I_\rho f(x)|}{C}\right) \le J(x) + C.$$

By Fubini's theorem and (7.7) we have

$$\begin{aligned}
\int_G J(x)\,dx &= \int_G \left(\int_{\{x\in G: |x-y|^{-\gamma}<|f(y)|\}} \frac{\ell_2(|x-y|^{-1})}{|x-y|^n}\,dx\right)|f(y)|\,dy \\
&= \int_G n\omega_n \left(\int_{d_G^{-1}}^{|f(y)|^{1/\gamma}} \frac{\ell_2(t)}{t}\,dt\right)|f(y)|\,dy \\
&\le C\int_G \ell_1(|f(y)|^{1/\gamma})|f(y)|\,dy \le C\int_G \Phi(|f(y)|)\,dy \le C,
\end{aligned}$$

which proves the conclusion. □

As special cases of Theorem 7.1, we can easily get the following corollaries.

Corollary 7.2. *Let* $0 < \alpha < n$, $p = n/(n-\alpha)$. *For* $\alpha_1 \in \mathbb{R}$ *and* $\beta_1 > 0$, *let*

$$\rho(r) = r^{\alpha}(\log(2 + r^{-1}))^{-\alpha_1},$$
$$\Phi(r) = r(\log(c + r))^{\beta_1},$$
$$\Psi(r) = r^{p}(\log(c + r))^{p(\alpha_1+\beta_1)-1},$$

where $c > e$ *is chosen so that* $\Phi, \Psi \in \mathcal{Y}$. *Then*

$$\|I_\rho f\|_{L^{\Psi}(G)} \le A\|f\|_{L^{\Phi}(G)}.$$

In fact, we have only to consider a function $\ell_2(r) = c_0(\log(c + r))^{\beta_1 - 1}$ with a suitable constant $c_0 > 0$.

Remark 7.1. In Corollary 7.2 we cannot take $\beta_1 = 0$. Actually, one can find $f \in L^1(\mathbb{B})$ but

$$\int_{\mathbb{B}} |I_\rho f(x)|^p (\log(1 + |I_\rho f(x)|))^{p\alpha_1 - 1}\, dx = \infty,$$

where $\mathbb{B} = B(0, 1)$.

To show this, for $0 < \gamma < 1/p$, let f be a nonnegative function on $\mathbb{B}$ such that

$$f(y) = |y|^{-n}(\log(1 + |y|^{-1}))^{-1}(\log(1 + \log(1 + |y|^{-1})))^{-\gamma-1}.$$

Then we have $f \in L^1(\mathbb{B})$ and, for $x \in \mathbb{B}$,

$$\begin{aligned} I_\rho f(x) &\ge \int_{B(0,|x|/2)} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy \\ &\ge C|x|^{\alpha-n}(\log(2 + 2|x|^{-1}))^{-\alpha_1} \\ &\quad \times \int_{B(0,|x|/2)} |y|^{-n}(\log(1 + |y|^{-1}))^{-1}(\log(1 + \log(1 + |y|^{-1})))^{-\gamma-1}\, dy \\ &\ge C|x|^{\alpha-n}(\log(1 + |x|^{-1}))^{-\alpha_1}(\log(1 + \log(1 + |x|^{-1})))^{-\gamma}. \end{aligned}$$

Hence it follows that

$$\begin{aligned} &\int_{\mathbb{B}} |I_\rho f(x)|^p (\log(1 + |I_\rho f(x)|))^{p\alpha_1 - 1}\, dx \\ \ge\ & C\int_{\mathbb{B}} |x|^{-n}(\log(1 + |x|^{-1}))^{-1}(\log(1 + \log(1 + |x|^{-1})))^{-\gamma p}\, dx = \infty \end{aligned}$$

when $\gamma < 1/p$.

Remark 7.2. Let α, α_1, β_1, p and Φ be as in Corollary 7.2. If $\gamma > p(\alpha_1 + \beta_1) - 1$, then one can find $f \in L^{\Phi}(\mathbb{B})$ but

$$\int_{\mathbb{B}} |I_\rho f(x)|^p (\log(1 + |I_\rho f(x)|))^{\gamma}\, dx = \infty.$$

For this purpose, let $\varepsilon > 0$ and f be a nonnegative function on $\mathbb{B}$ such that

$$f(y) = |y|^{-n}(\log(1 + |y|^{-1}))^{-\beta_1-\varepsilon-1}.$$

Then, as in Remark 7.1, we have

$$\int_{\mathbb{B}} f(y)(\log(1 + f(y)))^{\beta_1}\, dy < \infty$$

and

$$|I_\rho f(x)|^p(\log(1 + |I_\rho f(x)|))^\gamma \geq C|x|^{-n}(\log(1 + |x|^{-1})^{\gamma-p(\alpha_1+\beta_1+\varepsilon)}.$$

Hence, if $0 < \varepsilon < (\gamma - p(\alpha_1 + \beta_1) + 1)/p$, then

$$\int_{\mathbb{B}} |I_\rho f(x)|^p(\log(1 + |I_\rho f(x)|))^\gamma\, dx = \infty.$$

Corollary 7.3. *Let* $0 < \alpha < n$, $p = n/(n - \alpha)$*. For* $\alpha_1, \alpha_2 \in \mathbb{R}$ *and* $\beta_2 > 0$*, let*

$$\begin{aligned}
\rho(r) &= r^\alpha(\log(2 + r^{-1}))^{-\alpha_1}(\log\log(4 + r^{-1}))^{-\alpha_2},\\
\Phi(r) &= r(\log\log(c + r))^{\beta_2},\\
\Psi(r) &= r^p(\log(c + r))^{p\alpha_1-1}(\log\log(c + r))^{p(\alpha_2+\beta_2)-1},
\end{aligned}$$

where $c > e$ *is chosen so that* $\Phi, \Psi \in \mathcal{Y}$*. Then*

$$\|I_\rho f\|_{L^\Psi(G)} \leq A\|f\|_{L^\Phi(G)}.$$

In fact, we have only to consider a function

$$\ell_2(r) = c_0(\log(c + \log(c + t)))^{\beta_2-1}(\log(c + t))^{-1}$$

with a suitable constant $c_0 > 0$.

Remark 7.3. For the boundedness of I_α and Sobolev's embeddings of Riesz potentials in Orlicz spaces, see Edmunds, Gurka and Opic [4] and Cianchi [3].

7.2. **Logarithmic potentials.** A function $\theta : (0, +\infty) \to (0, +\infty)$ is said to be almost increasing if there exists a constant $C > 0$ such that

$$\theta(r) \leq C\theta(s) \quad \text{for} \quad r \leq s.$$

Consider the logarithmic potential

$$I_\rho f(x) = \int_G \frac{\rho(|x - y|)}{|x - y|^n} f(y)dy,$$

where $\rho \in \mathcal{G}$ is of the form $\rho(r) = \ell(r^{-1})^{-1}$ with $\ell \in \mathcal{L}$ satisfying

$$\int_0^1 \frac{\rho(t)}{t}\,dt < \infty. \tag{7.9}$$

For logarithmic potentials, we have the following.

Theorem 7.4. *Let $\rho \in \mathcal{G}$ be of the form $\rho(r) = \ell(r^{-1})^{-1}$ with $\ell \in \mathcal{L}$ satisfying* (7.9). *Let $\Phi \in \mathcal{Y}$ be of the form*

$$\Phi(r) = r\ell_1(r),$$

where $\ell_1 \in \mathcal{L}$. Let $\ell_2, m_1, m_2, m_3, m_4$ be functions in $\mathcal{L}$ such that

(i) *ℓm_1, ℓ_1/m_2, ℓ/m_3 and $\ell_1 m_4$ are almost increasing;*

(ii) $\displaystyle\int_{d_G^{-1}}^{r} \frac{m_1(t)}{t}\,dt \le c_1 m_2(r) \quad$ *for* $\quad d_G^{-1} \le r < \infty$*;*

(iii) $\displaystyle\int_r^{\infty} \frac{1}{m_3(t)t}\,dt \le \frac{c_2}{m_4(r)} \quad$ *for* $\quad d_G^{-1} \le r < \infty$*;*

(iv) $\displaystyle\frac{m_2(r)}{m_1(r)} + \frac{m_3(r)}{m_4(r)} \le \ell_2(r) \quad$ *for* $\quad d_G^{-1} \le r < \infty$*,*

where c_1, c_2 are positive constants. Take a function $\Psi \in \mathcal{Y}$ satisfying

$$\Psi(r) \le r\ell(r)\ell_1(r)\ell_2(r)^{-1} \quad \textit{for} \quad 0 \le r < \infty.$$

Then there exists a constant $A > 0$ such that

$$\|I_\rho f\|_{L^\Psi(G)} \le A\|f\|_{L^\Phi(G)}.$$

Proof. We may assume that $\|f\|_{L^\Phi(G)} = 1$. Then

$$\int_G \Phi(|f(y)|)\,dy \le 1.$$

Let $0 < \delta < 1$. For $x \in G$ and $0 < r < d_G$, write

$$G = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4,$$

where

$$\begin{aligned}
E_0 &= \{y \in B(x,r) : |f(y)| \le r^{-\delta}\},\\
E_1 &= \{y \in B(x,r) : |f(y)| > r^{-\delta},\ |f(y)| > |x-y|^{-\delta}\},\\
E_2 &= \{y \in B(x,r) : |f(y)| > r^{-\delta},\ |f(y)| \le |x-y|^{-\delta}\},\\
E_3 &= \{y \in G \setminus B(x,r) : |f(y)| \le |x-y|^{-\delta}\},\\
E_4 &= \{y \in G \setminus B(x,r) : |f(y)| > |x-y|^{-\delta}\}.
\end{aligned}$$

Then

$$\int_{E_0} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \le r^{-\delta} \int_{B(x,r)} \frac{\rho(|x-y|)}{|x-y|^n}\,dy \le Cr^{-\delta},$$
$$\int_{E_3} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \le \int_{G\setminus B(x,r)} \frac{\rho(|x-y|)}{|x-y|^n}|x-y|^{-\delta}\,dy \le Cr^{-\delta}.$$

Noting that ℓ_1 is nondecreasing by our assumption that Φ is convex and $\Phi(0) = 0$, we see by (7.3) and (7.4) that

$$\int_{E_4} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \le \int_{E_4} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\frac{\ell_1(|f(y)|)}{\ell_1(|x-y|^{-\delta})}\,dy$$
$$\le \int_{G\setminus B(x,r)} \frac{C\rho(|x-y|)}{|x-y|^n \ell_1(|x-y|^{-1})}\Phi(|f(y)|)\,dy$$
$$\le \frac{C\rho(r)}{r^n \ell_1(r^{-1})} \int_{G\setminus B(x,r)} \Phi(|f(y)|)\,dy \le \frac{C}{r^n \ell(r^{-1})\ell_1(r^{-1})}.$$

Since $r^{-\delta} \le C\{r^n \ell(r^{-1})\ell_1(r^{-1})\}^{-1}$ by (7.4), we have

$$\int_{E_0\cup E_3\cup E_4} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \le \frac{C}{r^n \ell(r^{-1})\ell_1(r^{-1})}.$$

Next let us consider the integral over $E_1 \cup E_2$. Set

$$J(x) = J_1(x) + J_2(x),$$

where

$$J_1(x) = \int_{\tilde{E}_1} \frac{m_1(|x-y|^{-1})}{|x-y|^n}\frac{\Phi(|f(y)|)}{m_2(|f(y)|)}\,dy,$$
$$J_2(x) = \int_{\tilde{E}_2} \frac{m_4(|f(y)|)\Phi(|f(y)|)}{|x-y|^n m_3(|x-y|^{-1})}\,dy$$

with

$$\tilde{E}_1 = \{y \in G : |f(y)| > |x-y|^{-\delta}\},$$
$$\tilde{E}_2 = \{y \in G : |f(y)| \le |x-y|^{-\delta}\}.$$

We insist by assumption (iv) that

$$\int_{E_1} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy$$
$$\le C\int_{E_1} \frac{1}{|x-y|^n \ell(|x-y|^{-1})}\frac{m_1(|x-y|^{-1})}{m_1(|x-y|^{-1})}|f(y)|\frac{\ell_1(|f(y)|)/m_2(|f(y)|)}{\ell_1(r^{-\delta})/m_2(r^{-\delta})}\,dy$$
$$\le \frac{C}{\ell(r^{-1})m_1(r^{-1})}\frac{m_2(r^{-1})}{\ell_1(r^{-1})}J_1(x) \le \frac{C\ell_2(r^{-1})}{\ell(r^{-1})\ell_1(r^{-1})}J_1(x),$$

since ℓm_1 and ℓ_1/m_2 are almost increasing, and

$$\begin{aligned}
&\int_{E_2} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \\
&\le C\int_{E_2} \frac{1}{|x-y|^n\ell(|x-y|^{-1})}\frac{m_3(|x-y|^{-1})}{m_3(|x-y|^{-1})}|f(y)|\frac{\ell_1(|f(y)|)m_4(|f(y)|)}{\ell_1(r^{-\delta})m_4(r^{-\delta})}\,dy \\
&\le C\frac{m_3(r^{-1})}{\ell(r^{-1})}\frac{1}{\ell_1(r^{-1})m_4(r^{-1})}J_2(x) \le \frac{C\ell_2(r^{-1})}{\ell(r^{-1})\ell_1(r^{-1})}J_2(x),
\end{aligned}$$

since ℓ/m_3 and $\ell_1 m_4$ are almost increasing. Noting that

$$\ell_2(t) \ge \frac{m_3(t)}{m_4(t)} \ge c_2^{-1}m_3(t)\int_t^{2t}\frac{1}{m_3(s)s}\,ds \ge C$$

for $d_G^{-1} \le t < \infty$ by assumption (iii), we find

$$|I_\rho f(x)| \le \frac{C}{\ell(r^{-1})\ell_1(r^{-1})}\left(\frac{1}{r^n}+\ell_2(r^{-1})J(x)\right) \le \frac{C\ell_2(r^{-1})}{\ell(r^{-1})\ell_1(r^{-1})}\left(\frac{1}{r^n}+J(x)\right).$$

Let

$$r = \min\{J(x)^{-1/n}, d_G\}.$$

If $r = J(x)^{-1/n}$, then we have by (7.3) and (7.6)

$$|I_\rho f(x)| \le \frac{C\ell_2(J(x))}{\ell(J(x))\ell_1(J(x))}J(x) \le C\Psi^{-1}(J(x)).$$

If $r = d_G$, then $J(x) \le {d_G}^{-n}$ and

$$|I_\rho f(x)| \le C.$$

Hence

$$\Psi\left(\frac{|I_\rho f(x)|}{C}\right) \le J(x) + C.$$

To end the proof, we have only to see from Fubini's theorem and assumptions (ii), (iii) that

$$\begin{aligned}
\int_G J_1(x)\,dx &= \int_G\left(\int_{\{x\in G:|x-y|^{-\delta}<|f(y)|\}}\frac{m_1(|x-y|^{-1})}{|x-y|^n}\,dx\right)\frac{\Phi(|f(y)|)}{m_2(|f(y)|)}\,dy \\
&= \int_G n\omega_n\left(\int_{d_G^{-1}}^{|f(y)|^{1/\delta}}\frac{m_1(t)}{t}\,dt\right)\frac{\Phi(|f(y)|)}{m_2(|f(y)|)}\,dy \\
&\le C\int_G m_2(|f(y)|^{1/\delta})\frac{\Phi(|f(y)|)}{m_2(|f(y)|)}\,dy \le C\int_G \Phi(|f(y)|)\,dy \le C,
\end{aligned}$$

and

$$
\begin{aligned}
&\int_G J_2(x)\,dx \\
= &\int_G \left(\int_{\{x\in G: |x-y|^{-\delta} \geq |f(y)|\}} \frac{1}{m_3(|x-y|^{-1})|x-y|^n}\,dx \right) \Phi(|f(y)|) m_4(|f(y)|)\,dy \\
= &\int_G n\omega_n \left(\int_{|f(y)|^{1/\delta}}^{\infty} \frac{1}{m_3(t)t}\,dt \right) \Phi(|f(y)|) m_4(|f(y)|)\,dy \\
\leq &\; C \int_G \frac{1}{m_4(|f(y)|^{1/\delta})} \Phi(|f(y)|) m_4(|f(y)|)\,dy \leq C \int_G \Phi(|f(y)|)\,dy \leq C.
\end{aligned}
$$

Thus the conclusion follows. □

As special cases of Theorem 7.4, we can easily get the following corollaries.

Corollary 7.5. *For $\alpha_1 > 0$ and $\beta_1 > 0$, let*

$$
\begin{aligned}
\rho(r) &= (\log(2+r^{-1}))^{-\alpha_1-1}, \\
\Phi(r) &= r(\log(c+r))^{\beta_1}, \\
\Psi(r) &= r(\log(c+r))^{\alpha_1+\beta_1},
\end{aligned}
$$

where $c > 0$ is chosen so that $\Phi, \Psi \in \mathcal{Y}$. Then

$$
\|I_\rho f\|_{L^\Psi(G)} \leq A \|f\|_{L^\Phi(G)}.
$$

In fact, for $0 < \varepsilon < \min\{\alpha_1, \beta_1\}$, we have only to consider

$$
\begin{gathered}
\ell(r) = (\log r)^{\alpha_1+1}, \quad \ell_1(r) = (\log r)^{\beta_1}, \quad \ell_2(r) = \log r, \\
m_1(r) = (\log r)^{\varepsilon-1}, \quad m_2(r) = (\log r)^{\varepsilon}, \\
m_3(r) = (\log r)^{\varepsilon+1}, \quad m_4(r) = (\log r)^{\varepsilon}
\end{gathered}
$$

for large $r > 0$.

Corollary 7.6. *For $\alpha_1 > 0$ and $\beta_2 > 0$, let*

$$
\begin{aligned}
\rho(r) &= (\log(2+r^{-1}))^{-\alpha_1-1}, \\
\Phi(r) &= r(\log\log(c+r))^{\beta_2}, \\
\Psi(r) &= r(\log(c+r))^{\alpha_1}(\log\log(c+r))^{\beta_2-1},
\end{aligned}
$$

where $c > 0$ is chosen so that $\Phi, \Psi \in \mathcal{Y}$. Then

$$
\|I_\rho f\|_{L^\Psi(G)} \leq A \|f\|_{L^\Phi(G)}.
$$

For this, letting $0 < \varepsilon < \min\{\alpha_1, \beta_2\}$, we have only to take

$$\ell(r) = (\log r)^{\alpha_1+1}, \quad \ell_1(r) = (\log\log r)^{\beta_2}, \quad \ell_2(r) = (\log r)(\log\log r),$$
$$m_1(r) = (\log r)^{-1}(\log\log r)^{\varepsilon-1}, \quad m_2(r) = (\log\log r)^{\varepsilon},$$
$$m_3(r) = (\log r)^{\varepsilon+1}, \quad m_4(r) = (\log r)^{\varepsilon}$$

for large $r > 0$.

Corollary 7.7. *For $\alpha_2 > 0$, $\beta_1 > 0$ and $\beta_2 \in \mathbb{R}$, let*

$$\rho(r) = (\log(2 + r^{-1}))^{-1}(\log\log(4 + r^{-1}))^{-\alpha_2-1},$$
$$\Phi(r) = r(\log(c + r))^{\beta_1}(\log\log(c + r))^{\beta_2},$$
$$\Psi(r) = r(\log(c + r))^{\beta_1}(\log\log(c + r))^{\alpha_2+\beta_2},$$

where $c > 0$ is chosen so that $\Phi, \Psi \in \mathcal{Y}$. Then

$$\|I_\rho f\|_{L^\Psi(G)} \le A\|f\|_{L^\Phi(G)}.$$

To show this, letting $0 < \varepsilon < \min\{\alpha_2, \beta_1\}$, we may consider

$$\ell(r) = (\log r)(\log\log r)^{\alpha_2+1}, \quad \ell_1(r) = (\log\ r)^{\beta_1}(\log\log r)^{\beta_2},$$
$$\ell_2(r) = (\log r)(\log\log r),$$
$$m_1(r) = (\log r)^{\varepsilon-1}, \quad m_2(r) = (\log r)^{\varepsilon},$$
$$m_3(r) = (\log r)(\log\log r)^{\varepsilon+1}, \quad m_4(r) = (\log\log r)^{\varepsilon}$$

for large $r > 0$.

Corollary 7.8. *For $\alpha_2 > 0$ and $\beta_2 > 0$, let*

$$\rho(r) = (\log(2 + r^{-1}))^{-1}(\log\log(4 + r^{-1}))^{-\alpha_2-1},$$
$$\Phi(r) = r(\log\log(c + r))^{\beta_2},$$
$$\Psi(r) = r(\log\log(c + r))^{\alpha_2+\beta_2},$$

where $c > 0$ is chosen so that $\Phi, \Psi \in \mathcal{Y}$. Then

$$\|I_\rho f\|_{L^\Psi(G)} \le A\|f\|_{L^\Phi(G)}.$$

To show this, letting $0 < \varepsilon < \min\{\alpha_2, \beta_2\}$, we may consider

$$\ell(r) = (\log r)(\log\log r)^{\alpha_2+1}, \quad \ell_1(r) = (\log\log r)^{\beta_2},$$
$$\ell_2(r) = (\log r)(\log\log r),$$
$$m_1(r) = (\log r)^{-1}(\log\log r)^{\varepsilon-1}, \quad m_2(r) = (\log\log r)^{\varepsilon},$$
$$m_3(r) = (\log r)(\log\log r)^{\varepsilon+1}, \quad m_4(r) = (\log\log r)^{\varepsilon},$$

for large $r > 0$.

Corollary 7.9. *For $\alpha_1 > 0$ and $\beta_1 > 0$, let*

$$\begin{aligned}\rho(r) &= (\log(2 + r^{-1}))^{-\alpha_1-1},\\ \Phi(r) &= r,\\ \Psi(r) &= r(\log(c + r))^{\alpha_1}(\log\log(c + r))^{-\beta_1-1},\end{aligned}$$

where $c > 0$ is chosen so that $\Phi, \Psi \in \mathcal{Y}$. Then

$$\|I_\rho f\|_{L^\Psi(G)} \le A\|f\|_{L^1(G)}.$$

To show this, letting $\varepsilon > 0$, we may consider

$$\ell(r) = (\log r)^{\alpha_1+1}, \quad \ell_1(r) = 1, \quad \ell_2(r) = (\log r)(\log\log r)^{\beta_1+1},$$
$$m_1(r) = (\log r)^{-1}(\log\log r)^{-\beta_1-1}, \quad m_2(r) = 1,$$
$$m_3(r) = (\log r)(\log\log r)^{\varepsilon+1}, \quad m_4(r) = (\log\log r)^{\varepsilon}$$

for large $r > 0$.

References

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J. **42** (1975), 765–778.

[2] F. Chiarenza and M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat. Appl. **7** (7) (1987), 273–279.

[3] A. Cianchi, *Strong and weak type inequalities for some classical operators in Orlicz spaces*, J. London Math. Soc. **60** (1999), 187–202.

[4] D. E. Edmunds, P. Gurka and B. Opic, *Double exponential integrability of convolution operators in generalized Lorentz-Zygmund spaces*, Indiana Univ. Math. J. **44** (1995), no. 1, 19–43

[5] Eridani, *On the boundedness of a generalized fractional integral on generalized Morrey spaces*, Tamkang J. Math. **33** (2002), 335–340.

[6] Eridani, H. Gunawan and E. Nakai, *On generalized fractional integral operators*, Sci. Math. Jpn. **60** (2004), 539–550.

[7] H. Gunawan, *A note on the generalized fractional integral operators*, J. Indonesian Math. Soc. (MIHMI) **9** (1) (2003), 39–43.

[8] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

[9] K. Kurata, S. Nishigaki and S. Sugano, *Boundedness of integral operators on generalized Morrey spaces and its application to Schrödinger operators*, Proc. Amer. Math. Soc. **128** (2000), no. 4, 1125–1134.

[10] Y. Mizuta, E. Nakai, T. Ohno and T. Shimomura, *An elementary proof of Sobolev embeddings for Riesz potentials of functions in Morrey spaces $L^{1,\nu,\beta}(G)$*, Hiroshima Math. J. **38** (2008), 461–472.

[11] Y. Mizuta and T. Shimomura, *Exponential integrability for Riesz potentials of functions in Orlicz classes*, Hiroshima Math. J. **28** (1998), 355–371.

[12] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.

[13] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166** (1994), 95–103.
[14] E. Nakai, *On generalized fractional integrals*, Taiwanese J. Math. **5** (2001), 587–602.
[15] E. Nakai, *On generalized fractional integrals in the Orlicz spaces on spaces of homogeneous type*, Sci. Math. Jpn. **54** (2001), 473–487.
[16] E. Nakai, *On generalized fractional integrals on the weak Orlicz spaces,* BMO_φ*, the Morrey spaces and the Campanato spaces*, Function spaces, interpolation theory and related topics (Lund, 2000), de Gruyter, Berlin, 2002, 389–401.
[17] E. Nakai, *Generalized fractional integrals on Orlicz-Morrey spaces*, Banach and Function Spaces (Kitakyushu, 2003), Yokohama Publishers, Yokohama, 2004, 323–333.
[18] E. Nakai, *Orlicz-Morrey spaces and the Hardy-Littlewood maximal function*, Studia Math, **188** (2008), 193–221.
[19] E. Nakai and H. Sumitomo, *On generalized Riesz potentials and spaces of some smooth functions*, Sci. Math. Jpn. **54** (2001), 463–472.
[20] P. A. Olsen, *Fractional integration, Morrey spaces and a Schrödinger equation*, Comm. Partial Differential Equations **20** (1995), no. 11-12, 2005–2055.
[21] R. O'Neil, *Fractional integration in Orlicz spaces. I*, Trans. Amer. Math. Soc. **115** (1965), 300–328.
[22] J. Peetre, *On the theory of* $\mathcal{L}_{p,\lambda}$ *spaces*, J. Funct. Anal. **4** (1969), 71–87.
[23] J. Serrin, *A remark on Morrey potential*, Contemporary Math. **426** (2007), 307-315.
[24] S. Spanne, *Some function spaces defined using the mean oscillation over cubes*, Ann. Scuola Norm. Sup. Pisa **19** (3) (1965), 593–608.
[25] S. Sugano and H. Tanaka, *Boundedness of fractional integral operators on generalized Morrey spaces*, Sci. Math. Jpn. Online, **8** (2003), 233–242.
[26] N. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473-483.

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