Approximate identities and Young type inequalities in variable Lebesgue-Orlicz spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$

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Abstract

Our aim in this paper is to deal with approximate identities in generalized Lebesgue spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$. As a related topic, we also study Young type inequalities for convolution with respect to norms in such spaces.

1 Introduction

Following Cruz-Uribe and Fiorenza [2], we consider two variable exponents $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ and $q(\cdot) : \mathbb{R}^n \to \mathbb{R}$, which are continuous functions. Letting $\Phi_{p(\cdot),q(\cdot)}(x,t) = t^{p(x)}(\log(c_0 + t))^{q(x)}$, we define the space $L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$ of all measurable functions $f$ on an open set $\Omega$ such that

$$\int_{\Omega} \Phi_{p(\cdot),q(\cdot)} \left(y, \frac{|f(y)|}{\lambda} \right) dy < \infty$$

for some $\lambda > 0$; here we assume

$(\Phi)$ $\Phi_{p(\cdot),q(\cdot)}(x,\cdot)$ is convex on $[0, \infty)$ for every fixed $x \in \mathbb{R}^n$.

Note that $(\Phi)$ holds for some $c_0 \geq e$ if and only if there is a positive constant $K$ such that

$$K(p(x) - 1) + q(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n$$

(1.1) (see Appendix). Further, we see from $(\Phi)$ that $t^{-1}\Phi_{p(\cdot),q(\cdot)}(x,t)$ is nondecreasing in $t$. 

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We define the norm
\[ \|f\|_{\Phi_p,q(\cdot),\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi_p(q(\cdot), \left( y, \frac{|f(y)|}{\lambda} \right)) \, dy \leq 1 \right\} \]
for \( f \in L^p(\log L)^q(\Omega) \). Note that \( L^p(\log L)^q(\Omega) \) is a Musielak–Orlicz space [9]. Such spaces have been studied in [2, 8, 10]. In case \( q(\cdot) = 0 \) on \( \mathbb{R}^n \), \( L^p(\log L)^q(\Omega) \) is denoted by \( L^p(\Omega) \) ([7]).

We assume throughout the article that our variable exponents \( p(\cdot) \) and \( q(\cdot) \) are continuous functions on \( \mathbb{R}^n \) satisfying:

1. \( 1 \leq p_- := \inf_{x \in \mathbb{R}^n} p(x) \leq \sup_{x \in \mathbb{R}^n} p(x) =: p_+ < \infty; \)
2. \( |p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x - y|)} \) whenever \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n; \)
3. \( |p(x) - p(y)| \leq \frac{C}{\log(e + |x|)} \) whenever \( |y| \geq |x|/2; \)
4. \( -\infty < q_- := \inf_{x \in \mathbb{R}^n} q(x) \leq \sup_{x \in \mathbb{R}^n} q(x) =: q_+ < \infty; \)
5. \( |q(x) - q(y)| \leq \frac{C}{\log(e + \log(e + 1/|x - y|))} \) whenever \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \)

for a positive constant \( C \).

We choose \( p_0 \geq 1 \) as follows: we take \( p_0 = p_- \) if \( t^{-p_0} \Phi_{p(\cdot),q(\cdot)}(x,t) \) is uniformly almost increasing in \( t; \) more precisely, if there exists \( C > 0 \) such that \( s^{-p_0} \Phi_{p(\cdot),q(\cdot)}(x,s) \leq Ct^{-p_0} \Phi_{p(\cdot),q(\cdot)}(x,t) \) whenever \( 0 < s < t \) and \( x \in \mathbb{R}^n \). Otherwise we choose \( 1 \leq p_0 < p_- \). Then note that \( t^{-p_0} \Phi_{p(\cdot),q(\cdot)}(x,t) \) is uniformly almost increasing in \( t \) in any case.

Let \( \phi \) be an integrable function on \( \mathbb{R}^n \). For each \( t > 0 \), define the function \( \phi_t \) by \( \phi_t(x) = t^{-n}\phi(x/t) \). Note that by a change of variables, \( \|\phi_t\|_{L^1,\mathbb{R}^n} = \|\phi\|_{L^1,\mathbb{R}^n} \). We say that the family \( \{\phi_t\} \) is an approximate identity if \( \int_{\mathbb{R}^n} \phi(x) \, dx = 1 \). Define the radial majorant of \( \phi \) to be the function
\[ \hat{\phi}(x) = \sup_{|y| \geq |x|} |\phi(y)|. \]
If \( \hat{\phi} \) is integrable, we say that the family \( \{\phi_t\} \) is of potential-type.

Theorem A. Let \( \{\phi_t\} \) be an approximate identity. Suppose that either:

1. \( \{\phi_t\} \) is of potential-type, or
2. \( \phi \in L^{p(\cdot)'}(\mathbb{R}^n) \) and has compact support.
Then
\[ \sup_{0 < t \leq 1} \| \phi_t * f \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)} \]
and
\[ \lim_{t \to +0} \| \phi_t * f - f \|_{L^p(\mathbb{R}^n)} = 0 \]
for all \( f \in L^p(\mathbb{R}^n) \).

Our aim in this note is to extend their result to the space \( L^p(\log L)^q(\Omega) \) of two variable exponents.

**Theorem 1.1.** Let \( \{ \phi_t \} \) be a potential-type approximate identity. If \( f \in L^p(\log L)^q(\mathbb{R}^n) \), then \( \{ \phi_t * f \} \) converges to \( f \) in \( L^p(\log L)^q(\mathbb{R}^n) \):

\[ \lim_{t \to 0} \| \phi_t * f - f \|_{\Phi^p,q(\mathbb{R}^n)} = 0. \]

**Theorem 1.2.** Let \( \{ \phi_t \} \) be an approximate identity. Suppose that \( \phi \in L^{(p_0)^\prime}(\mathbb{R}^n) \) and has compact support. If \( f \in L^p(\log L)^q(\mathbb{R}^n) \), then \( \{ \phi_t * f \} \) converges to \( f \) in \( L^p(\log L)^q(\mathbb{R}^n) \):

\[ \lim_{t \to 0} \| \phi_t * f - f \|_{\Phi^p,q(\mathbb{R}^n)} = 0. \]

We show by an example that the conditions on \( \phi \) are necessary; see Remarks 3.5 and 3.6 below.

Finally, in Section 4, we give some Young type inequalities for convolution with respect to the norms in \( L^p(\log L)^q(\mathbb{R}^n) \).

## 2 The case of potential-type

Throughout this paper, let \( C \) denote various positive constants independent of the variables in question.

Let us begin with the following result due to Stein [11].

**Lemma 2.1.** Let \( 1 \leq p < \infty \) and \( \{ \phi_t \} \) be a potential-type approximate identity. Then for every \( f \in L^p(\mathbb{R}^n) \), \( \{ \phi_t * f \} \) converges to \( f \) in \( L^p(\mathbb{R}^n) \).

We denote by \( B(x, r) \) the open ball centered at \( x \in \mathbb{R}^n \) and with radius \( r > 0 \). For a measurable set \( E \), we denote by \( |E| \) the Lebesgue measure of \( E \).

The following is due to Lemma 2.6 in [8].

**Lemma 2.2.** Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) with \( \| f \|_{\Phi^p,q(\mathbb{R}^n)} \leq 1 \) such that \( f(x) \geq 1 \) or \( f(x) = 0 \) for each \( x \in \mathbb{R}^n \). Set

\[ J = J(x, r, f) = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y)dy \]
and
\[ L = L(x, r, f) = \frac{1}{|B(x, r)|} \int_{B(x, r)} \Phi_{p(\cdot), q(\cdot)}(y, f(y)) dy. \]

Then
\[ J \leq CL^{1/p(x)}(\log(c_0 + L))^{-q(x)/p(x)}, \]
where \( C > 0 \) does not depend on \( x, r, f \).

Further we need the following result.

**Lemma 2.3.** Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) such that \((1 + |y|)^{-n-1} \leq f(y) \leq 1 \) or \( f(y) = 0 \) for each \( y \in \mathbb{R}^n \). Set
\[ J = J(x, r, f) = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy \]
and
\[ L = L(x, r, f) = \frac{1}{|B(x, r)|} \int_{B(x, r)} \Phi_{p(\cdot), q(\cdot)}(y, f(y)) dy. \]

Then
\[ J \leq C \{ L^{1/p(x)} + (1 + |x|)^{-n-1} \}, \]
where \( C > 0 \) does not depend on \( x, r, f \).

**Proof.** We have by Jensen’s inequality
\[ J \leq \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y)^{p(x)} dy \right)^{1/p(x)} \]
\[ \leq \left( \frac{1}{|B(x, r)|} \int_{B(x, r) \cap B(0, |x|/2)} f(y)^{p(x)} dy \right)^{1/p(x)} + \left( \frac{1}{|B(x, r)|} \int_{B(x, r) \setminus B(0, |x|/2)} f(y)^{p(x)} dy \right)^{1/p(x)} \]
\[ = J_1 + J_2, \]
We see from (p3) that
\[ J_1 \leq C \left( \frac{1}{|B(x, r)|} \int_{B(x, r) \cap B(0, |x|/2)} f(y)^{p(y)} dy \right)^{1/p(x)}. \]

Similarly, setting \( E_2 = \{ y \in \mathbb{R}^n : f(y) \geq (1 + |x|)^{-n-1} \} \), we see from (p3) that
\[ J_2 \leq C \left( \frac{1}{|B(x, r)|} \int_{B(x, r) \setminus B(0, |x|/2) \setminus E_2} f(y)^{p(y)} dy \right)^{1/p(x)} \]
\[ + \left( \frac{1}{|B(x, r)|} \int_{B(x, r) \setminus B(0, |x|/2) \setminus E_2} (1 + |x|)^{-p(x)(n+1)} dy \right)^{1/p(x)} \]
\[ \leq C \left\{ \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y)^{p(y)} dy \right)^{1/p(x)} + (1 + |x|)^{-(n+1)} \right\}. \]

Since \( f(y) \leq 1, f(y)^{p(y)} \leq C \Phi_{p(\cdot), q(\cdot)}(y, f(y)) \). Hence, we have the required estimate. \( \square \)
By using Lemmas 2.2 and 2.3, we show the following theorem.

**Theorem 2.4.** If \( \{\phi_t\} \) is of potential-type, then
\[
\|\phi_t \ast f\|_{p(t),q(t)} \leq C \|\hat{\phi}\|_{L^1,R^n} \|f\|_{p(t),q(t)} R^n
\]
for all \( t > 0 \) and \( f \in L^{p(t)}(\log L)^{q(t)}(R^n) \).

**Proof.** Suppose \( \|\hat{\phi}\|_{L^1,R^n} = 1 \) and take a nonnegative measurable function \( f \) on \( R^n \) such that \( \|f\|_{p(t),q(t)} \leq 1 \). Write
\[
f = \chi_{\{y \in R^n: f(y) \geq 1\}} + \chi_{\{y \in R^n: (1+|y|)^{-n-1} \leq f(y) < 1\}} + \chi_{\{y \in R^n: f(y) \leq (1+|y|)^{-n-1}\}}
\]
where \( \chi_E \) denotes the characteristic function of a measurable set \( E \subset R^n \).

Since \( \phi_t \) is a radial function, we write \( \hat{\phi}_t(r) \) for \( \hat{\phi}_t(x) \) when \( |x| = r \). First note that
\[
|\phi_t \ast f(x)| \leq \int_{R^n} \hat{\phi}_t(|x-y|) f_1(y) dy
\]
so that Jensen’s inequality and Lemma 2.2 yield
\[
\Phi_{p(t),q(t)}(x, |\phi_t \ast f_1(x)|) \leq \int_0^\infty \Phi_{p(t),q(t)} \left( x, \frac{1}{|B(x,r)|} \int_{B(x,r)} f_1(y) dy \right) |B(x,r)| d(-\hat{\phi}_t(r))
\]
\[
\leq C \int_0^\infty \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} \Phi_{p(t),q(t)}(y, f_1(y)) dy \right) |B(x,r)| d(-\hat{\phi}_t(r))
\]
\[
= C(\hat{\phi}_t \ast g)(x),
\]
where \( g(y) = \Phi_{p(t),q(t)}(y, f(y)) \). The usual Young inequality for convolution gives
\[
\int_{R^n} \Phi_{p(t),q(t)}(x, |\phi_t \ast f_1(x)|) dx \leq C \int_{R^n} (\hat{\phi}_t \ast g)(x) dx
\]
\[
\leq C\|\hat{\phi}_t\|_{L^1,R^n} \|g\|_{L^1,R^n} \leq C.
\]

Similarly, noting that \( \frac{1}{|B(x,r)|} \int_{B(x,r)} f_2(y) dy \leq 1 \) and applying Lemma 2.3, we derive the same result for \( f_2 \).

Finally, noting that \( |\phi_t \ast f_3| \leq C \|\phi_t\|_{L^1,R^n} \leq C \), we obtain
\[
\int_{R^n} \Phi_{p(t),q(t)}(x, |\phi_t \ast f_3(x)|) dx \leq C \int_{R^n} |\phi_t \ast f_3(x)| dx
\]
\[
\leq C\|\phi_t\|_{L^1,R^n} \|f_3\|_{L^1,R^n} \leq C,
\]
as required. \( \square \)
We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Given \( \varepsilon > 0 \), we find a bounded function \( g \) in \( L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n) \) with compact support such that \( \|f - g\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} < \varepsilon \). By Theorem 2.4 we have

\[
\|\phi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \\
\leq \|\phi_t * (f - g)\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} + \|\phi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} + \|f - g\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \\
\leq C\varepsilon + \|\phi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.
\]

Since \( |\phi_t * g| \leq \|g\|_{L^\infty(\mathbb{R}^n)} \),

\[
\|\phi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq C'\|\phi_t * g - g\|_{L^1(\mathbb{R}^n)} \rightarrow 0
\]

by Lemma 2.1. (Here \( C' \) depends on \( \|g\|_{L^\infty(\mathbb{R}^n)} \)). Hence

\[
\lim sup_{t \rightarrow 0} \|\phi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq C\varepsilon,
\]

which completes the proof.\( \square \)

As another application of Lemmas 2.2 and 2.3, we can prove the following result, which is an extension of [4, Theorem 1.5] and [8, Theorem 2.7] (see also [6]).

Let \( Mf \) be the Hardy-Littlewood maximal function of \( f \).

**Proposition 2.5.** Suppose \( p_- > 1 \). Then the operator \( M \) is bounded from \( L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n) \) to \( L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n) \).

**Proof.** Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) such that \( \|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq 1 \) and write \( f = f_1 + f_2 + f_3 \) as in the proof of Theorem 2.4. Take \( 1 < p_1 < p_- \) and apply Lemmas 2.2 and 2.3 with \( p(\cdot) \) and \( q(\cdot) \) replaced by \( p(\cdot)/p_1 \) and \( q(\cdot)/p_1 \), respectively. Then

\[
\Phi_{p(\cdot),q(\cdot)}(x, Mf_1(x)) \leq C[Mg_1(x)]^{p_1}
\]

and

\[
\Phi_{p(\cdot),q(\cdot)}(x, Mf_2(x)) \leq C \left\{ [Mg_1(x)]^{p_1} + (1 + |x|)^{-n-1} \right\},
\]

where \( g_1(y) = \Phi_{p(\cdot)/p_1,q(\cdot)/p_1}(y, f(y)) \). As to \( f_3 \), we have

\[
\Phi_{p(\cdot),q(\cdot)}(x, Mf_3(x)) \leq C[Mf_3(x)]^{p_1}.
\]

Then the boundedness of the maximal operator in \( L^{p_1}(\mathbb{R}^n) \) proves the proposition.\( \square \)

**Remark 2.6.** If \( p_- > 1 \), then the function \( \Phi_{p(\cdot),q(\cdot)} \) is a proper \( N \)-function and our Proposition 2.5 implies that this function is of class \( \mathcal{A} \) in the sense of Diening [5] (see [5, Lemma 3.2]). It would be an interesting problem to see whether “class \( \mathcal{A} \)” is also a sufficient condition or not for the boundedness of \( M \) on \( L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n) \).
3 The case of compact support

We know the following result due to Zo [12]; see also [1, Theorem 2.2].

**Lemma 3.1.** Let $1 \leq p < \infty$, $1/p + 1/p' = 1$ and $\{\phi_t\}$ be an approximate identity. Suppose that $\phi \in L^p(\mathbb{R}^n)$ has compact support. Then for every $f \in L^p(\mathbb{R}^n)$, $\{\phi_t \ast f\}$ converges to $f$ pointwise almost everywhere.

Set

$$\tilde{p}(x) = p(x)/p_0 \quad \text{and} \quad \tilde{q}(x) = q(x)/p_0;$$

recall that $p_0 \in [1, p_-]$ is chosen such that $t^{-p_0} \Phi_{p_0}^{\cdot, q_0}(\cdot, t)$ is uniformly almost increasing in $t$.

For a proof of Theorem 1.2, the following is a key lemma.

**Lemma 3.2.** Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^{p_0}(\mathbb{R}^n)} \leq 1$ such that $f(x) \geq 1$ or $f(x) = 0$ for each $x \in \mathbb{R}^n$ and let $\phi$ have compact support in $B(0, R)$ with $\|\phi\|_{L^{p_0}(\mathbb{R}^n)} \leq 1$. Set

$$F = F(x, t, f) = |\phi_t \ast f(x)|$$

and

$$G = G(x, t, f) = \int_{\mathbb{R}^n} |\phi_t(x - y)| \Phi_{\tilde{p}(\cdot), \tilde{q}(\cdot)}(y, f(y)) dy.$$

Then

$$F \leq CG^{1/\tilde{p}(x)}(\log(c_0 + G))^{-\tilde{q}(x)/\tilde{p}(x)}$$

for all $0 < t \leq 1$, where $C > 0$ depends on $R$.

**Proof.** Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^{p_0}(\mathbb{R}^n)} \leq 1$ such that $f(x) \geq 1$ or $f(x) = 0$ for each $x \in \mathbb{R}^n$ and let $\phi$ have compact support in $B(0, R)$ with $\|\phi\|_{L^{p_0}(\mathbb{R}^n)} \leq 1$. By Hölder’s inequality, we have

$$G \leq \|\phi_t\|_{L^{p_0}(\mathbb{R}^n)} \left( \int_{\mathbb{R}^n} \Phi_{p_0}^{\cdot, q_0}(y, f(y)) dy \right)^{1/p_0} \leq t^{-n/p_0}.$$

First consider the case when $G \geq 1$. Since $G \leq t^{-n/p_0}$, for $y \in B(x, tR)$ we have by (p2)

$$G^{-p(y)} \leq G^{-p(y) + C/\log(c_0 + (tR)^{-1})} \leq CG^{-p(x)}$$

and by (q2)

$$(\log(c_0 + G))^{q(y)} \leq C(\log(c_0 + G))^{q(x)}.$$
Hence it follows from the choice of $p_0$ that
\[
F \leq G^{1/p(x)} (\log(c_0 + G))^{-\tilde{q}(x)/\tilde{p}(x)} \int_{\mathbb{R}^n} |\phi_t(x-y)| dy \\
+ C \int_{\mathbb{R}^n} |\phi_t(x-y)| f(y) \left\{ \frac{f(y)}{G^{1/p(x)} (\log(c_0 + G))^{-\tilde{q}(x)/\tilde{p}(x)}} \right\} \tilde{q}(y) \tilde{p}(y)^{-1} \times \left\{ \frac{\log(c_0 + G^{1/p(x)} (\log(c_0 + G))^{-\tilde{q}(x)/\tilde{p}(x)})}{\log(c_0 + f(y))} \right\} dy \\
\leq C G^{1/p(x)} (\log(c_0 + G))^{-\tilde{q}(x)/\tilde{p}(x)}.
\]
(cf. the proof of [8, Lemma 2.6]).

In the case $G < 1$, noting from the choice of $p_0$ that $f(y) \leq C \Phi p(y, q(y), y, f(y))$ for $y \in \mathbb{R}^n$, we find
\[
F \leq CG \leq C G^{1/p(x)} \leq C G^{1/p(x)} (\log(c_0 + G))^{-\tilde{q}(x)/\tilde{p}(x)}.
\]
Now the result follows.

**Lemma 3.3.** Suppose that $\|\phi\|_{L^1(\mathbb{R}^n)} \leq 1$. Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi p(y, q(y), \mathbb{R}^n)} \leq 1$. Set
\[
I = I(x, t, f) = \int_{\{y \in \mathbb{R}^n : |y| > |x|/2\}} |\phi_t(x-y)| f(y) dy
\]
and
\[
H = H(x, t, f) = \int_{\mathbb{R}^n} |\phi_t(x-y)| \Phi p(y, q(y), y, f(y)) dy.
\]
If $A > 0$ and $H \leq H_0$, then
\[
I \leq C (H^{1/p(x)} + |x|^{-A/p(x)})
\]
for $|x| > 1$ and $0 < t \leq 1$, where $C > 0$ depends on $A$ and $H_0$.

**Proof.** Suppose that $\|\phi\|_{L^1(\mathbb{R}^n)} \leq 1$. Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi p(y, q(y), \mathbb{R}^n)} \leq 1$.

Let $|x| > 1$. In the case $H_0 \geq H \geq |x|^{-A}$ with $A > 0$, we have by (p3)
\[
H^{-p(y)} \leq CH^{-p(x)} \leq CH^{-p(x)}
\]
for $|y| \geq |x|/2$. Hence we find by (Φ)
\[
I \leq C \left\{ H^{1/p(x)} + \int_{\{y \in \mathbb{R}^n : |y| > |x|/2\}} |\phi_t(x-y)| f(y) \times \left\{ \frac{f(y)}{H^{1/p(x)}} \right\} p(y)^{-1} \times \left\{ \frac{\log(c_0 + f(y))}{\log(c_0 + H^{1/p(x)})} \right\} q(y) dy \right\} \leq CH^{1/p(x)}.
\]
Next note from (p3) that

\[ |x|^{p(y)} \leq |x|^{p(x) + C/\log(e + |x|)} \leq C|x|^{p(x)} \]

for \( |y| \geq |x|/2 \). Hence, when \( H \leq |x|^{-A} \), we obtain by (\( \Phi \))

\[ I \leq C \left\{ |x|^{-A/p(x)} + \int_{\{y \in \mathbb{R}^n : |y| > |x|/2\}} \left| \phi_t(x - y) \right| f(y) \right\}
\times \left\{ \left| f(y) \right|^{p(y)-1} \left( \frac{\log(c_0 + f(y))}{\log(c_0 + |x|^{-A/p(x)})} \right)^{q(y)} dy \right\}
\leq C|x|^{-A/p(x)}, \]

which completes the proof. \( \square \)

**Theorem 3.4.** Suppose that \( \phi \in L^{(p_0)' \mathbb{R}^n} \) has compact support in \( B(0, R) \). Then

\[ \| \phi_t * f \|_{L^{p_0}' \mathbb{R}^n} \leq C \| \phi \|_{L^{p_0}' \mathbb{R}^n} \| f \|_{L^{(p_0)' \mathbb{R}^n}} \]

for all \( 0 < t \leq 1 \) and \( f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n) \), where \( C > 0 \) depends on \( R \).

**Proof.** Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) such that \( \|f\|_{L^{p_0}' \mathbb{R}^n} \leq 1 \) and let \( \phi \) have compact support in \( B(0, R) \) with \( \|\phi\|_{L^{p_0}' \mathbb{R}^n} \leq 1 \). Write

\[ f = f\chi_{\{y \in \mathbb{R}^n : f(y) \geq 1\}} + f\chi_{\{y \in \mathbb{R}^n : f(y) < 1\}} = f_1 + f_2. \]

We have by Lemma 3.2,

\[ |\phi_t * f_1(x)| \leq C(|\phi_t| * g(x))^{p_0/p(x)}(\log(c_0 + |\phi_t| * g(x)))^{-q(x)/p(x)}, \]

where \( g(y) = \Phi^{(p_0)}(y, f(y)) = \Phi^{(p_0)}(y, f(y))^{1/p_0} \), so that

\[ \Phi^{(p_0)}(x, |\phi_t * f_1(x)|) \leq C(|\phi_t| * g(x))^{p_0}. \quad (3.1) \]

Hence, since \( g \in L^{p_0}(\mathbb{R}^n) \), the usual Young inequality for convolution gives

\[ \int_{\mathbb{R}^n} \Phi^{(p_0)}(x, |\phi_t * f_1(x)|)dx \leq C \int_{\mathbb{R}^n} (|\phi_t| * g(x))^{p_0}dx \leq C \left( \|\phi_t\|_{L^{1, \mathbb{R}^n}} \|g\|_{L^{p_0, \mathbb{R}^n}} \right)^{p_0} \leq C. \]

Next we are concerned with \( f_2 \). Write

\[ f_2 = f_2\chi_{B(0, R)} + f_2\chi_{B(0, R)^c} = f'_2 + f''_2. \]

Since \( |\phi_t * f_2(x)| \leq C \) on \( \mathbb{R}^n \), we have

\[ \int_{B(0, 2R)} \Phi^{(p_0)}(x, |\phi_t * f_2(x)|)dx \leq C. \]
Further, noting that $\phi_t * f'_2 = 0$ outside $B(0, 2R)$, we find
\[ \int_{\mathbb{R}^n} \Phi_{p(\cdot), q(\cdot)}(x, |\phi_t * f'_2(x)|)dx \leq C. \]

Therefore it suffices to prove
\[ \int_{\mathbb{R}^n \setminus B(0, 2R)} \Phi_{p(\cdot), q(\cdot)}(x, |\phi_t * f''_2(x)|)dx \leq C. \]

Thus, in the rest of the proof, we may assume that $0 \leq f < 1$ on $\mathbb{R}^n$ and $f = 0$ on $B(0, R)$. Note that
\[ \int_{B(0, |x|/2)} \phi_t(x - y)f(y)dy = 0 \]
for $|x| > 2R$. Hence, applying Lemma 3.3, we have
\[ |\phi_t * f(x)|^{p(x)} \leq C(|\phi_t| * h(x) + |x|^{-A}) \]
for $|x| > 2R$, where $h(y) = \Phi_{p(\cdot), q(\cdot)}(y, f(y))$. Thus the integration yields
\[ \int_{\mathbb{R}^n \setminus B(0, 2R)} |\phi_t * f(x)|^{p(x)}dx \leq C, \]
which completes the proof.

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Given $\varepsilon > 0$, choose a bounded function $g$ with compact support such that $\|f - g\|_{\Phi_{p(\cdot), q(\cdot)}(\mathbb{R}^n)} < \varepsilon$. As in the proof of Theorem 1.1, using Theorem 3.4 this time, we have
\[ \|\phi_t * f - f\|_{\Phi_{p(\cdot), q(\cdot)}(\mathbb{R}^n)} \leq C\varepsilon + \|\phi_t * g - g\|_{\Phi_{p(\cdot), q(\cdot)}(\mathbb{R}^n)}. \]

Obviously, $g \in L^p(\mathbb{R}^n)$. Hence by Lemma 3.1, $\phi_t * g \to g$ almost everywhere in $\mathbb{R}^n$. Since there is a compact set $S$ containing all the supports of $\phi_t * g$,
\[ \|\phi_t * g - g\|_{\Phi_{p(\cdot), q(\cdot)}(\mathbb{R}^n)} \leq C'\|\phi_t * g - g\|_{L^{p+1}(\mathbb{R}^n)}, \]
with $C'$ depending on $|S|$, and the Lebesgue convergence theorem implies $\|\phi_t * g - g\|_{L^{p+1}(\mathbb{R}^n)} \to 0$ as $t \to \infty$. Hence
\[ \limsup_{t \to 0} \|\phi_t * f - f\|_{\Phi_{p(\cdot), q(\cdot)}(\mathbb{R}^n)} \leq C\varepsilon, \]
which completes the proof.
Remark 3.5. In Theorem 1.2 (and in Theorem A), the condition $\phi \in L^{(p_\cdot')}(\mathbb{R}^n)$ cannot be weakened to $\phi \in L^q(\mathbb{R}^n)$ for $1 \leq q < (p_\cdot')$. In fact, for given $p_1 > 1$ and $1 \leq q < (p_1)'$, we can find a smooth exponent $p(\cdot)$ on $\mathbb{R}^n$ such that $p_- = p_1$, $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\phi \in L^q(\mathbb{R}^n)$ having compact support for which

$$\|\phi * f\|_{L^p(\cdot), \mathbb{R}^n} = \infty.$$ 

For this, let $a \in \mathbb{R}^n$ be a fixed point with $|a| > 1$ and let $p_2$ satisfy

$$\frac{1}{(p_1)'} + \frac{1}{p_2} < \frac{1}{q}.$$ 

Then choose a smooth exponent $p(\cdot)$ on $\mathbb{R}^n$ such that

$$p(x) = p_1 \text{ for } x \in B(0, 1/2), \quad p(x) = p_2 \text{ for } x \in B(a, 1/2),$$

$p_- = p_1$ and $p(x) =$ const. outside $B(0, |a| + 1)$. Take

$$\phi_j = j^{n/q} \chi_{B(a,j^{-1})} \quad \text{and} \quad f_j = j^{n/p_1} \chi_{B(0,j^{-1})}, \quad j = 2, 3, \ldots.$$ 

Then

$$\|\phi_j\|_{L^q, \mathbb{R}^n} = C < \infty \quad \text{and} \quad \|f_j\|_{L^{p(\cdot), \mathbb{R}^n}} = \|f_j\|_{L^{p_1, B(0,1/2)}} = C < \infty.$$ 

Note that if $x \in B(a,j^{-1})$ then

$$\phi_j * f_j(x) = j^{n/q+n/p_1} |B(a,j^{-1}) \cap B(x,j^{-1})| \geq C j^{n/q+n/p_1} j^{-n},$$

so that

$$\int_{\mathbb{R}^n} \{\phi_j * f_j(x)\}^{p(x)} dx \geq \int_{B(a,j^{-1})} \{\phi_j * f_j(x)\}^{p(x)} dx \geq C j^{p_2(n/q+n/p_1)-n} = C j^{p_2 n(1/q-1/(p_1)'-1/p_2)}.$$ 

Now consider

$$\phi = \sum_{j=2}^{\infty} j^{-2} \phi_{2j} \quad \text{and} \quad f = \sum_{j=2}^{\infty} j^{-2} f_{2j}.$$ 

Then $\phi \in L^q(\mathbb{R}^n)$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$. On the other hand,

$$\int_{\mathbb{R}^n} \{\phi * f(x)\}^{p(x)} dx \geq j^{-4} \int_{\mathbb{R}^n} \{\phi_{2j} * f_{2j}(x)\}^{p(x)} dx \geq C j^{-4j^{p_2 n(1/q-1/(p_1)'-1/p_2)}} \to \infty$$

as $j \to \infty$. Hence, $\|\phi * f\|_{L^p(\cdot), \mathbb{R}^n} = \infty.$
Remark 3.6. Cruz-Uribe and Fiorenza [1] gave an example showing that it can occur
\[ \limsup_{t \to 0} \| \phi_t \ast f \|_{L^p(R)} = \infty \]
for \( f \in L^{p(\cdot)}(R) \) when \( \phi \) does not have compact support.

By modifying their example, we can also find \( p(\cdot) \) and \( \phi \in L^{(p-\cdot)'}(R) \), whose support is not compact, such that
\[ \| \phi \ast f \|_{L^p(R)} \leq C \| f \|_{L^p(R)} \]
does not hold, namely there exists \( f_N (N = 1, 2, \ldots) \) such that \( \| f_N \|_{L^p(R)} \leq 1 \) and
\[ \lim_{N \to \infty} \| \phi \ast f_N \|_{L^p(R)} = \infty. \]

For this purpose, choose \( p_1 > 1, \ p_2 > p_1 \) and \( a > 1 \) such that
\[ -\frac{p_1}{p_2} - ap_1 + 2 > 0 \]
and let \( p(\cdot) \) be a smooth variable exponent on \( R \) such that
\[ p(x) = p_1 \text{ for } x \leq 0, \quad p(x) = p_2 \text{ for } x \geq 1 \]
and \( p_1 \leq p(x) \leq p_2 \) for \( 0 < x < 1 \). Set \( \phi = \sum_{j=1}^{\infty} \chi_j \), where \( \chi_j = \chi_{[-j, -j+j^{-a}]} \). Then
\[ \int_{R} \phi(x)^q dx = \sum_{j=1}^{\infty} \int_{-j}^{-j+j^{-a}} \chi_j(x)^q dx = \sum_{j} j^{-a} \leq C(a) < \infty \]
for any \( q > 0 \). Further set \( f_N = N^{-1/p_2} \chi_{[1,N+1]} \). Note that for \( 1-j+j^{-a} < x < 0 \) and \( j \leq N \)
\[ \chi_j \ast f_N(x) \geq \int_{x+j-j^{-a}}^{x+j} \chi_j(x-y)f_N(y)dy = N^{-1/p_2} j^{-a}, \]
so that
\[
\int_{R} \{ \phi \ast f_N(x) \}^{p(x)} dx \geq \int_{-\infty}^{0} \left\{ \sum_{j=1}^{\infty} \chi_j \ast f_N(x) \right\}^{p_1} dx \\
\geq \sum_{j=2}^{N} \int_{-j-j^{-a}}^{0} \{ \chi_j \ast f_N(x) \}^{p_1} dx \\
\geq N^{-p_1/p_2} \sum_{j=2}^{N} j^{-ap_1} (j - j^{-a} - 1) \\
\geq CN^{-p_1/p_2-ap_1+2} \to \infty \quad (N \to \infty). \]
4 Young type inequalities

Cruz-Uribe and Fiorenza [1] conjectured that Theorem A remains true if \( \phi \) satisfies the additional condition

\[
|\phi(x - y) - \phi(x)| \leq \frac{|y|}{|x|^{n+1}} \quad \text{when } |x| > 2|y|.
\] (4.1)

Noting that this condition implies

\[
\sup_{x, z \in B(0, 2^{j+1}) \setminus B(0, 2^j)} |\phi(x) - \phi(z)| \leq C 2^{-nj},
\]
we see that \( \lim_{|x| \to \infty} \phi(x) = 0 \) since \( \phi \in L^1(\mathbb{R}^n) \) and

\[
|\phi(x)| \leq C|x|^{-n}.
\] (4.2)

if \( \phi \) satisfies (4.1). In this connection we show

**Theorem 4.1.** Let \( p_+ > 1 \). Suppose that \( \phi \in L^1(\mathbb{R}^n) \cap L^{(p_0)'}(B(0, R)) \) and \( \phi \) satisfies (4.2) for \( |x| \geq R \). Then

\[
\|\phi * f\|_{\Phi_{p(-)},q(-),\mathbb{R}^n} \leq C(\|\phi\|_{L^1,\mathbb{R}^n} + \|\phi\|_{L^{(p_0)'},B(0,R)})\|f\|_{\Phi_{p(-)},q(-),\mathbb{R}^n}
\]

for all \( f \in L^{p(-)}(\log L)^{q(-)}(\mathbb{R}^n) \).

**Remark 4.2.** Theorem 4.1 does not imply an inequality

\[
\|\phi_t * f\|_{\Phi_{p(-)},q(-),\mathbb{R}^n} \leq C\|f\|_{\Phi_{p(-)},q(-),\mathbb{R}^n}
\]

with a constant \( C \) independent of \( t \in (0, 1] \) even if \( \phi \) satisfies (4.2) for all \( x \), because \( \{\|\phi_t\|_{L^{(p_0)'},B(0,R)}\}_{0 < t \leq 1} \) is not bounded.

**Proof of Theorem 4.1.** Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) such that \( \|f\|_{\Phi_{p(-)},q(-),\mathbb{R}^n} \leq 1 \). Suppose that \( \phi \) satisfies (4.2) for \( |x| \geq R \) and \( \|\phi\|_{L^1,\mathbb{R}^n} + \|\phi\|_{L^{(p_0)'},B(0,R)} \leq 1 \). Decompose \( \phi = \phi' + \phi'' \), where \( \phi' = \phi \chi_{B(0,R)} \). We first note by Theorem 1.2 that

\[
\|\phi' * f\|_{\Phi_{p(-)},q(-),\mathbb{R}^n} \leq C.
\]

Hence it suffices to show that

\[
\|\phi'' * f\|_{\Phi_{p(-)},q(-),\mathbb{R}^n} \leq C.
\]

For this purpose, write

\[
f = f \chi_{\{y \in \mathbb{R}^n : f(y) \geq 1\}} + f \chi_{\{y \in \mathbb{R}^n : f(y) < 1\}} = f_1 + f_2,
\]

where

f_1 = \int_{\{y \in \mathbb{R}^n : f(y) \geq 1\}} f(y) dy
\]

and

f_2 = \int_{\{y \in \mathbb{R}^n : f(y) < 1\}} f(y) dy.

as before. Then we have by (4.2) and (Φ)

\[ |\phi'' * f_1(x)| \leq C \int_{\mathbb{R}^n \setminus B(x, R)} |x - y|^{-n} f_1(y) \, dy \]

\[ \leq CR^{-n} \int_{\mathbb{R}^n} f_1(y) \, dy \]

\[ \leq CR^{-n} \int_{\mathbb{R}^n} \Phi_{p,q}(y, f(y)) \, dy \leq C. \]

Noting that \(|\phi'' * f_2| \leq 1\), we obtain

\[ \int_{B(0,R)} \Phi_{p,q}(x, \phi'' * f(x)) \, dx \leq C. \]

Next, let \(h(y) = \Phi_{p,q}(y, f(y))\). Then

\[ |\phi''| * h(x) \leq CR^{-n} \int_{\mathbb{R}^n} h(y) \, dy \leq CR^{-n}. \]

If \(x \in \mathbb{R}^n \setminus B(0, R)\), then we have by (4.2) and Lemma 3.3

\[ |\phi'' * f(x)| \leq \int_{B(0,|x|/2)} |\phi''(x - y)| f(y) \, dy + \int_{\mathbb{R}^n \setminus B(0,|x|/2)} |\phi''(x - y)| f(y) \, dy \]

\[ \leq C \left\{ |x|^{-n} \int_{B(|x|,|x|/2)} f(y) \, dy + \left( |\phi''| * h(x) \right)^{1/p(x)} + |x|^{-A/p(x)} \right\} \]

\[ \leq C \left\{ Mf(x) + \left( |\phi''| * h(x) \right)^{1/p(x)} + |x|^{-A/p(x)} \right\} \]

with \(A > n\). Now it follows from Proposition 2.5 that

\[ \int_{\mathbb{R}^n \setminus B(0, R)} \Phi_{p,q}(x, |\phi'' * f(x)|) \, dx \]

\[ \leq C \left\{ \int_{\mathbb{R}^n \setminus B(0, R)} \Phi_{p,q}(x, Mf(x)) \, dx \right\}

+ \int_{\mathbb{R}^n} |\phi| * h(x) \, dx + \int_{\mathbb{R}^n \setminus B(0, R)} |x|^{-A} \, dx \]

\[ \leq C, \]

as required. \(\square\)

**Theorem 4.3.** Let \(1 - p_-/p_+ \leq \theta < 1, 1 < \tilde{p} < p_-\),

\[ \frac{1}{s} = 1 - \frac{\theta}{\tilde{p}} \quad \text{and} \quad \frac{1}{r(x)} = \frac{1 - \theta}{p(x)}. \]

Take \(\nu = p_- / \tilde{p}\) if \(t^{-\nu - \tilde{p}} \Phi_{p,q}(x, t)\) is uniformly almost increasing in \(t\); otherwise choose \(1 \leq \nu < p_- / \tilde{p}\). Suppose that \(\phi \in L^1(\mathbb{R}^n) \cap L^s(\mathbb{R}^n) \cap L^{\nu'}(B(0, R))\) and \(\phi\) satisfies

\[ |\phi(x)| \leq C|x|^{-n/s} \]

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for $|x| \geq R$. Then
\[ \| \phi * f \|_{L^r, \mathbb{R}^n} \leq C(\| \phi \|_{L^1, \mathbb{R}^n} + \| \phi \|_{L^s, \mathbb{R}^n} + \| \phi \|_{L^{s'}, B(0, R)}) \|f\|_{\Phi_{\phi, \eta}} \mathbb{R}^n \]
for all $f \in L^{p'}(\log L)^{q'}(\mathbb{R}^n)$.

**Proof.** Suppose that $\| \phi \|_{L^1, \mathbb{R}^n} + \| \phi \|_{L^s, \mathbb{R}^n} + \| \phi \|_{L^{s'}, B(0, R)} \leq 1$ and $\phi$ satisfies
\[ |\phi(x)| \leq C|x|^{-n/s} \]
for $|x| \geq R$. Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$ such that $\| f \|_{\Phi_{\phi, \eta}} \mathbb{R}^n \leq 1$, and decompose
\[ f = f_1 + f_2, \]
where $f_1 = f \chi_{\{x \in \mathbb{R}^n : f(x) \geq 1\}}$. Let
\[ \frac{1}{r} = \frac{1 - \theta}{p_-} \quad \text{and} \quad \frac{1}{s_1} = 1 + \frac{1}{r} - \frac{1}{p_+}. \]

By our assumption, $s_1 \geq 1$. It follows from Young’s inequality for convolution that
\[ \| \phi * f_2 \|_{L^r, \mathbb{R}^n} \leq \| \phi \|_{L^{s_1}, \mathbb{R}^n} \| f_2 \|_{L^{p_1}, \mathbb{R}^n}. \]

Here note that $1 \leq s_1 < s$, so that $\| \phi \|_{L^{s_1}, \mathbb{R}^n} \leq \| \phi \|_{L^1, \mathbb{R}^n} + \| \phi \|_{L^s, \mathbb{R}^n} \leq 1$. Since $0 \leq f_2 < 1$, $\| f_2 \|_{L^{p_1}, \mathbb{R}^n} \leq C \| f \|_{\Phi_{\phi, \eta}} \mathbb{R}^n \leq C$. Thus, noting that $|\phi * f_2| \leq 1$ and
\[ \frac{1}{r(x)} - \frac{1}{r} = \frac{1 - \theta}{p(x)} - \frac{1 - \theta}{p_-} \leq 0, \]
we see that
\[ \| \phi * f_2 \|_{\Phi_{\phi, \eta}} \mathbb{R}^n \leq C \| \phi * f_2 \|_{L^r, \mathbb{R}^n} \leq C. \quad (4.3) \]

On the other hand, we have by Hölder’s inequality
\[
|\phi * f_1(x)| \leq \left( \int_{\mathbb{R}^n} |\phi(x - y)|^s f_1(y)^{p} dy \right)^{1 - \theta/p} \left( \int_{\mathbb{R}^n} |\phi(x - y)|^s dy \right)^{-1/p} \\
\times \left( \int_{\mathbb{R}^n} |f_1(y)|^{p_1} dy \right)^{\theta/p} \\
\leq C \left( |\phi|^s * f_1^p(x) \right)^{(1 - \theta)/p} \quad (4.4)
\]

Noting that $|\phi|^s \in L^1(\mathbb{R}^n) \cap L^{s'}(B(0, R))$, $|\phi|^s$ satisfies (4.2) for $|x| \geq R$ and $\|f_1^p\|_{\Phi_{\phi, \eta}} \mathbb{R}^n \leq C$, we find by Theorem 4.1
\[ \|\phi^s * f_1^p\|_{\Phi_{\phi, \eta}} \mathbb{R}^n \leq C. \]
Since (4.4) implies
\[ \Phi_{r,q}(x, \phi * f_1(x)) \leq C \Phi_{p,q}(x, |\phi|^s f_1^p(x)), \]
it follows that
\[ \|\phi * f_1\|_{\Phi_{r,q}, R^n} \leq C. \]
Thus, together with (4.3), we obtain
\[ \|\phi * f\|_{\Phi_{r,q}, R^n} \leq C, \]
as required.

**Remark 4.4.** Cruz-Uribe and Fiorenza [1] conjectured that Theorem A remains true if \( \phi \) satisfies the additional condition (4.1).

If \( p_- > 1 \), this conjecture was shown to be true by D. Cruz-Uribe, A. Fiorenza, J.M. Martell and C. Pérez in [3], using an extrapolation theorem ([3, Theorem 1.3 or Corollary 1.11]). Using our Proposition 2.5, we can prove the following extension of [3, Theorem 1.3]:

**Proposition 4.5.** Let \( \mathcal{F} \) be a family of ordered pairs \((f, g)\) of nonnegative measurable functions on \( \mathbb{R}^n \). Suppose that for some \( 0 < p_0 < p_- \),
\[ \int_{\mathbb{R}^n} f(x)^{p_0} w(x) \, dx \leq C_0 \int_{\mathbb{R}^n} g(x)^{p_0} w(x) \, dx \]
for all \((f, g)\) \( \in \mathcal{F} \) and for all \( A_1 \)-weights \( w \), where \( C_0 \) depends only on \( p_0 \) and the \( A_1 \)-constant of \( w \). Then
\[ \|f\|_{\Phi_{p,q}, R^n} \leq C \|g\|_{\Phi_{p,q}, R^n} \]
for all \((f, g)\) \( \in \mathcal{F} \) such that \( g \in L^{p_0}(\log L)^{q_0}(\mathbb{R}^n) \).

Then, as in [3, p. 249], we can prove:

**Theorem 4.6.** Assume that \( p_- > 1 \). If \( \phi \) is an integrable function on \( \mathbb{R}^n \) satisfying (4.1), then
\[ \|\phi_t * f\|_{\Phi_{p,q}, R^n} \leq C \|f\|_{\Phi_{p,q}, R^n} \]
for all \( t > 0 \) and \( f \in L^{p_0}(\log L)^{q_0}(\mathbb{R}^n) \). If in addition \( \int \phi(x) \, dx = 1 \), then
\[ \lim_{t \to 0} \|\phi_t * f - f\|_{\Phi_{p,q}, R^n} = 0. \]
5 Appendix

For \( p \geq 1, q \in \mathbb{R} \) and \( c \geq e \), we consider the function
\[
\Phi(t) = \Phi(p, q, c; t) = t^p (\log(c + t))^q, \quad t \in [0, \infty).
\]

In this appendix, we give a proof of the following elementary result:

**Theorem 5.1.** Let \( X \) be a non-empty set and let \( p(\cdot) \) and \( q(\cdot) \) be real valued functions on \( X \) such that \( 1 \leq p(x) \leq p_0 < \infty \) for all \( x \in X \). Then, the following (1) and (2) are equivalent to each other:

1. There exists \( c_0 \geq e \) such that \( \Phi(p(x), q(x), c_0; \cdot) \) is convex on \([0, \infty)\) for every \( x \in X \);
2. There exists \( K > 0 \) such that \( K(p(x) - 1) + q(x) \geq 0 \) for all \( x \in X \).

This theorem may be well known; however the authors fail to find any literature containing this result.

This theorem is a corollary to the following

**Proposition 5.2.** (1) If
\[
(1 + \log c)(p - 1) + q \geq 0,
\]
then \( \Phi \) is convex on \([0, \infty)\).

(2) Given \( p_0 > 1 \) and \( c \geq e \), there exists \( K = K(p_0, c) > 0 \) such that \( \Phi \) is not convex on \([0, \infty)\) whenever \( 1 \leq p \leq p_0 \) and \( q < -K(p - 1) \).

**Proof.** By elementary calculation we have
\[
\Phi''(t) = t^{p-2}(c + t)^{-2}(\log(c + t))^{q-2}G(t)
\]
with
\[
G(t) = p(p-1)(c+t)^2(\log(c+t))^2 + 2pqt(c+t)\log(c+t) - qt^2\log(c+t) + q(q-1)t^2
\]
for \( t > 0 \). \( \Phi(t) \) is convex on \([0, \infty)\) if and only if \( G(t) \geq 0 \) for all \( t \in (0, \infty) \).

1. If \( q \geq 0 \), then
\[
G(t) \geq qt(2p(c + t) - t) \log(c + t) - qt^2 \geq qt(2pc + 2(p - 1)t) \geq 0
\]
for all \( t \in (0, \infty) \), so that \( \Phi \) is convex on \([0, \infty)\).

If \(-1 + \log c)(p - 1) \leq q < 0 \), then
\[
G(t) = p \left\{ \sqrt{p - 1} (c + t) \log(c + t) + \frac{q}{\sqrt{p - 1}} t \right\}^2
\]
\[\quad - \frac{pq^2}{p - 1} t^2 - qt^2 \log(c + t) + q(q - 1)t^2 \]
\[\geq (-q)t^2 \left( \frac{pq}{p - 1} + \log c - (q - 1) \right) \]
\[= (-q)t^2 \left( \frac{q}{p - 1} + \log c + 1 \right) \geq 0
\]
for all \( t \in (0, \infty) \), so that \( \Phi \) is convex on \([0, \infty)\).

(2) If \( p = 1 \) and \( q < 0 \), then
\[
G(t) = qt((t + 2c) \log(c + t) + (q - 1)t) \to -\infty
\]
as \( t \to \infty \). Hence \( \Phi \) is not convex on \([0, \infty)\).

Next, let \( 1 < p \leq p_0 \) and \( q = -k(p - 1) \) with \( k > 0 \). Then
\[
\frac{G(t)}{p - 1} = p((c + t) \log(c + t) - kt)^2 + k(\log(c + t) - k + 1)t^2
\leq p_0((c + t) \log(c + t) - kt)^2 + k(\log(c + t) - k + 1)t^2.
\]
Let \( \lambda = 1 - 1/(2p_0) \). Then \( 0 < \lambda < 1 \). If \( k > (\log c)/\lambda \), there is (unique) \( t_k > 0 \) such that \( \log(c + t_k) = \lambda k \). Note that \( t_k/k \to \infty \) as \( k \to \infty \). We have
\[
\frac{G(t_k)}{p - 1} \leq p_0((c + t_k)\lambda k - kt_k)^2 + k(\lambda k - k + 1)t_k^2
= kt_k^2 \left\{ (p_0(1 - \lambda) - 1)(1 - \lambda)k + 1 - 2p_0c\lambda(1 - \lambda)\frac{k}{t_k} + p_0c^2\lambda^2\frac{k}{t_k^2} \right\}.
\]
Since \( p_0(1 - \lambda) - 1 = -1/2 \), it follows that there is \( K = K(c, p_0) > (\log c)/\lambda \) such that \( G(t_k) < 0 \) whenever \( k \geq K \). Hence \( \Phi \) is not convex if \( 1 < p \leq p_0 \) and \( q \leq -K(p - 1) \).

\[ \square \]

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