Sobolev’s inequalities and vanishing integrability for Riesz potentials of functions in the generalized Lebesgue space $L^{p(\cdot)}(\log L)^{q(\cdot)}$

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Abstract

Our aim in this paper is to deal with Sobolev’s inequalities for Riesz potentials of functions belonging to $L^{p(\cdot)}(\log L)^{q(\cdot)}$. To do so, we study the boundedness of Hardy-Littlewood maximal functions and apply the Hedberg’s trick. As an application, we treat vanishing integrability for Riesz potentials.

1 Introduction

In recent years, the generalized Lebesgue spaces have attracted more and more attention, in connection with the study of elasticity, fluid mechanics and differential equations with $p(\cdot)$-growth; see for example Orlicz [29], Kováčik-Rákosník [22], Edmunds-Rákosník [7] and Růžička [30].

In this paper, following Cruz-Uribe and Fiorenza [4], we consider continuous functions $p(\cdot): \mathbb{R}^n \to [1, \infty)$ and $q(\cdot): \mathbb{R}^n \to \mathbb{R}$, which are called variable exponents. In the present paper, we always assume that $p(\cdot)$ and $q(\cdot)$ are bounded on $\mathbb{R}^n$ and

$$p_- = \inf_{x \in \mathbb{R}^n} p(x) > 1.$$  \hfill (1.1)

Our typical examples of $p(\cdot)$ and $q(\cdot)$ are the exponents satisfying the following log-Hölder conditions:

$$|p(x) - p(y)| \leq \frac{a \log(e + \log(e + |x - y|^{-1}))}{\log(e + |x - y|^{-1})} + \frac{b}{\log(e + |x - y|^{-1})}$$

and

$$|q(x) - q(y)| \leq \frac{c \log(e + \log(e + \log(e + |x - y|^{-1})))}{\log(e + \log(e + |x - y|^{-1}))} + \frac{d}{\log(e + \log(e + |x - y|^{-1}))}$$

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whenever $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, where $a, b, c, d \geq 0$ are constants. In [14], Harjulehto and Hästö discussed the continuity of Sobolev functions, and in the paper by Hästö [19], he studied the integrability of maximal functions. For further related results, we refer the reader to [10], [11], [12] and [26].

By condition (1.1), one can find a constant $c_0 \geq e$ such that

$$t^{p(x)}(\log(c_0 + t))^{\rho(x)}$$

is a convex function of $t$ for each fixed $x \in \mathbb{R}^n$. (1.2)

We define the space $L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$ of all measurable functions $f$ on an open set $G$ such that

$$\int_G \left( \frac{|f(y)|}{\lambda} \right)^{p(y)} \left( \log \left( c_0 + \frac{|f(y)|}{\lambda} \right) \right)^{q(y)} dy < \infty$$

for some $\lambda > 0$. We define the norm

$$\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} = \inf \left\{ \lambda > 0 : \int_G \left( \frac{|f(y)|}{\lambda} \right)^{p(y)} \left( \log \left( c_0 + \frac{|f(y)|}{\lambda} \right) \right)^{q(y)} dy \leq 1 \right\}$$

for $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$. In case $q = 0$ on $G$, $L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$ is denoted by $L^{p(\cdot)}(G)$ for simplicity.

For $0 < \alpha < n$, we define the Riesz potential of order $\alpha$ for a locally integrable function $f$ on $\mathbb{R}^n$ by

$$U_\alpha f(x) = \int_{\mathbb{R}^n} |x - y|^{\alpha-n} f(y) dy.$$  

Here it is natural to assume that

$$\int_{\mathbb{R}^n} (1 + |y|)^{\alpha-n} |f(y)| dy < \infty,$$

which is equivalent to the condition that $U_\alpha |f| \neq \infty$ (see [25, Theorem 1.1, Chapter 2]).

Let $B(x, r)$ denote the open ball centered at $x$ with radius $r$. For a locally integrable function $f$ on an open set $G$, we consider the maximal function $Mf$ defined by

$$Mf(x) = \sup_B \frac{1}{|B|} \int_{B \cap G} |f(y)| dy,$$

where the supremum is taken over all balls $B = B(x, r)$ and $|B|$ denotes the volume of $B$. Diening [5] was the first to prove the local boundedness of maximal functions in the Lebesgue spaces of variable exponents satisfying the log-Hölder condition.

Our first aim in this paper is to obtain Sobolev’s inequality for Riesz potentials of functions in $L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$. To do so, we apply Hedberg’s trick [20] by use of the boundedness of maximal functions. Our result (see Theorem 2.8 below) is given in Section 2, which is an extension of Almeida-Samko [3], Diening [6], Futamura-Mizuta [10], Futamura-Mizuta-Shimomura [11, 12], Harjulehto-Hästö-Pere [18], Kokilashvili-Samko [21], Mizuta-Shimomura [27] and Samko-Vakulov [31].
For a measurable function $u$ on $\mathbb{R}^n$, we define the integral mean over a measurable set $E \subset \mathbb{R}^n$ of positive measure by
\[
\int_E u(x) \, dx = \frac{1}{|E|} \int_E u(x) \, dx,
\]
where $|E|$ denotes the Lebesgue measure of $E$. For a locally integrable function $f$ on $\mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ is called a Lebesgue point for $f$ if
\[
\lim_{r \to 0^+} \int_{B(x_0, r)} |f(x) - f(x_0)| \, dx = 0.
\]

Our second aim in this paper is to show that every point except in a small set is a Lebesgue point for $U_\alpha f$ with $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$. In the classical case, we refer the reader to [1], [24], [25], [32] and [34]. We aim to extend the results by Fiorenza [8], Futamura-Mizuta [10], Futamura-Mizuta-Shimomura [11] and Harjulehto-Hästö [13] in the variable exponent case.

A famous Trudinger inequality [33] insists that Sobolev functions in $W^{1,n}$ satisfy finite exponential integrability. Adams and Hurri-Syrjänen [2, Theorem 1.6] and Mizuta and Shimomura [28, Theorems 3.2, 4.5 and 5.2] have recently established the vanishing exponential integrability for Riesz potentials $U_\alpha f$ with $f \in L^{n/\alpha}(\mathbb{R}^n)$. In connection with these results, we study the vanishing exponential integrability for $U_\alpha f$; we in fact show (in Theorem 5.5 below) that
\[
\lim_{r \to 0^+} \int_{B(x_0, r)} \left\{ \exp \left( A |U_\alpha f(x) - U_\alpha f(x_0)|^{a^2} \times (\log(1 + |U_\alpha f(x) - U_\alpha f(x_0)|)^{b^2} \right) - 1 \right\} \, dx = 0
\]
for all $A > 0$ and all $x_0$ except in a small set, where $a^2 > 0$ and $b^2$ are suitable constants determined by $p(\cdot)$ and $q(\cdot)$.

2 Sobolev’s inequality

Throughout this paper, let $C$ denote various constants independent of the variables in question and $C(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$.

We say that a positive nondecreasing function $\varphi$ on the interval $[0, \infty)$ satisfies $(\varphi)$ if there exist $\varepsilon_1 > 0$ and $0 < r_1 < 1$ such that

\[
(\varphi) \quad (\log(1/r))^{-\varepsilon_1} \varphi(1/r) \text{ is nondecreasing on } (0, r_1).
\]

Similarly, we say that a positive nondecreasing function $\psi$ on the interval $[0, \infty)$ satisfies $(\psi)$ if there exist $\varepsilon_2 > 0$ and $0 < r_2 < 1/e$ such that

\[
(\psi) \quad (\log(\log(1/r)))^{-\varepsilon_2} \psi(1/r) \text{ is nondecreasing on } (0, r_2).
\]
Consider positive nondecreasing functions \( \varphi \) satisfying \((\varphi)\) and \( \psi \) satisfying \((\psi)\). Set 
\[
\varepsilon_0 = \max\{\varepsilon_1, \varepsilon_2\}.
\]
For the sake of convenience, we assume that
\[
(\varphi') \quad \varphi(t) \geq e^{\varepsilon_0} \text{ for all } t > 0,
\]
\[
(\psi') \quad \psi(t) \geq e^{\varepsilon_0} \text{ for all } t > 0.
\]
Set \( \omega(r) = \frac{\log \varphi(1/r)}{\log(1/r)} \) and \( \eta(r) = \frac{\log \psi(1/r)}{\log(\log(1/r))} \).

First we give the following results, which can be derived by conditions \((\varphi)\) and \((\varphi')\).

**Lemma 2.1** ([25, Lemma 3.1, Section 5.3], [26, Lemmas 2.1 and 2.2]).

(i) \( \varphi(r) \) is of log-type, that is, there exists \( C > 0 \) such that
\[
C^{-1} \varphi(r) \leq \varphi(r^2) \leq C \varphi(r) \quad \text{whenever } r > 0.
\]
(ii) For \( \gamma > 0 \), there exists \( C > 0 \) such that
\[
t^{-\gamma} \varphi(t) \leq Cs^{-\gamma} \varphi(s) \quad \text{whenever } t \geq s > 0.
\]
(iii) There exists \( 0 < \tilde{r}_1 < r_1 \) such that \( \omega(r) \) is nondecreasing on \([0, \tilde{r}_1]\).

Further, we see from conditions \((\psi)\) and \((\psi')\) that \( \psi \) satisfies (i), (ii) and
(iv) there exists \( 0 < \tilde{r}_2 < r_2 \) such that \( \eta(r) \) is nondecreasing on \([0, \tilde{r}_2]\).

Condition (2.1) implies the doubling condition on \( \varphi \), that is, there exists a constant \( C > 1 \) such that
\[
\varphi(r) \leq \varphi(2r) \leq C \varphi(r) \quad \text{whenever } r > 0.
\]
In what follows, set 
\[
r_0 = \min\{\tilde{r}_1, \tilde{r}_2\}.
\]
If \( r > r_0 \), then we set
\[
\omega(r) = \omega(r_0) \quad \text{and} \quad \eta(r) = \eta(r_0).
\]
Our typical example of \( \varphi \) is of the form
\[
\varphi(r) = a(\log(\beta_0 + r))^{b}(\log(\beta_0 + \log(\beta_0 + r)))^c,
\]
where \( a > 0, \, b \geq 0, \, c \in \mathbb{R} \) and \( \beta_0 \geq e \) are chosen so that \( \varphi(r) \) is nondecreasing on \([0, \infty)\); similarly, that of \( \psi \) is of the form

\[
\psi(r) = a(\log(\beta_0 + \log(\beta_0 + r)))^b(\log(\beta_0 + \log(\beta_0 + \log(\beta_0 + r))))^c.
\]

Note that if \( b = 0 \), then \( c \geq 0 \).

For a variable exponent \( p(\cdot) \) on \( \mathbb{R}^n \), set

\[
p_- = \inf_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p_+ = \sup_{x \in \mathbb{R}^n} p(x).
\]

Now we consider continuous exponents \( p(\cdot) \) and \( q(\cdot) \) on \( \mathbb{R}^n \) such that

\begin{align*}
(p1) \quad 1 &< p_- \leq p_+ < \infty; \\
(p2) \quad |p(x) - p(y)| &\leq \omega(|x - y|) \quad \text{whenever } x, y \in \mathbb{R}^n. \\
(q1) \quad -\infty &< q_- \leq q_+ < \infty; \\
(q2) \quad |q(x) - q(y)| &\leq \eta(|x - y|) \quad \text{whenever } x, y \in \mathbb{R}^n.
\end{align*}

Recall that the generalized Lebesgue space \( L^{p(\cdot)}(\log L)^{q(\cdot)}(G) \) given in the introduction is a Banach space with the norm \( \| \cdot \|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \). For \( 0 < \alpha < n \), we consider the Riesz potential \( U_\alpha f \) of \( f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(G) \) defined by

\[
U_\alpha f(x) = \int_G |x - y|^{\alpha - n} f(y) dy.
\]

Our first aim is to determine the space

\[
\{U_\alpha f : f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(G)\}.
\]

In our discussions below, it is convenient to note the following result.

**Lemma 2.2** If \( r > 0 \) and \( t > 0 \), then

\[
\varphi(rt) \leq C \varphi(r) \varphi(t),
\]

where \( C \) is the constant appearing in (2.1).

For this, it suffices to note that

\[
\varphi(rt) \leq \max \{ \varphi(r^2), \varphi(t^2) \} \leq \max \{ C \varphi(r), C \varphi(t) \} \leq C \varphi(r) \varphi(t)
\]

since \( \varphi \) is nondecreasing and \( \varphi(t) \geq 1 \).
Corollary 2.3 Set \( \kappa(y, t) = t(\log(e + t))^{y_1} \varphi(t)^{y_2} \psi(t)^{y_3} \) for \( y = (y_1, y_2, y_3) \) and \( t \geq 0 \). Then

\[
\kappa(y, at) \leq \tau(y, a) \kappa(y, t)
\]

whenever \( a, t > 0 \), where

\[
\tau(y, a) = a \max \left\{ (C \log(e + a))^{y_1}, (C \log(e + a^{-1}))^{-y_1} \right\} \\
\times \max \left\{ (C \varphi(a))^{y_2}, (C \varphi(a^{-1}))^{-y_2} \right\} \max \left\{ (C \psi(a))^{y_3}, (C \psi(a^{-1}))^{-y_3} \right\}.
\]

For \( A > n \) we set

\[
\Phi_A(x, t) = \kappa(q(x)/p(x), -A/p(x)^2, -1/p(x), t)^{p(x)}.
\]

By Corollary 2.3 and conditions \((\varphi')\), \((\psi')\), \((p1)\) and \((q1)\), we see that

\[
\Phi_A(x, at) \leq C \tau(x, a)^{p(x)} \Phi_A(x, t)
\]

whenever \( a, t > 0 \) and \( x \in \mathbb{R}^n \), where

\[
\tau(x, a) = a \max \left\{ \left( \log(e + a) \right)^{q(x)/p(x)}, \left( \log(e + a^{-1}) \right)^{-q(x)/p(x)} \right\} \\
\times \varphi(a^{-1})^{A/p(x)^2} \psi(a^{-1})^{1/p(x)}.
\]

We see that

\[
\lim_{a \to 0^+} \sup_{x \in \mathbb{R}^n} \tau(x, a) = 0
\]

and \( \Phi_A(x, \cdot) \) satisfies the doubling condition for each fixed \( x \in \mathbb{R}^n \); more precisely,

\[
C^{-1} \Phi_A(x, t) \leq \Phi_A(x, 2t) \leq C \Phi_A(x, t)
\]

for all \( t > 0 \) and \( x \in \mathbb{R}^n \).

From now on let \( G \) be a bounded open set in \( \mathbb{R}^n \). Denote by \( \Phi_A(G) \) the family of all measurable functions \( u \) on \( G \) such that

\[
\int_G \Phi_A(x, |u(x)|/\lambda)dx < \infty
\]

for some \( \lambda > 0 \) and define

\[
\|u\|_{\Phi_A(G)} = \inf \left\{ \lambda > 0 : \int_G \Phi_A(x, |u(x)|/\lambda)dx \leq 1 \right\}
\]

for \( u \in \Phi_A(G) \).

Lemma 2.4 There exists \( C > 0 \) such that

\[
\int_G \Phi_A(x, |u(x)|)dx \leq C \|u\|_{\Phi_A(G)}
\]

for all measurable functions \( u \in \Phi_A(G) \) with \( \|u\|_{\Phi_A(G)} \leq 1 \).
Proof. If \( \|u\|_{\Phi_A(G)} \leq 1 \), then we can find \( \lambda > 0 \) such that \( \|u\|_{\Phi_A(G)} \leq \lambda < 2 \) and
\[
\int_G \Phi_A(x, |u(x)|/\lambda)dx \leq 1.
\]
By inequality (2.3) we find
\[
\int_G \Phi_A(x, |u(x)|)dx \leq \sup_{x \in G} \tau(x, \lambda)^p(x) \int_G \Phi_A(x, |u(x)|/\lambda)dx
\leq \sup_{x \in G} \tau(x, \lambda)^p(x)
\leq C\lambda.
\]
Letting \( \lambda \to \|u\|_{\Phi_A(G)} \) yields the required inequality. \( \square \)

Lemma 2.5 \( \cdot \|_{\Phi_A(G)} \) is a quasi-norm, that is, for \( u, v \in \Phi_A(G) \) and a real number \( k \),

(i) \( \|u\|_{\Phi_A(G)} = 0 \) if and only if \( u = 0 \);

(ii) \( \|k\cdot u\|_{\Phi_A(G)} = |k|\|u\|_{\Phi_A(G)} \);

(iii) \( \|u + v\|_{\Phi_A(G)} \leq C \left( \|u\|_{\Phi_A(G)} + \|v\|_{\Phi_A(G)} \right) \).

Proof. First we note that (i) follows from Lemma 2.4. Since (ii) is trivial, it suffices to show (iii). For this purpose, we take \( \lambda_j \) \((j = 1, 2)\) such that \( \|u_j\|_{\Phi_A(G)} \leq \lambda_j < 2\|u_j\|_{\Phi_A(G)} \) and
\[
\int_G \Phi_A(x, |u_j(x)|/\lambda_j)dx \leq 1.
\]
We note from (2.3) that
\[
\Phi_A(x, s) \leq C\Phi_A(x, t) \tag{2.6}
\]
for all \( x \in G \) and \( 0 < s < t \). Hence, with the aid of (2.5), we obtain
\[
\int_G \Phi_A(x, a(|u_1(x) + u_2(x)|/\lambda_1 + \lambda_2))dx
\leq C \int_G \{ \Phi_A(x, a|u_1(x)|/\lambda_1) + \Phi_A(x, a|u_2(x)|/\lambda_2) \} dx
\leq C \sup_{x \in G} \tau(x, a)^p(x) \left\{ \int_G \Phi_A(x, |u_1(x)|/\lambda_1)dx + \int_G \Phi_A(x, |u_2(x)|/\lambda_2)dx \right\}
\leq C \sup_{x \in G} \tau(x, a)^p(x).
\]
Now, in view of (2.4), we take \( a > 0 \) so small that
\[
\int_G \Phi_A(x, a(|u_1(x) + u_2(x)|)/(\lambda_1 + \lambda_2))dx \leq 1.
\]
Then we obtain
\[ \|u_1 + u_2\|_{\Phi_A(G)} \leq n^{-1}(\lambda_1 + \lambda_2), \]
which proves (iii), as required. 

Next we show the boundedness of the maximal operator from \(L^{p}(\log L)^{q}(G)\) into \(\Phi_A(G)\). For this purpose, we need the following result.

**Lemma 2.6 (cf. [27, Lemma 2.4]).** Let \(f\) be a nonnegative measurable function on \(G\) with \(\|f\|_{L^{p}(\log L)^{q}(G)} \leq 1\) such that \(f(x) \geq 1\) or \(f(x) = 0\) for each \(x \in G\). Set
\[ I = I(x, r, f) = \frac{1}{|B(x, r)|} \int_{B(x, r) \cap G} f(y) \, dy \]
and
\[ J = J(x, r, f) = \frac{1}{|B(x, r)|} \int_{B(x, r) \cap G} g(y) \, dy, \]
where \(g(y) = f(y)^{p(y)}(\log(c_0 + f(y)))^{q(y)}\). Then
\[ I \leq CJ^{1/p(x)}(\log(e + J))^{-q(x)/p(x)}\varphi(J)^{n/p(x^2)}\psi(J)^{1/p(x)}. \]

**Proof.** Let \(f\) be a nonnegative measurable function on \(G\) with \(\|f\|_{L^{p}(\log L)^{q}(G)} \leq 1\) such that \(f(x) \geq 1\) or \(f(x) = 0\) for each \(x \in G\). First consider the case when \(J \geq 1\). Note that
\[ J^{\omega(CJ^{-1/n})} \leq C\varphi(J)^n \]
and
\[ \varphi(J)\omega(CJ^{-1/n}) \leq C. \]
Further note that
\[ (\log J)^{\omega(CJ^{-1/n})} \leq C\psi(J). \]
Set
\[ k = CJ^{1/p(x)}(\log(e + J))^{-q(x)/p(x)}\varphi(J)^{n/p(x^2)}\psi(J)^{1/p(x)}. \]
Then we have
\[ I \leq k + \frac{C}{|B(x, r)|} \int_{B(x, r)} f(y) \left( \frac{f(y)}{k} \right)^{p(y)-1} \left( \frac{\log(c_0 + f(y))}{\log(c_0 + k)} \right)^{q(y)} \, dy. \]
Since \(\|f\|_{L^{p}(\log L)^{q}(G)} \leq 1\), we find
\[ J \leq \frac{1}{|B(x, r)|} \int_{G} g(y) \, dy \leq \frac{1}{|B(x, r)|}. \]
Hence we obtain for \(y \in B(x, r)\),
\[ k^{-p(y)} \leq \left\{ CJ^{1/p(x)}(\log(e + J))^{-q(x)/p(x)}\varphi(J)^{n/p(x^2)}\psi(J)^{1/p(x)} \right\}^{-p(x)+\omega(r)} \]
\[ \leq \left\{ CJ^{1/p(x)}(\log(e + J))^{-q(x)/p(x)}\varphi(J)^{n/p(x^2)}\psi(J)^{1/p(x)} \right\}^{-p(x)+\omega(CJ^{-1/n})} \]
\[ \leq CJ^{-1}(\log(e + J))^{q(x)}\psi(J)^{-1} \]
and
\[(\log(c_0 + k))^{-q(y)} \leq \{C \log(e + J)\}^{-q(x) + \eta(r)} \leq \{C \log(e + J)\}^{-q(x) + \eta(CJ^{-1/n})} \leq C(\log(e + J))^{-q(x)\psi(J)}.
\]

Consequently it follows that
\[I \leq CJ^{1/p(x)}(\log(e + J))^{-q(x)/p(x)}\varphi(J)^{n/p(x)^2}\psi(J)^{1/p(x)}.
\]

In the case \(J \leq 1\), using Lemma 2.1 (ii), we find
\[I \leq CJ \leq CJ^{1/p(x)}(\log(e + J))^{-q(x)/p(x)}\varphi(J)^{n/p(x)^2}\psi(J)^{1/p(x)}.
\]

Now the result follows.

Now we are ready to show the boundedness of the maximal operator \(\mathcal{M}\), as an extension of Diening [5] and Cruz-Uribe and Fiorenza [4].

**Theorem 2.7** The maximal operator \(\mathcal{M}\) is bounded from \(L^{p(·)}(\log L)^{q(·)}(G)\) to \(\Phi_A(G)\) for all \(A > n\).

**Proof.** Let \(f\) be a nonnegative measurable function on \(G\) with \(\|f\|_{L^{p(·)}(\log L)^{q(·)}(G)} \leq 1\). Write
\[f = f \chi_{\{y : f(y) \geq 1\}} + f \chi_{\{y : f(y) < 1\}} = f_1 + f_2,
\]
where \(\chi_E\) denotes the characteristic function of \(E\). Then, since \(Mf_2 \leq 1\) on \(G\), we see from Lemmas 2.6 and 2.1 that
\[Mf(x)^{p(x)}(\log(e + Mf(x)))^{q(x)}\varphi(Mf(x))^{-n/p(x)}\psi(Mf(x))^{-1} \leq C + CMg(x),
\]
where \(g(y) = f(y)^{p(y)}(\log(c_0 + f(y)))^{q(y)}\). Now take \(p_1\) such that \(1 < p_1 < p^-\). Then, applying the above inequality with \(p(x), \varphi(r), q(x)\) and \(\psi(r)\) replaced by \(p(x)/p_1, \varphi(r)^{1/p_1}, q(x)/p_1\) and \(\psi(r)^{1/p_1}\) respectively, we obtain
\[
\{Mf(x)^{p(x)}(\log(e + Mf(x)))^{q(x)}\varphi(Mf(x))^{-n/p_1}\psi(Mf(x))^{-1}\}^{1/p_1} \leq C + CMg_1(x),
\]
where \(g_1(y) = f(y)^{p(y)/p_1}(\log(c_0 + f(y)))^{q(y)/p_1} = g(y)^{1/p_1}\), so that
\[\Phi_A(x, Mf(x)) \leq C + CMg_1(x)^{p_1}
\]
with \(A = np_1\). Hence, by the well-known boundedness of the maximal operator, we see that
\[
\int_G \Phi_A(x, Mf(x))dx \leq C,
\]
as required. \(\square\)
By applying the boundedness of the maximal operator and Hedberg’s trick [20], we establish the Sobolev type inequality for Riesz potentials, as an extension of the authors [27, Theorem 3.5] (see also Almeida-Samko [3], Diening [6], Futamura-Mizuta [10], Futamura-Mizuta-Shimomura [11, 12], Harjulehto-Hästö-Pere [18], Kokilashvili-Samko [21] and Samko-Vakulov [31]).

If \( p_+ < n/\alpha \), then we let

\[
1/p^+(x) = 1/p(x) - \alpha/n.
\]

For \( A > n \), setting

\[
\tilde{\Phi}_A(x, t) = \kappa(q(x)/p(x), -A/p(x)^2, -1/p(x), t)^{p(x)},
\]

we define the family \( \Phi_A(G) \) and the corresponding quasi-norm \( \| \cdot \|_{\Phi_A(G)} \) (see the proof of Lemma 2.5).

**Theorem 2.8** Suppose \( p_+(G) = \sup_{x \in G} p(x) < n/\alpha \). If \( A > n \), then

\[
\| U_{\alpha} f \|_{\Phi_A(G)} \leq C \| f \|_{L^{p(x)}(\log L)^{q(x)}(G)}
\]

for \( f \in L^{p(x)}(\log L)^{q(x)}(G) \).

To show this, we need the following estimate for Riesz potentials.

**Lemma 2.9** Let \( f \) be a nonnegative measurable function on \( G \) with \( \| f \|_{L^{p(x)}(\log L)^{q(x)}(G)} \leq 1 \). Then

\[
\int_{G \setminus B(x, \delta)} |x - y|^{-n/p(x)} f(y) dy \leq C \int_{G} f(y) dy \leq C \int_{G} g(y) dy \leq C,
\]

for all \( x \in G \) and \( 0 < \delta < r_0 \), where \( C \) is a positive constant independent of \( x, \delta \) and \( f \).

**Proof.** Let \( f \) be a nonnegative measurable function on \( G \) with \( \| f \|_{L^{p(x)}(\log L)^{q(x)}(G)} \leq 1 \) and \( 0 < \delta < r_0 \). First note that

\[
\int_{G \setminus B(x, \delta)} |x - y|^{-n/p(x)} f(y) dy \leq C \int_{G} f(y) dy \leq C \int_{G} g(y) dy \leq C,
\]

where \( g(y) = f(y)^{p(y)} (\log (c_0 + f(y)))^{q(y)} \) as in Lemma 2.6. Next set

\[
k = |x - y|^{-n/p(x)} (\log (1/|x - y|))^{-q(x)/p(x)} \varphi(|x - y|^{-1})^{n/p(x)} \psi(|x - y|^{-1})^{1/p(x)}.
\]

Then we have

\[
\int_{B(x, \delta) \setminus B(x, \delta)} |x - y|^{-n/p(x)} f(y) dy \leq \int_{B(x, \delta) \setminus B(x, \delta)} k |x - y|^{-n/p(x)} dy
\]

\[
+ C \int_{B(x, \delta) \setminus B(x, \delta)} |x - y|^{-n/p(x)} f(y) \left( \frac{f(y)}{k} \right)^{p(y) - 1} \left( \frac{\log (c_0 + f(y))}{\log (c_0 + k)} \right)^{q(y)} dy.
\]

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Here note that
\[ k^{-p(y)} \leq C|x - y|^n (\log(1/|x - y|))^{q(x)} \psi(|x - y|^{-1})^{-1} \]
and
\[ (\log(c_0 + k))^{-q(y)} \leq C(\log(1/|x - y|))^{-q(x)} \psi(|x - y|^{-1}) \]
for \( y \in B(x, r_0) \setminus B(x, \delta) \), so that
\[
\int_{B(x, r_0) \setminus B(x, \delta)} |x - y|^{\alpha-n} f(y) dy \\
\leq C \delta^{\alpha-n/p(x)} (\log(1/\delta))^{-q(x)/p(x)} \varphi(\delta^{-1})^{n/p(x)} \psi(\delta^{-1})^{1/p(x)} \\
+ C \int_{B(x, r_0) \setminus B(x, \delta)} |x - y|^{\alpha-n/p(x)} (\log(1/|x - y|))^{-q(x)/p(x)} \\
\times \varphi(|x - y|^{-1})^{n/p(x)} g(y) dy \\
\leq C \delta^{\alpha-n/p(x)} (\log(1/\delta))^{-q(x)/p(x)} \varphi(\delta^{-1})^{n/p(x)} \psi(\delta^{-1})^{1/p(x)} \\
+ C \delta^{\alpha-n/p(x)} (\log(1/\delta))^{-q(x)/p(x)} \varphi(\delta^{-1})^{n/p(x)} \psi(\delta^{-1})^{1/p(x)} \\
\int_{B(x, r_0) \setminus B(x, \delta)} g(y) dy \\
\leq C \delta^{\alpha-n/p(x)} (\log(1/\delta))^{-q(x)/p(x)} \varphi(\delta^{-1})^{n/p(x)} \psi(\delta^{-1})^{1/p(x)},
\]
as required. \( \square \)

**Proof of Theorem 2.8.** Let \( f \) be a nonnegative measurable function on \( G \) with \( \|f\|_{L^p((\log L)^{q(x)}(G))} \leq 1 \). By Lemma 2.9, we find
\[
U_{\alpha} f(x) = \int_{B(x, \delta)} |x - y|^{\alpha-n} f(y) dy + \int_{G \setminus B(x, \delta)} |x - y|^{\alpha-n} f(y) dy \\
\leq C \delta^{\alpha} Mf(x) + C \delta^{\alpha-n/p(x)} (\log(1/\delta))^{-q(x)/p(x)} \varphi(\delta^{-1})^{n/p(x)} \psi(\delta^{-1})^{1/p(x)}.
\]
Considering
\[
\delta = Mf(x)^{-p(x)/n} (\log(e + Mf(x)))^{-q(x)/n} \varphi(Mf(x))^{1/p(x)} \psi(Mf(x))^{1/n}
\]
when \( Mf(x) \) is large enough, we establish
\[
U_{\alpha} f(x) \leq C Mf(x)^{1-\alpha p(x)/n} (\log(e + Mf(x)))^{-\alpha q(x)/n} \varphi(Mf(x))^{\alpha/p(x)} \psi(Mf(x))^{\alpha/n} + C.
\]
If \( A = n + \varepsilon > n \), then we find
\[
\tilde{\Phi}_A(x, U_{\alpha} f(x)) \leq C \Phi_B(x, Mf(x)) + C
\]
for \( x \in G \), where \( B = n + \varepsilon n/(n - \alpha p^{-}) < n + \varepsilon p^2(x)/p(x) \). Thus it follows from Theorem 2.7 that
\[
\int_{G} \tilde{\Phi}_A(x, U_{\alpha} f(x)) dx \leq C,
\]
as required. \( \square \)
Remark 2.10 Theorems 2.7 and 2.8 are shown to be valid if conditions \((\varphi')\) and \((\psi')\) can be replaced by
\[(\varphi'') \lim_{t \to 0} \varphi(t) \geq 1 ;
(\psi'') \lim_{t \to 0} \psi(t) \geq 1 ,\]
since we may consider \(e^{\varepsilon_0} \varphi\) and \(e^{\varepsilon_0} \psi\) instead of \(\varphi\) and \(\psi\) respectively.

In the later use, we need the following result, which can be proved in the same manner as Lemma 2.4.

**Corollary 2.11** Suppose \(p_+(G) < n/\alpha\). If \(A > n\), then
\[
\int_G \tilde{\Phi}_A(x, U_\alpha f(x)) dx \leq C \|f\|_{L^p(\log L)^q(G)}
\]
for all measurable functions \(f\) on \(G\) such that \(\|f\|_{L^p(\log L)^q(G)} \leq 1\).

3 Mean continuity I

First we introduce a notion of capacity as an extension of Meyers [23] and the first author [25]. For a set \(E \subset \mathbb{R}^n\) and an open set \(G \subset \mathbb{R}^n\), we define
\[
C_{\alpha,p(\cdot),q(\cdot)}(E;G) = \inf_f \int_G f(y)^{p(y)}(\log(c_0 + f(y)))^{q(y)} dy,
\]
where the infimum is taken over all nonnegative measurable functions \(f\) on \(\mathbb{R}^n\) such that \(f\) vanishes outside \(G\) and \(U_\alpha f(x) \geq 1\) for every \(x \in E\) (cf. Futamura-Mizuta-Shimomura [11], Harjulehto-Hästö [13], Harjulehto-Hästö-Koskenoja [15] and Harjulehto-Hästö-Koskenoja-Varonen [16]). Then, since \(t^{p(x)}(\log(c_0 + t))^{q(x)}\) is convex for each fixed \(x \in \mathbb{R}^n\) (see (1.2)), we see that \(C_{\alpha,p(\cdot),q(\cdot)}(\cdot;G)\) is a countably subadditive and nondecreasing capacity. We say that \(E\) is of \(C_{\alpha,p(\cdot),q(\cdot)}\)-capacity zero, written as \(C_{\alpha,p(\cdot),q(\cdot)}(E) = 0\), if
\[
C_{\alpha,p(\cdot),q(\cdot)}(E \cap G;G) = 0 \quad \text{for every bounded open set } G.
\]

We here mention the following fundamental properties of our capacity.

**Lemma 3.1** (cf. [11, Lemma 4.1]). For \(E \subset \mathbb{R}^n\), \(C_{\alpha,p(\cdot),q(\cdot)}(E) = 0\) if and only if there exists a nonnegative function \(f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)\) such that \(U_\alpha f \neq \infty\) but \(U_\alpha f(x) = \infty\) for every \(x \in E\).

**Lemma 3.2** (cf. [25, Corollary 1.2, Chapter 5]). If \(C_{\alpha,p(\cdot),q(\cdot)}(E;G) = 0\) for some bounded open set \(G\), then \(C_{\alpha,p(\cdot),q(\cdot)}(E) = 0\).
For \( x_0 \in \mathbb{R}^n \) and \( r > 0 \), set \( f_{x_0,r}(w) = r^\alpha f(x_0 + r w) \). Then note that

\[
U_\alpha f(x) = U_\alpha f_{x_0,r}(z) \quad \text{for} \quad x = x_0 + rz.
\]

Further set

\[
p_{x_0,r}(z) = p(x_0 + rz) \quad \text{and} \quad q_{x_0,r}(z) = q(x_0 + rz);
\]

see also Fiorenza-Rakotoson [9] for shifting the exponent. Then note that \( p_{x_0,r} \) satisfies (p1) and (p2) for \( r \leq 1 \) since \( \log \varphi(t)/\log(1/t) \) is nondecreasing on \( (0,r_0] \). Similarly, note that \( q_{x_0,r} \) satisfies (q1) and (q2) for \( r \leq 1 \).

Before showing our third theorem, we give the following result.

**Lemma 3.3** Let \( f \) be a nonnegative locally integrable function on \( \mathbb{R}^n \) such that

\[
\lim_{r \to 0+} \int_{B(x_0,r)} r^{\alpha p(y) - n} \max \left\{ 1, (\log(e + r^{-1}))^{-q(y)} \right\} f(y)^{p(y)}(\log(e + f(y)))^{q(y)} dy = 0.
\]

Then

\[
\lim_{r \to 0+} \left\| (f \chi_{B(x_0,r)})(x_0,r) \right\|_{L^{p_{x_0,r},r}(\log L)^{q_{x_0,r}}(\mathbb{R}^n)} = 0.
\]

**Proof.** As in [22, Theorem 2.4], it suffices to show that

\[
\lim_{r \to 0+} \int_{\mathbb{R}^n} \left( (f \chi_{B(x_0,r)})(x_0,r)(w) \right)^{p_{x_0,r}(w)}(\log(e + (f \chi_{B(x_0,r)})(x_0,r)(w)))^{q_{x_0,r}(w)} dw = 0.
\]

For this we have only to find

\[
\int_{\mathbb{R}^n} \left( (f \chi_{B(x_0,r)})(x_0,r)(w) \right)^{p_{x_0,r}(w)}(\log(e + (f \chi_{B(x_0,r)})(x_0,r)(w)))^{q_{x_0,r}(w)} dw
= \int_{\mathbb{R}^n} r^{\alpha}(f \chi_{B(x_0,r)})(x_0 + r w)^{p(x_0 + rw)}(\log(e + r^\alpha(f \chi_{B(x_0,r)})(x_0 + r w)))^{q(x_0 + rw)} dw
\leq C \int_{B(x_0,r)} r^{\alpha p(y) - n} \max \left\{ 1, (\log(e + r^{-1}))^{-q(y)} \right\} f(y)^{p(y)}(\log(e + f(y)))^{q(y)} dy.
\]

The required assertion is now proved. \( \square \)

We are now ready to show our third theorem concerning the vanishing Sobolev type integrability, which gives an extension of Meyers [24], Harjulehto-Hästö [13] and the authors [11, Theorem 4.5].

**Theorem 3.4** Suppose \( p_+ < n/\alpha \). Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) with \( \|f\|_{L^{p_+}(\log L)^{q_+}(\mathbb{R}^n)} \leq 1 \) and (1.3). Then

\[
\lim_{r \to 0+} \int_{B(x_0,r)} \tilde{\Phi}_A(x, |U_\alpha f(x) - U_\alpha f(x_0)|) dx = 0 \quad (3.1)
\]
holds for all \( x_0 \in \mathbb{R}^n \setminus (E_1 \cup E_2) \), where

\[
E_1 = \{ x \in \mathbb{R}^n : U_a f(x) = \infty \},
\]

\[
E_2 = \{ x \in \mathbb{R}^n : \limsup_{r \to 0^+} \int_{B(x,r)} r^{\alpha p(y)-n} \max \left\{ 1, (\log(e + r^{-1}))^{-\psi(y)} \right\} x f(y)^p(y) (\log(e + f(y)))^q(y) dy > 0 \}.
\]

By Lemma 3.1, we see that \( E_1 \) has \( C_{\alpha,p(\cdot),q(\cdot)} \)-capacity zero. In the next section we show examples of \( p(\cdot) \) and \( q(\cdot) \) for which \( E_2 \) has \( C_{\alpha,p(\cdot),q(\cdot)} \)-capacity zero, where \( \varphi \) and \( \psi \) are not necessarily constants.

**Proof of Theorem 3.4.** It suffices to show that (3.1) holds for \( x_0 \in \mathbb{R}^n \setminus (E_1 \cup E_2) \). Write

\[
U_\alpha f(x) - U_\alpha f(x_0) = \int_{B(x_0,2|x-x_0|)} |x-y|^{\alpha-n} f(y) dy + \int_{\mathbb{R}^n \setminus B(x_0,2|x-x_0|)} |x-y|^{\alpha-n} f(y) dy - U_\alpha f(x_0) = U_1(x) + U_2(x).
\]

If \( y \in \mathbb{R}^n \setminus B(x_0,2|x-x_0|) \), then \( |x-y| < 2|x-x_0| \). Since \( U_\alpha f(x_0) < \infty \), so that we can apply Lebesgue’s dominated convergence theorem to obtain

\[
\lim_{x \to x_0} U_2(x) = 0.
\]

Note here that

\[
U_1(x) \leq \int_{B(x_0,r)} |x-y|^{\alpha-n} f(y) dy = U_\alpha f_r(x)
\]

for \( x \in B(x_0,r/2) \), where \( f_r = f \chi_{B(x_0, r)} \). Hence, we have only to show that

\[
\lim_{r \to 0^+} \int_{B(x_0,r)} \tilde{\Phi}_A(x, U_\alpha f_r(x)) dx = 0.
\]

We may assume from Lemma 3.3 that \( \|(f_r)_{x_0,r}\|_{L^{p(x_0,r)}(\log L)^{\psi(x_0,r)}(\mathbb{R}^n)} \) is small when \( r \) is small. By Corollary 2.11, we have

\[
\int_{B(x_0,r)} \tilde{\Phi}_A(x, U_\alpha f_r(x)) dx = \int_{B(0,1)} \kappa(q(x_0 + rz)/p(x_0 + rz), -A/p(x_0 + rz)^2, -1/p(x_0 + rz), U_\alpha(f_r)_{x_0,r}(z)) p'(x_0 + rz) dz \leq C \|(f_r)_{x_0,r}\|_{L^{p(x_0,r)}(\log L)^{\psi(x_0,r)}(\mathbb{R}^n)},
\]

which together with Lemma 3.3 implies that the left hand side tends to zero as \( r \to 0^+ \). Thus the proof is completed. \( \square \)

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Lemma 3.5 Suppose both \( \varphi \) and \( \psi \) are constants. Then
\[
C_{\alpha,p,q}(B(x_0,r); B(x_0,r)) \leq Cr^{n-\alpha p(x_0)}(\log(e + r^{-1}))^q(x_0)
\]
for each \( x_0 \in \mathbb{R}^n \) and \( r > 0 \).

Proof. For \( x_0 \in \mathbb{R}^n \) and \( r > 0 \), define the potential
\[
u(x) = \int |x - y|^{\alpha - n} f(y) dy,
\]
where \( f(y) = r^{-\alpha} \chi_{B(x_0,r)} \). Then, since \( \nu(x) \geq C \) for \( x \in B(x_0,r) \), we have
\[
C_{\alpha,p,q}(B(x_0,r); B(x_0,r)) \leq C \int_{B(x_0,r)} r^{-\alpha p(y)}(\log(e + r^{-\alpha}))^q(y) dy
\]
\[
\leq Cr^{n-\alpha p(x_0)}(\log(e + r^{-1}))^q(x_0),
\]
which proves the lemma.

We can show the next lemma (see the proof of Lemma 4.3 below).

Lemma 3.6 (cf. [11, Lemma 4.4]). Suppose both \( \varphi \) and \( \psi \) are constants. Further suppose \( p_+ < n/\alpha \). If \( f \) is a nonnegative measurable function on \( \mathbb{R}^n \) with
\[
\|f\|_{L^p(\log L)^q(\mathbb{R}^n)} \leq 1,
\]
then
\[
\lim_{r \to 0^+} r^{\alpha p(x) - n} (\log(e + r^{-1}))^{-q(x)} \int_{B(x,r)} f(y)^p(y)(\log(e + f(y)))^q(y) dy = 0
\]
holds for all \( x \in \mathbb{R}^n \) except in a set \( E \subset \mathbb{R}^n \) with \( C_{\alpha,p,q}(E) = 0 \).

By Theorem 3.4, we can show the following result, which is an extension of [11, Corollary 4.6] (see also Harjulehto-H"ast"o [13, Theorem 4.12] for \( \alpha = 1 \)).

Proposition 3.7 Suppose both \( \varphi \) and \( \psi \) are constants. Further suppose \( p_+ < n/\alpha \) and \( q_+ \leq 0 \). Let \( f \) be a nonnegative measurable function on \( \mathbb{R}^n \) with
\[
\|f\|_{L^p(\log L)^q(\mathbb{R}^n)} \leq 1
\]
such that \( U_\alpha f \neq \infty \). Then
\[
\lim_{r \to 0^+} \int_{B(x_0,r)} \left\{ |U_\alpha f(x) - U_\alpha f(x_0)| (\log(e + |U_\alpha f(x) - U_\alpha f(x_0)|))^q(x)/p(x) \right\}^{p(x)} dx = 0
\]
holds for all \( x_0 \) except in a set \( E \subset \mathbb{R}^n \) with \( C_{\alpha,p,q}(E) = 0 \).

For, if \( q_+ \leq 0 \), then we have \( C_{\alpha,p,q}(E_2) = 0 \) by Lemma 3.6, so that Theorem 3.4 gives the present proposition.
4 Mean continuity II

In this section, let

\[ p(x) = p_0 - \omega(|x_n|) \quad \text{and} \quad q(x) = q_0 - \eta(|x_n|) \]

for \( x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n \), where \( p_0 > 1 \), \( q_0 \in \mathbb{R} \) and, further, \( p_- > 1 \). To show that \( p(\cdot) \) satisfies (p2), note that

\[
\frac{\log \phi(1/(s + t))}{\log(1/(s + t))} \leq \frac{\log \phi(1/(s + t))}{\log(1/s)} + \frac{\log \phi(1/(s + t))}{\log(1/t)}
\]

for all \( 0 < s, t < r_0 \), so that

\[
|\omega(s) - \omega(t)| \leq \omega(|s - t|),
\]

which implies (p2).

Similarly, noting that

\[
|\eta(s) - \eta(t)| \leq \eta(|s - t|),
\]

we insist that \( q(\cdot) \) satisfies (q2).

Let

\[
H = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1 : x_n = 0\}.
\]

**Lemma 4.1** Suppose \( 0 < r < r_0 \). If \( x_0 \in H \), then

\[
C_{\alpha,p(\cdot),q(\cdot)}(B(x_0, r); B(x_0, r)) \leq Cr^{-\alpha p_0} (\log(e + r^{-1}))^{\phi_0 \varphi(r^{-1})^{-a} \psi(r^{-1})^{-1}}.
\]

**Proof.** For a proof, we consider the set

\[
S_r = \{x \in \mathbb{R}^n : r/2 < |x_n| < r\}
\]

and define

\[
u(x) = \int_{(B(x_0, r) \setminus B(x_0, r/2)) \cap S_r} r^{-\alpha} |x - y|^{\alpha - n} dy.
\]

Then, since \( u(x) \geq C \) for \( x \in B(x_0, r) \), we have

\[
C_{\alpha,p(\cdot),q(\cdot)}(B(x_0, r); B(x_0, r)) \leq C \int_{(B(x_0, r) \setminus B(x_0, r/2)) \cap S_r} r^{-\alpha p(y)} (\log(e + r^{-1}))^{\phi(y)} dy
\]

\[
\leq C r^{-\alpha p_0} (\log(e + r^{-1}))^{\phi_0 \varphi(r^{-1})^{-a} \psi(r^{-1})^{-1}},
\]

which proves the lemma. \( \square \)
Lemma 4.2 If \(x_0 \in \mathbb{R}^n \setminus H\) and \(0 < r < \min\{r_0, |(x_0)_n|/2\}\), then

\[
C_{\alpha,p(h),q(\cdot)}(B(x_0,r); B(x_0,r)) \leq C(|(x_0)_n|r^{-\alpha\psi(r^{-1})}\log(e + r^{-1}))^{q(\cdot)}.
\]

Proof. First we show that

\[
|p(x) - p(y)| \leq \frac{C(|(x_0)_n|)}{\log(1/|x - y|)}
\]

(4.1)

for \(x, y \in B(x_0,r)\) with \(0 < r < \min\{r_0, |(x_0)_n|/2\}\). This is trivial when \(|(x_0)_n|/2 \geq r_0\). If \(|(x_0)_n|/2 < |y_n| < |x_n| < r_0\), then we have

\[
|p(x) - p(y)| = \left( \frac{\log \varphi(1/|x_n|) - \log \varphi(1/|y_n|)}{\log(1/|x_n|)} \right) + \left( \frac{\log \varphi(1/|x_n|)}{\log(1/|y_n|)} \right)
\]

\[
\leq \log \varphi(1/|x_n|) \left( \frac{1}{\log(1/|x_n|)} - \frac{1}{\log(1/|y_n|)} \right)
\]

\[
\leq \frac{\log \varphi(1/|x_n|)}{\log(1/|x - y|)}
\]

which proves (4.1).

Similarly note that

\[
|q(x) - q(y)| \leq \frac{C(|(x_0)_n|)}{\log(\log(1/|x - y|))}
\]

for \(x, y \in B(x_0,r)\) with \(0 < r < \min\{r_0, |(x_0)_n|/2\}\).

Now Lemma 3.5 gives the required result. \(\square\)

For \(r > 0\), set

\[
h(r; x) = \begin{cases} 
  r^{-\alpha\varphi_0}(\log(e + r^{-1}))^{q_0} \varphi(r^{-1})^{-\alpha\psi(r^{-1})} & \text{if } x \in H, \\
  r^{-\alpha\varphi(x)}(\log(e + r^{-1}))^{q(x)} & \text{if } x \in \mathbb{R}^n \setminus H. 
\end{cases}
\]

We show the following result.

Lemma 4.3 (cf. [11, Lemma 4.4]) Suppose \(p_0 < n/\alpha\). If \(f\) is a nonnegative measurable function on \(\mathbb{R}^n\) with \(\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1\), then

\[
\lim_{r \to 0^+} h(r; x)^{-1} \int_{B(x,r)} f(y)^{p(y)}(\log(e + f(y)))^{q(y)}dy = 0
\]

holds for all \(x \in \mathbb{R}^n\) except in a set \(E \subset \mathbb{R}^n\) with \(C_{\alpha,p(h),q(\cdot)}(E) = 0\).
Proof. First we prove that $C_{\alpha,p,q}(E \setminus H) = 0$. For each integer $j$, set $L_j = \{x \in \mathbb{R}^n : 2^{-j} < |x_n| \leq 2^{-j+1}\}$. For $\delta > 0$, consider the set

$$E_\delta = \left\{ x \in L_j : \limsup_{r \to 0+} h(r; x)^{-1} \int_{B(x,r)} f(y)^{p(y)}(\log(e + f(y)))^{q(y)} dy > \delta \right\}.$$ 

By subadditivity and Lemma 3.2, it suffices that $C_{\alpha,p,q}(E_\delta \cap B(0, R); B(0, 2R)) = 0$ for all $R > 1/2$. Let $0 < \varepsilon < 1/5 \cdot 2^{-j+1}$. For each $x \in E_\delta \cap B(0, R)$, we find $0 < r(x) < \varepsilon$ such that

$$h(r(x); x)^{-1} \int_{B(x,r(x))} f(y)^{p(y)}(\log(e + f(y)))^{q(y)} dy > \delta.$$ 

By the covering lemma (see [32, Lemma, p. 9]), there exists a disjoint family $\{B_i\}$ such that $B_i = B(x_i, r(x_i))$ and $\bigcup_i B(x_i, 5r(x_i)) \supseteq E_\delta \cap B(0, R)$. Then we have by Lemma 4.2

$$C_{\alpha,p,q}(E_\delta \cap B(0, R); B(0, 2R)) \leq \sum_i C_{\alpha,p,q}(B(x_i, 5r(x_i)); B(x_i, 5r(x_i))) \leq C(j) \sum_i h(r(x_i); x_i) \leq C(j) \delta^{-1} \int_{\bigcup_i B_i} f(y)^{p(y)}(\log(e + f(y)))^{q(y)} dy.$$ 

Since

$$\left| \bigcup_i B_i \right| \leq C \delta^{-1} r(x_i)^{\alpha p(x_i)}(\log(e + r(x_i)^{-1}))^{-q(x_i)} \int_{B_i} f(y)^{p(y)}(\log(e + f(y)))^{q(y)} dy \leq C \delta^{-1} \varepsilon^{\alpha p -}.$$ 

it follows from the absolute continuity of integral that

$$C_{\alpha,p,q}(E_\delta \cap B(0, R); B(0, 2R)) = 0.$$ 

Similarly, we can prove that $C_{\alpha,p,q}(E \cap H) = 0$ with the aid of Lemma 4.1, as required. \hfill \Box

Corollary 4.4 Suppose $p_0 < n/\alpha$ and $q_0 \leq 0$. If $f$ is a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^p(\log L)^q}(\mathbb{R}^n) \leq 1$, then

$$C_{\alpha,p,q}(E_2) = 0.$$ 

For this, in case $q_0 \leq 0$, note

$$\int_{B(x_0,r)} r^{\alpha p(y) - n}(\log(e + r^{-1}))^{-q(y)} f(y)^{p(y)}(\log(e + f(y)))^{q(y)} dy \leq C r^{\alpha p_0 - n}(\log(e + r^{-1}))^{-q_0 \varphi(r^{-1})^{\alpha \psi(r^{-1})}} \int_{B(x_0,r)} f(y)^{p(y)}(\log(e + f(y)))^{q(y)} dy.$$ 

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for $x_0 \in H$ and $0 < r \leq r_0$; and
\[
\int_{B(x_0, r)} r^{\alpha p(y) - n} (\log(e + r^{-1}))^{-q(y)} f(y)^p(y) (\log(e + f(y)))^q(y) dy 
\leq C(|(x_0)_n|) r^{\alpha p(x_0) - n} (\log(e + r^{-1}))^{-q(x_0)} \int_{B(x_0, r)} f(y)^p(y) (\log(e + f(y)))^q(y) dy
\]
for $x_0 \in \mathbb{R}^n \setminus H$ and $0 < r \leq \min\{r_0, |(x_0)_n|/2\}$. Hence $C_{\alpha, p(\cdot), q(\cdot)}(E_2) = 0$ by Lemma 3.6.

Now Theorem 3.4 gives the following result.

**Theorem 4.5** Suppose $p_0 < n/\alpha$ and $q_0 \leq 0$. Let $f$ be a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)} \leq 1$ such that $U_\alpha f \neq \infty$. Then
\[
\lim_{r \to 0^+} \int_{B(x_0, r)} \Phi_A(x, |U_\alpha f(x) - U_\alpha f(x_0)|) dx = 0
\]
holds for all $x_0 \in \mathbb{R}^n$ except in a set of $C_{\alpha, p(\cdot), q(\cdot)}$-capacity zero.

**Remark 4.6** For $p_0 < n/\alpha$ and $q_0 \leq 0$, set
\[
p(x) = \begin{cases} p_0 + \omega(x_n) & \text{if } x_n \geq 0, \\ p_0 & \text{if } x_n < 0 \end{cases}
\]
and
\[
q(x) = \begin{cases} q_0 + \eta(x_n) & \text{if } x_n \geq 0, \\ q_0 & \text{if } x_n < 0. \end{cases}
\]
If $p_+ < n/\alpha$ and $q_+ \leq 0$, then we can show that Theorem 4.5 is true for these exponents.

## 5 Vanishing exponential integrability

For a compact set $K$ in $G$, we define
\[
K(r) = \{ x \in G : \delta_K(x) < r \},
\]
where $\delta_K(x)$ denotes the distance of $x$ from $K$. For $\nu \geq 0$, we say that the Minkowski $(n - \nu)$-content of $K$ is finite if
\[
|K(r)| \leq Cr^\nu \quad \text{for small } r > 0.
\]
Note here that if $K$ is a singleton, then its Minkowski 0-content is finite, and if $K$ is a spherical surface, then its Minkowski $(n - 1)$-content is finite. As another examples of $K$, we may consider fractal type sets like Cantor sets or Koch curves. In this section, we consider variable exponents
\[
p(x) = p(\delta_K(x)) = p_0 + \omega(\delta_K(x))
\]
and
\[ q(x) = q(\delta_K(x)) = q_0 + \eta(\delta_K(x)) \]
for \( p_0 > 1 \) and \( q_0 \in \mathbb{R} \). Note that \( p(\cdot) \) and \( q(\cdot) \) satisfies (p2) and (q2), respectively.

We know the following result.

**Lemma 5.1** (cf. [26, Lemma 2.3]). Let \( K \) be a compact set in \( G \) whose Minkowski \((n - \nu)\)-content is finite. Then
\[
\int_G \delta_K(x)^{-\nu}(\log(1 + \delta_K(x)^{-1}))^{-a} \, dx < \infty
\]
for every \( a > 1 \).

**Lemma 5.2** (cf. [26, Lemma 2.4]). Suppose the Minkowski \((n - \nu)\)-content of \( K \) is finite. If \( f \) is a measurable function on \( G \) with \( \|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \leq 1 \), then
\[
\int_G |f(x)|^{p_0}(\log(e + |f(x)|))^{q_0}\varphi(|f(x)|)^{\nu/p_0}\psi(|f(x)|) \, dx \leq C.
\]

**Proof.** Consider the set
\[ G' = \{ x \in K(r_0) : |f(x)| < \delta(x)^{-\nu/p_0}(\log(1/\delta(x)))^{-a/p_0} \}, \]
where we will determine \( a \) later; here we set \( \delta(x) = \delta_K(x) \) for simplicity. If \( x \in G' \), then we have by (\( \varphi \)) and (\( \psi \))
\[
|f(x)|^{p_0}(\log(e + |f(x)|))^{q_0}\varphi(|f(x)|)^{\nu/p_0}\psi(|f(x)|) \\
\leq C\delta(x)^{-\nu}(\log(1/\delta(x)))^{-a}(\log(1/\delta(x)))^{q_0}\varphi(1/\delta(x))^{\nu/p_0}\psi(1/\delta(x)) \\
\leq C\delta(x)^{-\nu}(\log(1/\delta(x)))^{-a+q_0+\varepsilon_3},
\]
where \( \varepsilon_3 > \varepsilon_1\nu/p_0 \). If we take \( a \) so large that \( a > 1 + q_0 + \varepsilon_3 \), it follows from Lemma 5.1 that
\[
\int_{G'} |f(x)|^{p_0}(\log(e + |f(x)|))^{q_0}\varphi(|f(x)|)^{\nu/p_0}\psi(|f(x)|) \, dx \leq C.
\]

If \( x \notin G' \) and \( \delta(x) < r_0 \), then \( |f(x)| \geq \delta(x)^{-\nu/p_0}(\log(1/\delta(x)))^{-a/p_0} \), so that
\[
\delta(x) \geq C|f(x)|^{-\nu/p_0}(\log |f(x)|)^{-a/\nu}.
\]

Hence, in view of Lemma 2.1, we see that
\[
\frac{\log \varphi(1/\delta(x))}{\log(1/\delta(x))} \log |f(x)| \geq \frac{\log \varphi(C|f(x)|^{p_0/\nu}(\log |f(x)|)^{a/\nu})}{\log(C|f(x)|^{p_0/\nu}(\log |f(x)|)^{a/\nu})} \log |f(x)| \\
\geq \frac{\nu}{p_0} \left\{ \log(C\varphi(|f(x)|)) \right\} \\
= \frac{\nu}{p_0} \left\{ \log(C\varphi(|f(x)|)) \right\} \left( 1 - \frac{C \log(C \log |f(x)|)}{\log |f(x)| + C \log(C \log |f(x)|)} \right) \\
\geq \frac{\nu}{p_0} \log \varphi(|f(x)|) - C,
\]
20
which yields
\[ |f(x)|^{p(x) - p_0} = \exp \left( \frac{\log \varphi(1/\delta(x))}{\log(1/\delta(x))} \log |f(x)| \right) \geq \exp \left( \frac{\nu}{p_0} \log \varphi(|f(x)|) - C \right) \]
\[ = C \varphi(|f(x)|)^{\nu/p_0}. \]

Similarly, we have
\[ \frac{\log \psi(1/\delta(x))}{\log(1/\delta(x))} \log(\log |f(x)|) \geq \log \psi(|f(x)|) - C, \]
which yields
\[ (\log |f(x)|)^{q(x) - q_0} \geq C \psi(|f(x)|). \]

Thus it follows that
\[ \int_{K(r_0) \setminus G} |f(x)|^{p_0} (\log(e + |f(x)|))^{q_0} \varphi(|f(x)|)^{\nu/p_0} \psi(|f(x)|) dx \]
\[ \leq C \int_G |f(x)|^{p(x)} (\log(e + |f(x)|))^{q(x)} dx \leq C. \]

Finally, since \( p(x) \geq p_1 > p_0 \) when \( \delta(x) \geq r_0 \), we find
\[ \int_{G \setminus K(r_0)} |f(x)|^{p_0} (\log(e + |f(x)|))^{q_0} \varphi(|f(x)|)^{\nu/p_0} \psi(|f(x)|) dx \]
\[ \leq C \int_G |f(x)|^{p(x)} (\log(e + |f(x)|))^{q(x)} dx + C \leq C. \]

The required assertion is now proved.

From now on set \( p_0 = n/\alpha, q_0 \geq 0, \varphi(r) = c(\log(e + r))^a, \psi(r) = c(\log(\log(e + r)))^b \) and \( K = H \) for \( a, b \geq 0 \) and \( c > 0 \). For \( x_0 \in H \) and \( r_0 > 0 \), let \( B = B(x_0, r_0) \) be a ball in \( \mathbb{R}^n \). By Lemma 5.2, we have the following integrability for all measurable functions \( f \) on \( \mathbb{R}^n \) with \( \|f\|_{L^p(\log L)^{q(x)}(\mathbb{R}^n)} \leq 1 \) (see also [26]):

**Corollary 5.3** If \( f \) is a measurable function on \( \mathbb{R}^n \) with \( \|f\|_{L^p(\log L)^{q(x)}(\mathbb{R}^n)} \leq 1 \), then
\[ \int_B |f(x)|^{n/\alpha} (\log(e + |f(x)|))^{q_0 + \alpha n} (\log(\log(e + |f(x)|)))^b dx \leq C. \] (5.1)

We know the following vanishing exponential integrability for Riesz potentials of functions in Orlicz classes ([28]):
**Lemma 5.4** Let $a^\sharp = n^2/(n^2 - \alpha n - \alpha a^2 - \alpha q_0) > 0$ and $b^\sharp = \alpha b/(n^2 - \alpha n - \alpha a^2 - \alpha q_0)$. If $f$ is a nonnegative measurable function on $\mathbb{R}^n$ satisfying (1.3) and (5.1), then

$$
\lim_{r \to 0+} \int_{B(x_0, r)} \left\{ \exp \left( A |U_\alpha f(x) - U_\alpha f(x_0)|^{a^\sharp} \right) \times \left( \log(1 + |U_\alpha f(x) - U_\alpha f(x_0)|)^{b^\sharp} \right) - 1 \right\} dx = 0
$$

holds for all $A > 0$ and all $x_0 \in H \setminus E_f$, where

$$E_f = \{ x \in H : U_\alpha f(x) = \infty \}.$$

By Lemma 3.1 we see that $E_f$ has $C_{\alpha,p(\cdot),q(\cdot)}$-capacity zero.

Finally, in view of Lemmas 3.1 and 5.4 and Corollary 5.3, we give the vanishing exponential integrability for Riesz potentials with variable exponent, which is based on a constant exponent, Orlicz space result.

**Theorem 5.5** Let $a^\sharp = n^2/(n^2 - \alpha n - \alpha a^2 - \alpha q_0) > 0$ and $b^\sharp = \alpha b/(n^2 - \alpha n - \alpha a^2 - \alpha q_0)$. If $f$ is a nonnegative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)} \leq 1$ satisfying (1.3), then

$$
\lim_{r \to 0+} \int_{B(x_0, r)} \left\{ \exp \left( A |U_\alpha f(x) - U_\alpha f(x_0)|^{a^\sharp} \right) \times \left( \log(1 + |U_\alpha f(x) - U_\alpha f(x_0)|)^{b^\sharp} \right) - 1 \right\} dx = 0
$$

holds for all $A > 0$ and all $x_0 \in H$ except in a set of $C_{\alpha,p(\cdot),q(\cdot)}$-capacity zero.

**References**


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