Sobolev's inequalities and vanishing integrability for Riesz potentials of functions in the generalized Lebesgue space $L^{p(\cdot)}(\log L)^{q(\cdot)}$

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Abstract

Our aim in this paper is to deal with Sobolev's inequalities for Riesz potentials of functions belonging to $L^{p(\cdot)}(\log L)^{q(\cdot)}$. To do so, we study the boundedness of Hardy-Littlewood maximal functions and apply the Hedberg's trick. As an application, we treat vanishing integrability for Riesz potentials.

1 Introduction

In recent years, the generalized Lebesgue spaces have attracted more and more attention, in connection with the study of elasticity, fluid mechanics and differential equations with $p(\cdot)$-growth; see for example Orlicz [29], Kováčik-Rákosník [22], Edmunds-Rákosník [7] and Růžička [30].

In this paper, following Cruz-Uribe and Fiorenza [4], we consider continuous functions $p(\cdot) : \mathbf{R}^n \to [1, \infty)$ and $q(\cdot) : \mathbf{R}^n \to \mathbf{R}$, which are called variable exponents. In the present paper, we always assume that $p(\cdot)$ and $q(\cdot)$ are bounded on $\mathbf{R}^n$ and

$$p_- \equiv \inf_{x \in \mathbf{R}^n} p(x) > 1. \tag{1.1}$$

Our typical examples of $p(\cdot)$ and $q(\cdot)$ are the exponents satisfying the following log-Hölder conditions:

$$|p(x) - p(y)| \le \frac{a \log(e + \log(e + |x-y|^{-1}))}{\log(e + |x-y|^{-1})} + \frac{b}{\log(e + |x-y|^{-1})}$$

and

$$|q(x) - q(y)| \le \frac{c \log(e + \log(e + \log(e + |x-y|^{-1})))}{\log(e + \log(e + |x-y|^{-1}))} + \frac{d}{\log(e + \log(e + |x-y|^{-1}))}$$

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whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$, where $a, b, c, d \geq 0$ are constants. In [14], Harjulehto and Hästö discussed the continuity of Sobolev functions, and in the paper by Hästö [19], he studied the integrability of maximal functions. For further related results, we refer the reader to [10], [11], [12] and [26].

By condition (1.1), one can find a constant $c_0 \geq e$ such that

$$t^{p(x)}(\log(c_0 + t))^{q(x)} \text{ is a convex function of } t \text{ for each fixed } x \in \mathbf{R}^n. \tag{1.2}$$

We define the space $L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$ of all measurable functions f on an open set G such that

$$\int_G \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} \left(\log\left(c_0 + \frac{|f(y)|}{\lambda}\right)\right)^{q(y)} dy < \infty$$

for some $\lambda > 0$. We define the norm

$$\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} = \inf\left\{\lambda > 0 : \int_G \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} \left(\log\left(c_0 + \frac{|f(y)|}{\lambda}\right)\right)^{q(y)} dy \leq 1\right\}$$

for $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$. In case $q = 0$ on G, $L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$ is denoted by $L^{p(\cdot)}(G)$ for simplicity.

For $0 < \alpha < n$, we define the Riesz potential of order α for a locally integrable function f on $\mathbf{R}^n$ by

$$U_\alpha f(x) = \int_{\mathbf{R}^n} |x - y|^{\alpha - n} f(y) dy.$$

Here it is natural to assume that

$$\int_{\mathbf{R}^n} (1 + |y|)^{\alpha - n} |f(y)| dy < \infty, \tag{1.3}$$

which is equivalent to the condition that $U_\alpha |f| \not\equiv \infty$ (see [25, Theorem 1.1, Chapter 2]).

Let $B(x, r)$ denote the open ball centered at x with radius r. For a locally integrable function f on an open set G, we consider the maximal function Mf defined by

$$Mf(x) = \sup_B \frac{1}{|B|} \int_{B \cap G} |f(y)| dy,$$

where the supremum is taken over all balls $B = B(x, r)$ and $|B|$ denotes the volume of B. Diening [5] was the first to prove the local boundedness of maximal functions in the Lebesgue spaces of variable exponents satisfying the log-Hölder condition.

Our first aim in this paper is to obtain Sobolev's inequality for Riesz potentials of functions in $L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$. To do so, we apply Hedberg's trick [20] by use of the boundedness of maximal functions. Our result (see Theorem 2.8 below) is given in Section 2, which is an extension of Almeida-Samko [3], Diening [6], Futamura-Mizuta [10], Futamura-Mizuta-Shimomura [11, 12], Harjulehto-Hästö-Pere [18], Kokilashvili-Samko [21], Mizuta-Shimomura [27] and Samko-Vakulov [31].

For a measurable function u on $\mathbf{R}^n$, we define the integral mean over a measurable set $E \subset \mathbf{R}^n$ of positive measure by

$$\fint_E u(x)\, dx = \frac{1}{|E|} \int_E u(x)\, dx,$$

where $|E|$ denotes the Lebesgue measure of E. For a locally integrable function f on $\mathbf{R}^n$, $x_0 \in \mathbf{R}^n$ is called a Lebesgue point for f if

$$\lim_{r \to 0+} \fint_{B(x_0,r)} |f(x) - f(x_0)| dx = 0.$$

Our second aim in this paper is to show that every point except in a small set is a Lebesgue point for $U_\alpha f$ with $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$. In the classical case, we refer the reader to [1], [24], [25], [32] and [34]. We aim to extend the results by Fiorenza [8], Futamura-Mizuta [10], Futamura-Mizuta-Shimomura [11] and Harjulehto-Hästö [13] in the variable exponent case.

A famous Trudinger inequality [33] insists that Sobolev functions in $W^{1,n}$ satisfy finite exponential integrability. Adams and Hurri-Syrjänen [2, Theorem 1.6] and Mizuta and Shimomura [28, Theorems 3.2, 4.5 and 5.2] have recently established the vanishing exponential integrability for Riesz potentials $U_\alpha f$ with $f \in L^{n/\alpha}(\mathbf{R}^n)$. In connection with these results, we study the vanishing exponential integrability for $U_\alpha f$; we in fact show (in Theorem 5.5 below) that

$$\lim_{r \to 0+} \fint_{B(x_0,r)} \Big\{ \exp\Big(A |U_\alpha f(x) - U_\alpha f(x_0)|^{a^\sharp} \times (\log(1 + |U_\alpha f(x) - U_\alpha f(x_0)|))^{b^\sharp} \Big) - 1 \Big\} dx = 0$$

for all $A > 0$ and all x_0 except in a small set, where $a^\sharp > 0$ and $b^\sharp$ are suitable constants determined by $p(\cdot)$ and $q(\cdot)$.

2 Sobolev's inequality

Throughout this paper, let C denote various constants independent of the variables in question and $C(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$.

We say that a positive nondecreasing function φ on the interval $[0, \infty)$ satisfies (φ) if there exist $\varepsilon_1 > 0$ and $0 < r_1 < 1$ such that

(φ) $(\log(1/r))^{-\varepsilon_1} \varphi(1/r)$ is nondecreasing on $(0, r_1)$.

Similarly, we say that a positive nondecreasing function ψ on the interval $[0, \infty)$ satisfies (ψ) if there exist $\varepsilon_2 > 0$ and $0 < r_2 < 1/e$ such that

(ψ) $(\log(\log(1/r)))^{-\varepsilon_2} \psi(1/r)$ is nondecreasing on $(0, r_2)$.

Consider positive nondecreasing functions φ satisfying (φ) and ψ satisfying (ψ). Set

$$\varepsilon_0 = \max\{\varepsilon_1, \varepsilon_2\}.$$

For the sake of convenience, we assume that

(φ') $\varphi(t) \geq e^{\varepsilon_0}$ for all $t > 0$,

(ψ') $\psi(t) \geq e^{\varepsilon_0}$ for all $t > 0$.

Set $\omega(r) = \dfrac{\log \varphi(1/r)}{\log(1/r)}$ and $\eta(r) = \dfrac{\log \psi(1/r)}{\log(\log(1/r))}$.

First we give the following results, which can be derived by conditions (φ) and (φ').

Lemma 2.1 ([25, Lemma 3.1, Section 5.3], [26, Lemmas 2.1 and 2.2]).

(i) $\varphi(r)$ *is of log-type, that is, there exists* $C > 0$ *such that*

$$C^{-1}\varphi(r) \leq \varphi(r^2) \leq C\varphi(r) \qquad \text{whenever } r > 0. \tag{2.1}$$

(ii) *For* $\gamma > 0$, *there exists* $C > 0$ *such that*

$$t^{-\gamma}\varphi(t) \leq Cs^{-\gamma}\varphi(s) \qquad \text{whenever } t \geq s > 0.$$

(iii) *There exists* $0 < \tilde{r}_1 < r_1$ *such that* $\omega(r)$ *is nondecreasing on* $[0, \tilde{r}_1]$.

Further, we see from conditions (ψ) and (ψ') that ψ satisfies (i), (ii) and

(iv) there exists $0 < \tilde{r}_2 < r_2$ such that $\eta(r)$ is nondecreasing on $[0, \tilde{r}_2]$.

Condition (2.1) implies the doubling condition on φ, that is, there exists a constant $C > 1$ such that

$$\varphi(r) \leq \varphi(2r) \leq C\varphi(r) \qquad \text{whenever } r > 0. \tag{2.2}$$

In what follows, set

$$r_0 = \min\{\tilde{r}_1, \tilde{r}_2\}.$$

If $r > r_0$, then we set

$$\omega(r) = \omega(r_0) \qquad \text{and} \qquad \eta(r) = \eta(r_0).$$

Our typical example of φ is of the form

$$\varphi(r) = a(\log(\beta_0 + r))^b(\log(\beta_0 + \log(\beta_0 + r)))^c,$$

where $a > 0$, $b \geq 0$, $c \in \mathbf{R}$ and $\beta_0 \geq e$ are chosen so that $\varphi(r)$ is nondecreasing on $[0, \infty)$; similarly, that of ψ is of the form

$$\psi(r) = a(\log(\beta_0 + \log(\beta_0 + r)))^b (\log(\beta_0 + \log(\beta_0 + \log(\beta_0 + r))))^c.$$

Note that if $b = 0$, then $c \geq 0$.

For a variable exponent $p(\cdot)$ on $\mathbf{R}^n$, set

$$p_- = \inf_{x \in \mathbf{R}^n} p(x) \qquad \text{and} \qquad p_+ = \sup_{x \in \mathbf{R}^n} p(x).$$

Now we consider continuous exponents $p(\cdot)$ and $q(\cdot)$ on $\mathbf{R}^n$ such that

(p1) $1 < p_- \leq p_+ < \infty$;

(p2) $|p(x) - p(y)| \leq \omega(|x - y|)$ whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$.

(q1) $-\infty < q_- \leq q_+ < \infty$;

(q2) $|q(x) - q(y)| \leq \eta(|x - y|)$ whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$.

Recall that the generalized Lebesgue space $L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$ given in the introduction is a Banach space with the norm $\|\cdot\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)}$. For $0 < \alpha < n$, we consider the Riesz potential $U_\alpha f$ of $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$ defined by

$$U_\alpha f(x) = \int_G |x - y|^{\alpha - n} f(y) dy.$$

Our first aim is to determine the space

$$\{U_\alpha f : f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(G)\}.$$

In our discussions below, it is convenient to note the following result.

LEMMA 2.2 *If $r > 0$ and $t > 0$, then*

$$\varphi(rt) \leq C\varphi(r)\varphi(t),$$

where C is the constant appearing in (2.1).

For this, it suffices to note that

$$\varphi(rt) \leq \max\left\{\varphi(r^2), \varphi(t^2)\right\} \leq \max\left\{C\varphi(r), C\varphi(t)\right\} \leq C\varphi(r)\varphi(t)$$

since φ is nondecreasing and $\varphi(t) \geq 1$.

Corollary 2.3 *Set* $\kappa(y,t) = t(\log(e+t))^{y_1}\varphi(t)^{y_2}\psi(t)^{y_3}$ *for* $y = (y_1, y_2, y_3)$ *and* $t \geq 0$. *Then*

$$\kappa(y, at) \leq \tau(y,a)\kappa(y,t)$$

whenever $a, t > 0$, *where*

$$\begin{aligned} \tau(y,a) &= a \max\left\{(C\log(e+a))^{y_1}, (C\log(e+a^{-1}))^{-y_1}\right\} \\ &\quad \times \max\left\{(C\varphi(a))^{y_2}, (C\varphi(a^{-1}))^{-y_2}\right\} \max\left\{(C\psi(a))^{y_3}, (C\psi(a^{-1}))^{-y_3}\right\}. \end{aligned}$$

For $A > n$ we set

$$\Phi_A(x,t) = \kappa(q(x)/p(x), -A/p(x)^2, -1/p(x), t)^{p(x)}.$$

By Corollary 2.3 and conditions (φ'), (ψ'), (p1) and (q1), we see that

$$\Phi_A(x, at) \leq C\tau(x,a)^{p(x)}\Phi_A(x,t) \tag{2.3}$$

whenever $a, t > 0$ and $x \in \mathbf{R}^n$, where

$$\begin{aligned} \tau(x,a) &= a \max\left\{(\log(e+a))^{q(x)/p(x)}, (\log(e+a^{-1}))^{-q(x)/p(x)}\right\} \\ &\quad \times \varphi(a^{-1})^{A/p(x)^2}\psi(a^{-1})^{1/p(x)}. \end{aligned}$$

We see that

$$\lim_{a\to 0+} \sup_{x\in\mathbf{R}^n} \tau(x,a) = 0 \tag{2.4}$$

and $\Phi_A(x,\cdot)$ satisfies the doubling condition for each fixed $x \in \mathbf{R}^n$; more precisely,

$$C^{-1}\Phi_A(x,t) \leq \Phi_A(x,2t) \leq C\Phi_A(x,t) \tag{2.5}$$

for all $t > 0$ and $x \in \mathbf{R}^n$.

From now on let G be a bounded open set in $\mathbf{R}^n$. Denote by $\Phi_A(G)$ the family of all measurable functions u on G such that

$$\int_G \Phi_A(x, |u(x)|/\lambda)dx < \infty$$

for some $\lambda > 0$ and define

$$\|u\|_{\Phi_A(G)} = \inf\left\{\lambda > 0 : \int_G \Phi_A(x, |u(x)|/\lambda)dx \leq 1\right\}$$

for $u \in \Phi_A(G)$.

Lemma 2.4 *There exists* $C > 0$ *such that*

$$\int_G \Phi_A(x, |u(x)|)dx \leq C\|u\|_{\Phi_A(G)}$$

for all measurable functions $u \in \Phi_A(G)$ *with* $\|u\|_{\Phi_A(G)} \leq 1$.

PROOF. If $\|u\|_{\Phi_A(G)} \le 1$, then we can find $\lambda > 0$ such that $\|u\|_{\Phi_A(G)} \le \lambda < 2$ and

$$\int_G \Phi_A(x, |u(x)|/\lambda)dx \le 1.$$

By inequality (2.3) we find

$$\begin{aligned}\int_G \Phi_A(x, |u(x)|)dx &\le \sup_{x\in G} \tau(x,\lambda)^{p(x)} \int_G \Phi_A(x, |u(x)|/\lambda)dx \\ &\le \sup_{x\in G} \tau(x,\lambda)^{p(x)} \\ &\le C\lambda.\end{aligned}$$

Letting $\lambda \to \|u\|_{\Phi_A(G)}$ yields the required inequality. □

LEMMA 2.5 *$\|\cdot\|_{\Phi_A(G)}$ is a quasi-norm, that is, for $u, v \in \Phi_A(G)$ and a real number k,*

(i) $\|u\|_{\Phi_A(G)} = 0$ *if and only if* $u = 0$;

(ii) $\|ku\|_{\Phi_A(G)} = |k|\|u\|_{\Phi_A(G)}$;

(iii) $\|u + v\|_{\Phi_A(G)} \le C\left(\|u\|_{\Phi_A(G)} + \|v\|_{\Phi_A(G)}\right)$.

PROOF. First we note that (i) follows from Lemma 2.4. Since (ii) is trivial, it suffices to show (iii). For this purpose, we take λ_j $(j = 1, 2)$ such that $\|u_j\|_{\Phi_A(G)} \le \lambda_j < 2\|u_j\|_{\Phi_A(G)}$ and

$$\int_G \Phi_A(x, |u_j(x)|/\lambda_j)dx \le 1.$$

We note from (2.3) that

$$\Phi_A(x, s) \le C\Phi_A(x, t) \tag{2.6}$$

for all $x \in G$ and $0 < s < t$. Hence, with the aid of (2.5), we obtain

$$\begin{aligned}&\int_G \Phi_A(x, a(|u_1(x) + u_2(x)|)/(\lambda_1 + \lambda_2))dx \\ \le\ & C \int_G \left\{\Phi_A(x, a|u_1(x)|/\lambda_1) + \Phi_A(x, a|u_2(x)|/\lambda_2)\right\} dx \\ \le\ & C \sup_{x\in G} \tau(x, a)^{p(x)} \left\{\int_G \Phi_A(x, |u_1(x)|/\lambda_1)dx + \int_G \Phi_A(x, |u_2(x)|/\lambda_2)dx\right\} \\ \le\ & C \sup_{x\in G} \tau(x, a)^{p(x)}.\end{aligned}$$

Now, in view of (2.4), we take $a > 0$ so small that

$$\int_G \Phi_A(x, a(|u_1(x) + u_2(x)|)/(\lambda_1 + \lambda_2))dx \le 1.$$

Then we obtain

$$\|u_1 + u_2\|_{\Phi_A(G)} \le a^{-1}(\lambda_1 + \lambda_2),$$

which proves (iii), as required. $\square$

Next we show the boundedness of the maximal operator from $L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$ into $\Phi_A(G)$. For this purpose, we need the following result.

LEMMA 2.6 (cf. [27, Lemma 2.4]). *Let f be a nonnegative measurable function on G with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \le 1$ such that $f(x) \ge 1$ or $f(x) = 0$ for each $x \in G$. Set*

$$I = I(x, r, f) = \frac{1}{|B(x,r)|} \int_{B(x,r)\cap G} f(y)dy$$

and

$$J = J(x, r, f) = \frac{1}{|B(x,r)|} \int_{B(x,r)\cap G} g(y)dy,$$

where $g(y) = f(y)^{p(y)}(\log(c_0 + f(y)))^{q(y)}$. Then

$$I \le CJ^{1/p(x)}(\log(e+J))^{-q(x)/p(x)}\varphi(J)^{n/p(x)^2}\psi(J)^{1/p(x)}.$$

PROOF. Let f be a nonnegative measurable function on G with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \le 1$ such that $f(x) \ge 1$ or $f(x) = 0$ for each $x \in G$. First consider the case when $J \ge 1$. Note that

$$J^{\omega(CJ^{-1/n})} \le C\varphi(J)^n$$

and

$$\varphi(J)^{\omega(CJ^{-1/n})} \le C.$$

Further note that

$$(\log J)^{\eta(CJ^{-1/n})} \le C\psi(J).$$

Set

$$k = CJ^{1/p(x)}(\log(e+J))^{-q(x)/p(x)}\varphi(J)^{n/p(x)^2}\psi(J)^{1/p(x)}.$$

Then we have

$$I \ \le \ k + \frac{C}{|B(x,r)|} \int_{B(x,r)} f(y) \left(\frac{f(y)}{k}\right)^{p(y)-1} \left(\frac{\log(c_0 + f(y))}{\log(c_0 + k)}\right)^{q(y)} dy.$$

Since $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \le 1$, we find

$$J \le \frac{1}{|B(x,r)|} \int_G g(y)dy \le \frac{1}{|B(x,r)|}.$$

Hence we obtain for $y \in B(x,r)$,

$$\begin{aligned} k^{-p(y)} \ &\le \ \left\{CJ^{1/p(x)}(\log(e+J))^{-q(x)/p(x)}\varphi(J)^{n/p(x)^2}\psi(J)^{1/p(x)}\right\}^{-p(x)+\omega(r)} \\ &\le \ \left\{CJ^{1/p(x)}(\log(e+J))^{-q(x)/p(x)}\varphi(J)^{n/p(x)^2}\psi(J)^{1/p(x)}\right\}^{-p(x)+\omega(CJ^{-1/n})} \\ &\le \ CJ^{-1}(\log(e+J))^{q(x)}\psi(J)^{-1} \end{aligned}$$

and

$$\begin{aligned}(\log(c_0+k))^{-q(y)} &\le \{C\log(e+J)\}^{-q(x)+\eta(r)} \\ &\le \{C\log(e+J)\}^{-q(x)+\eta(CJ^{-1/n})} \\ &\le C(\log(e+J))^{-q(x)}\psi(J).\end{aligned}$$

Consequently it follows that

$$I \le CJ^{1/p(x)}(\log(e+J))^{-q(x)/p(x)}\varphi(J)^{n/p(x)^2}\psi(J)^{1/p(x)}.$$

In the case $J \le 1$, using Lemma 2.1 (ii), we find

$$I \le CJ \le CJ^{1/p(x)}(\log(e+J))^{-q(x)/p(x)}\varphi(J)^{n/p(x)^2}\psi(J)^{1/p(x)}.$$

Now the result follows. □

Now we are ready to show the boundedness of the maximal operator $\mathcal{M}$, as an extension of Diening [5] and Cruz-Uribe and Fiorenza [4].

THEOREM 2.7 *The maximal operator $\mathcal{M}$ is bounded from $L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$ to $\Phi_A(G)$ for all $A > n$.*

PROOF. Let f be a nonnegative measurable function on G with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \le 1$. Write

$$f = f\chi_{\{y:f(y)\ge 1\}} + f\chi_{\{y:f(y)<1\}} = f_1 + f_2,$$

where χ_E denotes the characteristic function of E. Then, since $Mf_2 \le 1$ on G, we see from Lemmas 2.6 and 2.1 that

$$Mf(x)^{p(x)}(\log(e+Mf(x)))^{q(x)}\varphi(Mf(x))^{-n/p(x)}\psi(Mf(x))^{-1} \le C + CMg(x),$$

where $g(y) = f(y)^{p(y)}(\log(c_0+f(y)))^{q(y)}$. Now take p_1 such that $1 < p_1 < p^-$. Then, applying the above inequality with $p(x), \varphi(r), q(x)$ and $\psi(r)$ replaced by $p(x)/p_1, \varphi(r)^{1/p_1}, q(x)/p_1$ and $\psi(r)^{1/p_1}$ respectively, we obtain

$$\begin{aligned}&\left\{Mf(x)^{p(x)}(\log(e+Mf(x)))^{q(x)}\varphi(Mf(x))^{-np_1/p(x)}\psi(Mf(x))^{-1}\right\}^{1/p_1} \\ \le\ & C + CMg_1(x),\end{aligned}$$

where $g_1(y) = f(y)^{p(y)/p_1}(\log(c_0+f(y)))^{q(y)/p_1} = g(y)^{1/p_1}$, so that

$$\Phi_A(x, Mf(x)) \le C + CMg_1(x)^{p_1}$$

with $A = np_1$. Hence, by the well-known boundedness of the maximal operator, we see that

$$\int_G \Phi_A(x, Mf(x))dx \le C,$$

as required. □

By applying the boundedness of the maximal operator and Hedberg's trick [20], we establish the Sobolev type inequality for Riesz potentials, as an extension of the authors [27, Theorem 3.5] (see also Almeida-Samko [3], Diening [6], Futamura-Mizuta [10], Futamura-Mizuta-Shimomura [11, 12], Harjulehto-Hästö-Pere [18], Kokilashvili-Samko [21] and Samko-Vakulov [31]).

If $p_+ < n/\alpha$, then we let

$$1/p^\sharp(x) = 1/p(x) - \alpha/n.$$

For $A > n$, setting

$$\widetilde{\Phi}_A(x,t) = \kappa(q(x)/p(x), -A/p(x)^2, -1/p(x), t)^{p^\sharp(x)},$$

we define the family $\widetilde{\Phi}_A(G)$ and the corresponding quasi-norm $\|\cdot\|_{\widetilde{\Phi}_A(G)}$ (see the proof of Lemma 2.5).

THEOREM 2.8 *Suppose* $p_+(G) = \sup_{x\in G} p(x) < n/\alpha$. *If* $A > n$, *then*

$$\|U_\alpha f\|_{\widetilde{\Phi}_A(G)} \le C\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)}$$

for $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(G)$.

To show this, we need the following estimate for Riesz potentials.

LEMMA 2.9 *Let* f *be a nonnegative measurable function on* G *with* $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \le 1$. *Then*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\delta^{\alpha-n/p(x)}(\log(1/\delta))^{-q(x)/p(x)}\varphi(\delta^{-1})^{n/p(x)^2}\psi(\delta^{-1})^{1/p(x)}$$

for all $x \in G$ *and* $0 < \delta < r_0$, *where* C *is a positive constant independent of* x, δ *and* f.

PROOF. Let f be a nonnegative measurable function on G with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \le 1$ and $0 < \delta < r_0$. First note that

$$\int_{G\setminus B(x,r_0)} |x-y|^{\alpha-n} f(y)dy \le C\int_G f(y)dy \le C + C\int_G g(y)dy \le C,$$

where $g(y) = f(y)^{p(y)}(\log(c_0 + f(y)))^{q(y)}$ as in Lemma 2.6. Next set

$$k = |x-y|^{-n/p(x)}(\log(1/|x-y|))^{-q(x)/p(x)}\varphi(|x-y|^{-1})^{n/p(x)^2}\psi(|x-y|^{-1})^{1/p(x)}.$$

Then we have

$$\int_{B(x,r_0)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le \int_{B(x,r_0)\setminus B(x,\delta)} k|x-y|^{\alpha-n}dy$$
$$+ C\int_{B(x,r_0)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)\left(\frac{f(y)}{k}\right)^{p(y)-1}\left(\frac{\log(c_0+f(y))}{\log(c_0+k)}\right)^{q(y)} dy.$$

Here note that

$$k^{-p(y)} \le C|x-y|^n(\log(1/|x-y|))^{q(x)}\psi(|x-y|^{-1})^{-1}$$

and

$$(\log(c_0+k))^{-q(y)} \le C(\log(1/|x-y|))^{-q(x)}\psi(|x-y|^{-1})$$

for $y \in B(x,r_0)\setminus B(x,\delta)$, so that

$$\begin{aligned}
&\int_{B(x,r_0)\setminus B(x,\delta)} |x-y|^{\alpha-n}f(y)dy \\
\le\ & C\delta^{\alpha-n/p(x)}(\log(1/\delta))^{-q(x)/p(x)}\varphi(\delta^{-1})^{n/p(x)^2}\psi(\delta^{-1})^{1/p(x)} \\
& + C\int_{B(x,r_0)\setminus B(x,\delta)} |x-y|^{\alpha-n/p(x)}(\log(1/|x-y|))^{-q(x)/p(x)} \\
& \quad \times\varphi(|x-y|^{-1})^{n/p(x)^2}\psi(|x-y|^{-1})^{1/p(x)}g(y)dy \\
\le\ & C\delta^{\alpha-n/p(x)}(\log(1/\delta))^{-q(x)/p(x)}\varphi(\delta^{-1})^{n/p(x)^2}\psi(\delta^{-1})^{1/p(x)} \\
& + C\delta^{\alpha-n/p(x)}(\log(1/\delta))^{-q(x)/p(x)}\varphi(\delta^{-1})^{n/p(x)^2}\psi(\delta^{-1})^{1/p(x)}\int_{B(x,r_0)\setminus B(x,\delta)} g(y)dy \\
\le\ & C\delta^{\alpha-n/p(x)}(\log(1/\delta))^{-q(x)/p(x)}\varphi(\delta^{-1})^{n/p(x)^2}\psi(\delta^{-1})^{1/p(x)},
\end{aligned}$$

as required. □

PROOF OF THEOREM 2.8. Let f be a nonnegative measurable function on G with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \le 1$. By Lemma 2.9, we find

$$\begin{aligned}
U_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n}f(y)dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n}f(y)dy \\
&\le C\delta^{\alpha}Mf(x) + C\delta^{\alpha-n/p(x)}(\log(1/\delta))^{-q(x)/p(x)}\varphi(\delta^{-1})^{n/p(x)^2}\psi(\delta^{-1})^{1/p(x)}.
\end{aligned}$$

Considering

$$\delta = Mf(x)^{-p(x)/n}(\log(e+Mf(x)))^{-q(x)/n}\varphi(Mf(x))^{1/p(x)}\psi(Mf(x))^{1/n}$$

when $Mf(x)$ is large enough, we establish

$$\begin{aligned}
U_\alpha f(x) \le\ & CMf(x)^{1-\alpha p(x)/n}(\log(e+Mf(x)))^{-\alpha q(x)/n}\varphi(Mf(x))^{\alpha/p(x)}\psi(Mf(x))^{\alpha/n} \\
& + C.
\end{aligned}$$

If $A = n+\varepsilon > n$, then we find

$$\widetilde{\Phi}_A(x, U_\alpha f(x)) \le C\Phi_B(x, Mf(x)) + C$$

for $x \in G$, where $B = n + \varepsilon n/(n-\alpha p^-) < n + \varepsilon p^\sharp(x)/p(x)$. Thus it follows from Theorem 2.7 that

$$\int_G \widetilde{\Phi}_A(x, U_\alpha f(x))dx \le C,$$

as required. □

Remark 2.10 Theorems 2.7 and 2.8 are shown to be valid if conditions (φ') and (ψ') can be replaced by

(φ'') $\lim_{t\to 0} \varphi(t) \geq 1$;

(ψ'') $\lim_{t\to 0} \psi(t) \geq 1$,

since we may consider $e^{\varepsilon_0}\varphi$ and $e^{\varepsilon_0}\psi$ instead of φ and ψ respectively.

In the later use, we need the following result, which can be proved in the same manner as Lemma 2.4.

Corollary 2.11 *Suppose $p_+(G) < n/\alpha$. If $A > n$, then*

$$\int_G \widetilde{\Phi}_A(x, U_\alpha f(x))dx \leq C\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)}$$

for all measurable functions f on G such that $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \leq 1$.

3 Mean continuity I

First we introduce a notion of capacity as an extension of Meyers [23] and the first author [25]. For a set $E \subset \mathbf{R}^n$ and an open set $G \subset \mathbf{R}^n$, we define

$$C_{\alpha,p(\cdot),q(\cdot)}(E;G) = \inf_f \int_G f(y)^{p(y)}(\log(c_0 + f(y)))^{q(y)}dy,$$

where the infimum is taken over all nonnegative measurable functions f on $\mathbf{R}^n$ such that f vanishes outside G and $U_\alpha f(x) \geq 1$ for every $x \in E$ (cf. Futamura-Mizuta-Shimomura [11], Harjulehto-Hästö [13], Harjulehto-Hästö-Koskenoja [15] and Harjulehto-Hästö-Koskenoja-Varonen [16]). Then, since $t^{p(x)}(\log(c_0+t))^{q(x)}$ is convex for each fixed $x \in \mathbf{R}^n$ (see (1.2)), we see that $C_{\alpha,p(\cdot),q(\cdot)}(\cdot;G)$ is a countably subadditive and nondecreasing capacity. We say that E is of $C_{\alpha,p(\cdot),q(\cdot)}$-capacity zero, written as $C_{\alpha,p(\cdot),q(\cdot)}(E) = 0$, if

$$C_{\alpha,p(\cdot),q(\cdot)}(E \cap G;G) = 0 \qquad \text{for every bounded open set } G.$$

We here mention the following fundamental properties of our capacity.

Lemma 3.1 (cf. [11, Lemma 4.1]). *For $E \subset \mathbf{R}^n$, $C_{\alpha,p(\cdot),q(\cdot)}(E) = 0$ if and only if there exists a nonnegative function $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$ such that $U_\alpha f \not\equiv \infty$ but $U_\alpha f(x) = \infty$ for every $x \in E$.*

Lemma 3.2 (cf. [25, Corollary 1.2, Chapter 5]). *If $C_{\alpha,p(\cdot),q(\cdot)}(E;G) = 0$ for some bounded open set G, then $C_{\alpha,p(\cdot),q(\cdot)}(E) = 0$.*

For $x_0 \in \mathbf{R}^n$ and $r > 0$, set $f_{x_0,r}(w) = r^\alpha f(x_0 + rw)$. Then note that

$$U_\alpha f(x) = U_\alpha f_{x_0,r}(z) \qquad \text{for } x = x_0 + rz.$$

Further set

$$p_{x_0,r}(z) = p(x_0 + rz) \text{ and } q_{x_0,r}(z) = q(x_0 + rz);$$

see also Fiorenza-Rakotoson [9] for shifting the exponent. Then note that $p_{x_0,r}$ satisfies (p1) and (p2) for $r \le 1$ since $\log \varphi(1/t)/\log(1/t)$ is nondecreasing on $(0, r_0]$. Similarly, note that $q_{x_0,r}$ satisfies (q1) and (q2) for $r \le 1$.

Before showing our third theorem, we give the following result.

LEMMA 3.3 *Let f be a nonnegative locally integrable function on $\mathbf{R}^n$ such that*

$$\lim_{r\to 0+} \int_{B(x_0,r)} r^{\alpha p(y)-n} \max\left\{1, (\log(e + r^{-1}))^{-q(y)}\right\} f(y)^{p(y)} (\log(e + f(y)))^{q(y)} dy = 0.$$

Then $\lim_{r\to 0+} \|(f\chi_{B(x_0,r)})_{x_0,r}\|_{L^{p_{x_0,r}(\cdot)}(\log L)^{q_{x_0,r}(\cdot)}(\mathbf{R}^n)} = 0$.

PROOF. As in [22, Theorem 2.4], it suffices to show that

$$\lim_{r\to 0+} \int_{\mathbf{R}^n} ((f\chi_{B(x_0,r)})_{x_0,r}(w))^{p_{x_0,r}(w)} (\log(e + (f\chi_{B(x_0,r)})_{x_0,r}(w)))^{q_{x_0,r}(w)} dw = 0.$$

For this we have only to find

$$\begin{aligned} & \int_{\mathbf{R}^n} ((f\chi_{B(x_0,r)})_{x_0,r}(w))^{p_{x_0,r}(w)} (\log(e + (f\chi_{B(x_0,r)})_{x_0,r}(w)))^{q_{x_0,r}(w)} dw \\ = & \int_{\mathbf{R}^n} (r^\alpha (f\chi_{B(x_0,r)})(x_0 + rw))^{p(x_0+rw)} (\log(e + r^\alpha (f\chi_{B(x_0,r)})(x_0 + rw)))^{q(x_0+rw)} dw \\ \le & \; C \int_{B(x_0,r)} r^{\alpha p(y)-n} \max\left\{1, (\log(e + r^{-1}))^{-q(y)}\right\} f(y)^{p(y)} (\log(e + f(y)))^{q(y)} dy. \end{aligned}$$

The required assertion is now proved. □

We are now ready to show our third theorem concerning the vanishing Sobolev type integrability, which gives an extension of Meyers [24], Harjulehto-Hästö [13] and the authors [11, Theorem 4.5].

THEOREM 3.4 *Suppose $p_+ < n/\alpha$. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)} \le 1$ and (1.3). Then*

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \widetilde{\Phi}_A(x, |U_\alpha f(x) - U_\alpha f(x_0)|) dx = 0 \tag{3.1}$$

holds for all $x_0 \in \mathbf{R}^n \setminus (E_1 \cup E_2)$, *where*

$$\begin{aligned} E_1 &= \{x \in \mathbf{R}^n : U_\alpha f(x) = \infty\}, \\ E_2 &= \Big\{x \in \mathbf{R}^n : \limsup_{r\to 0+} \int_{B(x,r)} r^{\alpha p(y)-n} \max\left\{1, (\log(e+r^{-1}))^{-q(y)}\right\} \\ &\qquad \times f(y)^{p(y)}(\log(e+f(y)))^{q(y)} dy > 0\Big\}. \end{aligned}$$

By Lemma 3.1, we see that E_1 has $C_{\alpha,p(\cdot),q(\cdot)}$-capacity zero. In the next section we show examples of $p(\cdot)$ and $q(\cdot)$ for which E_2 has $C_{\alpha,p(\cdot),q(\cdot)}$-capacity zero, where φ and ψ are not necessarily constants.

Proof of Theorem 3.4. It suffices to show that (3.1) holds for $x_0 \in \mathbf{R}^n \setminus (E_1 \cup E_2)$. Write

$$\begin{aligned} U_\alpha f(x) - U_\alpha f(x_0) &= \int_{B(x_0,2|x-x_0|)} |x-y|^{\alpha-n} f(y)dy \\ &\quad + \int_{\mathbf{R}^n \setminus B(x_0,2|x-x_0|)} |x-y|^{\alpha-n} f(y)dy - U_\alpha f(x_0) \\ &= U_1(x) + U_2(x). \end{aligned}$$

If $y \in \mathbf{R}^n \setminus B(x_0, 2|x-x_0|)$, then $|x_0 - y| \le 2|x-y|$, since $U_\alpha f(x_0) < \infty$, so that we can apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{x\to x_0} U_2(x) = 0.$$

Note here that

$$U_1(x) \le \int_{B(x_0,r)} |x-y|^{\alpha-n} f(y)\ dy \equiv U_\alpha f_r(x)$$

for $x \in B(x_0, r/2)$, where $f_r = f\chi_{B(x_0,r)}$. Hence, we have only to show that

$$\lim_{r\to 0+} ⨍_{B(x_0,r)} \widetilde{\Phi}_A(x, U_\alpha f_r(x))dx = 0.$$

We may assume from Lemma 3.3 that $\|(f_r)_{x_0,r}\|_{L^{p_{x_0,r}(\cdot)}(\log L)^{q_{x_0,r}(\cdot)}(\mathbf{R}^n)}$ is small when r is small. By Corollary 2.11, we have

$$\begin{aligned} ⨍_{B(x_0,r)} \widetilde{\Phi}_A(x, U_\alpha f_r(x))dx &= ⨍_{B(0,1)} \kappa(q(x_0+rz)/p(x_0+rz), -A/p(x_0+rz)^2, \\ &\qquad -1/p(x_0+rz), U_\alpha(f_r)_{x_0,r}(z))^{p^\sharp(x_0+rz)}dz \\ &\le C\|(f_r)_{x_0,r}\|_{L^{p_{x_0,r}(\cdot)}(\log L)^{q_{x_0,r}(\cdot)}(\mathbf{R}^n)}, \end{aligned}$$

which together with Lemma 3.3 implies that the left hand side tends to zero as $r \to 0+$. Thus the proof is completed. □

Lemma 3.5 *Suppose both φ and ψ are constants. Then*

$$C_{\alpha,p(\cdot),q(\cdot)}(B(x_0,r);B(x_0,r)) \le Cr^{n-\alpha p(x_0)}(\log(e+r^{-1}))^{q(x_0)}$$

for each $x_0 \in \mathbf{R}^n$ and $r > 0$.

Proof. For $x_0 \in \mathbf{R}^n$ and $r > 0$, define the potential

$$u(x) = \int |x-y|^{\alpha-n} f(y)dy,$$

where $f(y) = r^{-\alpha}\chi_{B(x_0,r)}$. Then, since $u(x) \ge C$ for $x \in B(x_0,r)$, we have

$$\begin{aligned} C_{\alpha,p(\cdot),q(\cdot)}(B(x_0,r);B(x_0,r)) &\le C\int_{B(x_0,r)} r^{-\alpha p(y)}(\log(e+r^{-\alpha}))^{q(y)}dy \\ &\le Cr^{n-\alpha p(x_0)}(\log(e+r^{-1}))^{q(x_0)}, \end{aligned}$$

which proves the lemma. □

We can show the next lemma (see the proof of Lemma 4.3 below).

Lemma 3.6 (cf. [11, Lemma 4.4]). *Suppose both φ and ψ are constants. Further suppose $p_+ < n/\alpha$. If f is a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)} \le 1$, then*

$$\lim_{r\to 0+} r^{\alpha p(x)-n}(\log(e+r^{-1}))^{-q(x)}\int_{B(x,r)} f(y)^{p(y)}(\log(e+f(y)))^{q(y)}dy = 0$$

holds for all $x \in \mathbf{R}^n$ except in a set $E \subset \mathbf{R}^n$ with $C_{\alpha,p(\cdot),q(\cdot)}(E) = 0$.

By Theorem 3.4, we can show the following result, which is an extension of [11, Corollary 4.6] (see also Harjulehto-Hästö [13, Theorem 4.12] for $\alpha = 1$).

Proposition 3.7 *Suppose both φ and ψ are constants. Further suppose $p_+ < n/\alpha$ and $q_+ \le 0$. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)} \le 1$ such that $U_\alpha f \not\equiv \infty$. Then*

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \left\{|U_\alpha f(x) - U_\alpha f(x_0)|(\log(e+|U_\alpha f(x)-U_\alpha f(x_0)|))^{q(x)/p(x)}\right\}^{p^\sharp(x)} dx = 0$$

holds for all x_0 except in a set $E \subset \mathbf{R}^n$ with $C_{\alpha,p(\cdot),q(\cdot)}(E) = 0$.

For, if $q_+ \le 0$, then we have $C_{\alpha,p(\cdot),q(\cdot)}(E_2) = 0$ by Lemma 3.6, so that Theorem 3.4 gives the present proposition.

4 Mean continuity II

In this section, let

$$p(x) = p_0 - \omega(|x_n|) \qquad \text{and} \qquad q(x) = q_0 - \eta(|x_n|)$$

for $x = (x_1, ..., x_{n-1}, x_n) \in \mathbf{R}^n$, where $p_0 > 1$, $q_0 \in \mathbf{R}$ and, further, $p_- > 1$. To show that $p(\cdot)$ satisfies (p2), note that

$$\begin{aligned}
\frac{\log \varphi(1/(s+t))}{\log(1/(s+t))} &\le \frac{\log \varphi(1/(s+t))}{\log(1/s)} + \frac{\log \varphi(1/(s+t))}{\log(1/t)} \\
&\le \frac{\log \varphi(1/s)}{\log(1/s)} + \frac{\log \varphi(1/t)}{\log(1/t)}
\end{aligned}$$

for all $0 < s, t < r_0$, so that

$$|\omega(s) - \omega(t)| \le \omega(|s-t|),$$

which implies (p2).

Similarly, noting that

$$|\eta(s) - \eta(t)| \le \eta(|s-t|),$$

we insist that $q(\cdot)$ satisfies (q2).

Let

$$H = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}^1 : x_n = 0\}.$$

LEMMA 4.1 *Suppose* $0 < r < r_0$. *If* $x_0 \in H$, *then*

$$C_{\alpha,p(\cdot),q(\cdot)}(B(x_0,r); B(x_0,r)) \le C r^{n-\alpha p_0} (\log(e + r^{-1}))^{q_0} \varphi(r^{-1})^{-\alpha} \psi(r^{-1})^{-1}.$$

PROOF. For a proof, we consider the set

$$S_r = \{x \in \mathbf{R}^n : r/2 < |x_n| < r\}$$

and define

$$u(x) = \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap S_r} r^{-\alpha} |x-y|^{\alpha-n} dy.$$

Then, since $u(x) \ge C$ for $x \in B(x_0,r)$, we have

$$\begin{aligned}
C_{\alpha,p(\cdot),q(\cdot)}(B(x_0,r); B(x_0,r)) &\le C \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap S_r} r^{-\alpha p(y)} (\log(e + r^{-\alpha}))^{q(y)} dy \\
&\le C r^{n-\alpha p_0} (\log(e + r^{-1}))^{q_0} \varphi(r^{-1})^{-\alpha} \psi(r^{-1})^{-1},
\end{aligned}$$

which proves the lemma. □

LEMMA 4.2 *If* $x_0 \in \mathbf{R}^n \setminus H$ *and* $0 < r < \min\{r_0, |(x_0)_n|/2\}$, *then*

$$C_{\alpha,p(\cdot),q(\cdot)}(B(x_0,r); B(x_0,r)) \le C(|(x_0)_n|) r^{n-\alpha p(x_0)} (\log(e + r^{-1}))^{q(x_0)}.$$

PROOF. First we show that

$$|p(x) - p(y)| \le \frac{C(|(x_0)_n|)}{\log(1/|x-y|)} \tag{4.1}$$

for $x, y \in B(x_0, r)$ with $0 < r < \min\{r_0, |(x_0)_n|/2\}$. This is trivial when $|(x_0)_n|/2 \ge r_0$. If $|(x_0)_n|/2 < |y_n| < |x_n| < r_0$, then we have

$$\begin{aligned} |p(x) - p(y)| &= \left(\frac{\log \varphi(1/|x_n|)}{\log(1/|x_n|)} - \frac{\log \varphi(1/|x_n|)}{\log(1/|y_n|)} \right) + \left(\frac{\log \varphi(1/|x_n|)}{\log(1/|y_n|)} - \frac{\log \varphi(1/|y_n|)}{\log(1/|y_n|)} \right) \\ &\le \log \varphi(1/|x_n|) \left(\frac{1}{\log(1/|x_n|)} - \frac{1}{\log(1/|y_n|)} \right) \\ &\le \frac{\log \varphi(1/|x_n|)}{\log(1/|x_n - y_n|)} \\ &\le \frac{C(|(x_0)_n|)}{\log(1/|x-y|)}, \end{aligned}$$

which proves (4.1).

Similarly note that

$$|q(x) - q(y)| \le \frac{C(|(x_0)_n|)}{\log(\log(1/|x-y|))}$$

for $x, y \in B(x_0, r)$ with $0 < r < \min\{r_0, |(x_0)_n|/2\}$.

Now Lemma 3.5 gives the required result. □

For $r > 0$, set

$$h(r; x) = \begin{cases} r^{n-\alpha p_0} (\log(e + r^{-1}))^{q_0} \varphi(r^{-1})^{-\alpha} \psi(r^{-1})^{-1} & \text{if } x \in H, \\ r^{n-\alpha p(x)} (\log(e + r^{-1}))^{q(x)} & \text{if } x \in \mathbf{R}^n \setminus H. \end{cases}$$

We show the following result.

LEMMA 4.3 (cf. [11, Lemma 4.4]) *Suppose* $p_0 < n/\alpha$. *If* f *is a nonnegative measurable function on* $\mathbf{R}^n$ *with* $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)} \le 1$, *then*

$$\lim_{r \to 0+} h(r; x)^{-1} \int_{B(x,r)} f(y)^{p(y)} (\log(e + f(y)))^{q(y)} dy = 0$$

holds for all $x \in \mathbf{R}^n$ *except in a set* $E \subset \mathbf{R}^n$ *with* $C_{\alpha,p(\cdot),q(\cdot)}(E) = 0$.

PROOF. First we prove that $C_{\alpha,p(\cdot),q(\cdot)}(E \setminus H) = 0$. For each integer j, set $L_j = \{x \in \mathbf{R}^n : 2^{-j} < |x_n| \le 2^{-j+1}\}$. For $\delta > 0$, consider the set

$$E_{\delta,j} = \left\{ x \in L_j : \limsup_{r \to 0+} h(r;x)^{-1} \int_{B(x,r)} f(y)^{p(y)} (\log(e + f(y)))^{q(y)} dy > \delta \right\}.$$

By subadditivity and Lemma 3.2, it suffices that $C_{\alpha,p(\cdot),q(\cdot)}(E_{\delta,j} \cap B(0,R); B(0,2R)) = 0$ for all $R > 1/2$. Let $0 < \varepsilon < 1/(5 \cdot 2^{|j|+1})$. For each $x \in E_{\delta,j} \cap B(0,R)$, we find $0 < r(x) < \varepsilon$ such that

$$h(r(x);x)^{-1} \int_{B(x,r(x))} f(y)^{p(y)} (\log(e + f(y)))^{q(y)} dy > \delta.$$

By the covering lemma (see [32, Lemma, p. 9]), there exists a disjoint family $\{B_i\}$ such that $B_i = B(x_i, r(x_i))$ and $\bigcup_i B(x_i, 5r(x_i)) \supset E_{\delta,j} \cap B(0,R)$. Then we have by Lemma 4.2

$$\begin{aligned} C_{\alpha,p(\cdot),q(\cdot)}(E_{\delta,j} \cap B(0,R); B(0,2R)) &\le \sum_i C_{\alpha,p(\cdot),q(\cdot)}(B(x_i, 5r(x_i)); B(x_i, 5r(x_i))) \\ &\le C(j) \sum_i h(r(x_i); x_i) \\ &\le C(j) \delta^{-1} \int_{\bigcup_i B_i} f(y)^{p(y)} (\log(e + f(y)))^{q(y)} dy. \end{aligned}$$

Since

$$\begin{aligned} \left| \bigcup_i B_i \right| &\le C \sum_i \delta^{-1} r(x_i)^{\alpha p(x_i)} (\log(e + r(x_i)^{-1}))^{-q(x_i)} \int_{B_i} f(y)^{p(y)} (\log(e + f(y)))^{q(y)} dy \\ &\le C \delta^{-1} \varepsilon^{\alpha p_-}, \end{aligned}$$

it follows from the absolute continuity of integral that

$$C_{\alpha,p(\cdot),q(\cdot)}(E_{\delta,j} \cap B(0,R); B(0,2R)) = 0.$$

Similarly, we can prove that $C_{\alpha,p(\cdot),q(\cdot)}(E \cap H) = 0$ with the aid of Lemma 4.1, as required. □

COROLLARY 4.4 *Suppose $p_0 < n/\alpha$ and $q_0 \le 0$. If f is a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)} \le 1$, then*

$$C_{\alpha,p(\cdot),q(\cdot)}(E_2) = 0.$$

For this, in case $q_0 \le 0$, note

$$\begin{aligned} &\int_{B(x_0,r)} r^{\alpha p(y) - n} (\log(e + r^{-1}))^{-q(y)} f(y)^{p(y)} (\log(e + f(y)))^{q(y)} dy \\ \le\ & C r^{\alpha p_0 - n} (\log(e + r^{-1}))^{-q_0} \varphi(r^{-1})^{\alpha} \psi(r^{-1}) \int_{B(x_0,r)} f(y)^{p(y)} (\log(e + f(y)))^{q(y)} dy \end{aligned}$$

for $x_0 \in H$ and $0 < r \le r_0$; and

$$\int_{B(x_0,r)} r^{\alpha p(y)-n}(\log(e+r^{-1}))^{-q(y)} f(y)^{p(y)}(\log(e+f(y)))^{q(y)}dy$$
$$\le\ C(|(x_0)_n|)r^{\alpha p(x_0)-n}(\log(e+r^{-1}))^{-q(x_0)}\int_{B(x_0,r)} f(y)^{p(y)}(\log(e+f(y)))^{q(y)}dy$$

for $x_0 \in \mathbf{R}^n \setminus H$ and $0 < r \le \min\{r_0, |(x_0)_n|/2\}$. Hence $C_{\alpha,p(\cdot),q(\cdot)}(E_2) = 0$ by Lemma 3.6.

Now Theorem 3.4 gives the following result.

THEOREM 4.5 *Suppose $p_0 < n/\alpha$ and $q_0 \le 0$. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)} \le 1$ such that $U_\alpha f \not\equiv \infty$. Then*

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \widetilde{\Phi}_A(x, |U_\alpha f(x) - U_\alpha f(x_0)|)dx = 0$$

holds for all $x_0 \in \mathbf{R}^n$ except in a set of $C_{\alpha,p(\cdot),q(\cdot)}$-capacity zero.

REMARK 4.6 For $p_0 < n/\alpha$ and $q_0 \le 0$, set

$$p(x) = \begin{cases} p_0 + \omega(x_n) & \text{if } x_n \ge 0, \\ p_0 & \text{if } x_n < 0 \end{cases} \quad \text{and } q(x) = \begin{cases} q_0 + \eta(x_n) & \text{if } x_n \ge 0, \\ q_0 & \text{if } x_n < 0. \end{cases}$$

If $p_+ < n/\alpha$ and $q_+ \le 0$, then we can show that Theorem 4.5 is true for these exponents.

5 Vanishing exponential integrability

For a compact set K in G, we define

$$K(r) = \{x \in G : \delta_K(x) < r\},$$

where $\delta_K(x)$ denotes the distance of x from K. For $\nu \ge 0$, we say that the Minkowski $(n-\nu)$-content of K is finite if

$$|K(r)| \le Cr^\nu \qquad \text{for small } r > 0.$$

Note here that if K is a singleton, then its Minkowski 0-content is finite, and if K is a spherical surface, then its Minkowski $(n-1)$-content is finite. As another examples of K, we may consider fractal type sets like Cantor sets or Koch curves. In this section, we consider variable exponents

$$p(x) = p(\delta_K(x)) = p_0 + \omega(\delta_K(x))$$

and

$$q(x) = q(\delta_K(x)) = q_0 + \eta(\delta_K(x))$$

for $p_0 > 1$ and $q_0 \in \mathbf{R}$. Note that $p(\cdot)$ and $q(\cdot)$ satisfies (p2) and (q2), respectively.

We know the following result.

LEMMA 5.1 (cf. [26, Lemma 2.3]). *Let K be a compact set in G whose Minkowski $(n-\nu)$-content is finite. Then*

$$\int_G \delta_K(x)^{-\nu}(\log(1+\delta_K(x)^{-1}))^{-a}\,dx < \infty$$

for every $a > 1$.

LEMMA 5.2 (cf. [26, Lemma 2.4]). *Suppose the Minkowski $(n-\nu)$-content of K is finite. If f is a measurable function on G with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(G)} \le 1$, then*

$$\int_G |f(x)|^{p_0}(\log(e+|f(x)|))^{q_0}\varphi(|f(x)|)^{\nu/p_0}\psi(|f(x)|)dx \le C.$$

PROOF. Consider the set

$$G' = \{x \in K(r_0) : |f(x)| < \delta(x)^{-\nu/p_0}(\log(1/\delta(x)))^{-a/p_0}\},$$

where we will determine a later; here we set $\delta(x) = \delta_K(x)$ for simplicity. If $x \in G'$, then we have by (φ) and (ψ)

$$\begin{aligned}
&|f(x)|^{p_0}(\log(e+|f(x)|))^{q_0}\varphi(|f(x)|)^{\nu/p_0}\psi(|f(x)|)\\
\le\ & C\delta(x)^{-\nu}(\log(1/\delta(x)))^{-a}(\log(1/\delta(x)))^{q_0}\varphi(1/\delta(x))^{\nu/p_0}\psi(1/\delta(x))\\
\le\ & C\delta(x)^{-\nu}(\log(1/\delta(x)))^{-a+q_0+\varepsilon_3},
\end{aligned}$$

where $\varepsilon_3 > \varepsilon_1\nu/p_0$. If we take a so large that $a > 1 + q_0 + \varepsilon_3$, it follows from Lemma 5.1 that

$$\int_{G'} |f(x)|^{p_0}(\log(e+|f(x)|))^{q_0}\varphi(|f(x)|)^{\nu/p_0}\psi(|f(x)|)dx \le C.$$

If $x \notin G'$ and $\delta(x) < r_0$, then $|f(x)| \ge \delta(x)^{-\nu/p_0}(\log(1/\delta(x)))^{-a/p_0}$, so that

$$\delta(x) \ge C|f(x)|^{-p_0/\nu}(\log|f(x)|)^{-a/\nu}.$$

Hence, in view of Lemma 2.1, we see that

$$\begin{aligned}
\frac{\log\varphi(1/\delta(x))}{\log(1/\delta(x))}\log|f(x)| &\ge \frac{\log\varphi(C|f(x)|^{p_0/\nu}(\log|f(x)|)^{a/\nu})}{\log(C|f(x)|^{p_0/\nu}(\log|f(x)|)^{a/\nu})}\log|f(x)|\\
&\ge \frac{\nu}{p_0}\left\{\frac{\log(C\varphi(|f(x)|))}{\log|f(x)| + C\log(C\log|f(x)|)}\log|f(x)|\right\}\\
&= \frac{\nu}{p_0}\left\{\log(C\varphi(|f(x)|))\left(1 - \frac{C\log(C\log|f(x)|)}{\log|f(x)| + C\log(C\log|f(x)|)}\right)\right\}\\
&\ge \frac{\nu}{p_0}\log\varphi(|f(x)|) - C,
\end{aligned}$$

which yields

$$
\begin{aligned}
|f(x)|^{p(x)-p_0} &= \exp\left(\frac{\log\varphi(1/\delta(x))}{\log(1/\delta(x))}\log|f(x)|\right) \geq \exp\left(\frac{\nu}{p_0}\log\varphi(|f(x)|) - C\right) \\
&= C\varphi(|f(x)|)^{\nu/p_0}.
\end{aligned}
$$

Similarly, we have

$$
\frac{\log\psi(1/\delta(x))}{\log(\log(1/\delta(x)))}\log(\log|f(x)|) \geq \log\psi(|f(x)|) - C,
$$

which yields

$$
(\log|f(x)|)^{q(x)-q_0} \geq C\psi(|f(x)|).
$$

Thus it follows that

$$
\begin{aligned}
&\int_{K(r_0)\setminus G'} |f(x)|^{p_0}(\log(e+|f(x)|))^{q_0}\varphi(|f(x)|)^{\nu/p_0}\psi(|f(x)|)dx \\
\leq\ & C\int_G |f(x)|^{p(x)}(\log(e+|f(x)|))^{q(x)}\,dx \leq C.
\end{aligned}
$$

Finally, since $p(x) \geq p_1 > p_0$ when $\delta(x) \geq r_0$, we find

$$
\begin{aligned}
&\int_{G\setminus K(r_0)} |f(x)|^{p_0}(\log(e+|f(x)|))^{q_0}\varphi(|f(x)|)^{\nu/p_0}\psi(|f(x)|)dx \\
\leq\ & C\int_G |f(x)|^{p(x)}(\log(e+|f(x)|))^{q(x)}\,dx + C \leq C.
\end{aligned}
$$

The required assertion is now proved. □

From now on set $p_0 = n/\alpha, q_0 \geq 0, \varphi(r) = c(\log(e+r))^a, \psi(r) = c(\log(\log(e+r)))^b$ and $K = H$ for $a, b \geq 0$ and $c > 0$. For $x_0 \in H$ and $r_0 > 0$, let $\mathbf{B} = B(x_0, r_0)$ be a ball in $\mathbf{R}^n$. By Lemma 5.2, we have the following integrability for all measurable functions f on $\mathbf{R}^n$ with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)} \leq 1$ (see also [26]).

Corollary 5.3 *If f is a measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)} \leq 1$, then*

$$
\int_{\mathbf{B}} |f(x)|^{n/\alpha}(\log(e+|f(x)|))^{q_0+a\alpha/n}(\log(\log(e+|f(x)|)))^b dx \leq C. \tag{5.1}
$$

We know the following vanishing exponential integrability for Riesz potentials of functions in Orlicz classes ([28]):

LEMMA 5.4 *Let* $a^\sharp = n^2/(n^2 - \alpha n - a\alpha^2 - \alpha n q_0) > 0$ *and* $b^\sharp = \alpha n b/(n^2 - \alpha n - a\alpha^2 - \alpha n q_0)$*. If* f *is a nonnegative measurable function on* $\mathbf{R}^n$ *satisfying* (1.3) *and* (5.1)*, then*

$$\lim_{r\to 0+} ⨍_{B(x_0,r)} \Big\{\exp\Big(A|U_\alpha f(x) - U_\alpha f(x_0)|^{a^\sharp} \times (\log(1+|U_\alpha f(x) - U_\alpha f(x_0)|))^{b^\sharp}\Big) - 1\Big\} dx = 0$$

holds for all $A > 0$ *and all* $x_0 \in H \setminus E_f$*, where*

$$E_f = \{x \in H : U_\alpha f(x) = \infty\}.$$

By Lemma 3.1 we see that E_f has $C_{\alpha,p(\cdot),q(\cdot)}$-capacity zero.

Finally, in view of Lemmas 3.1 and 5.4 and Corollary 5.3, we give the vanishing exponential integrability for Riesz potentials with variable exponent, which is based on a constant exponent, Orlicz space result.

THEOREM 5.5 *Let* $a^\sharp = n^2/(n^2 - \alpha n - a\alpha^2 - \alpha n q_0) > 0$ *and* $b^\sharp = \alpha n b/(n^2 - \alpha n - a\alpha^2 - \alpha n q_0)$*. If* f *is a nonnegative measurable function on* $\mathbf{R}^n$ *with* $\|f\|_{L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)} \le 1$ *satisfying* (1.3)*, then*

$$\lim_{r\to 0+} ⨍_{B(x_0,r)} \Big\{\exp\Big(A|U_\alpha f(x) - U_\alpha f(x_0)|^{a^\sharp} \times (\log(1+|U_\alpha f(x) - U_\alpha f(x_0)|))^{b^\sharp}\Big) - 1\Big\} dx = 0$$

holds for all $A > 0$ *and all* $x_0 \in H$ *except in a set of* $C_{\alpha,p(\cdot),q(\cdot)}$*-capacity zero.*

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