Trudinger's inequality for Riesz potentials of functions in Musielak-Orlicz spaces

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Abstract

In this paper we are concerned with Trudinger's inequality for Riesz potentials of functions in Musielak-Orlicz spaces.

1 Introduction

A famous Trudinger inequality ([19]) insists that Sobolev functions in $W^{1,N}(G)$ satisfy finite exponential integrability, where G is an open bounded set in $\mathbf{R}^N$ (see also [1], [3], [17], [20]). Great progress on Trudinger type inequalities has been made for Riesz potentials of order α $(0 < \alpha < N)$ in the limiting case $\alpha p = N$ (see e.g. [5], [6], [7], [8]). In [2], [14] and [16], Trudinger type exponential integrability was studied on Orlicz spaces, as extensions of [5], [6] and [8].

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions (see [4]). Trudinger type exponential integrability on variable exponent Lebesgue spaces $L^{p(\cdot)}$ was investigated in [9], [10] and [11]. For the two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$, see [13]. These spaces are special cases of so-called Musielak-Orlicz spaces ([18]).

Our aim in this paper is to give a general version of Trudinger type exponential integrability for Riesz potentials of functions in Musielak-Orlicz spaces as an extension of the above results. By treating such general setting, we can obtain new results (Corollary 4.2) which have not been found in the literature.

2 Preliminaries

Let G be a bounded open set in $\mathbf{R}^N$. Let d_G=diam G.

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We consider a function

$$\Phi(x,t) = t\phi(x,t) : G \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions (Φ1) – (Φ4):

(Φ1) $\phi(\,\cdot\,,t)$ is measurable on G for each $t \geq 0$ and $\phi(x,\,\cdot\,)$ is continuous on $[0,\infty)$ for each $x \in G$;

(Φ2) there exists a constant $A_1 \geq 1$ such that

$$A_1^{-1} \leq \phi(x,1) \leq A_1 \quad \text{for all } x \in G;$$

(Φ3) $\phi(x,\cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \geq 1$ such that

$$\phi(x,t) \leq A_2\phi(x,s) \quad \text{for all } x \in G \quad \text{whenever } 0 \leq t < s;$$

(Φ4) there exists a constant $A_3 \geq 1$ such that

$$\phi(x,2t) \leq A_3\phi(x,t) \quad \text{for all } x \in G \text{ and } t > 0.$$

Note that (Φ2), (Φ3) and (Φ4) imply

$$0 < \inf_{x\in G} \phi(x,t) \leq \sup_{x\in G} \phi(x,t) < \infty$$

for each $t > 0$.

If $\Phi(x,\cdot)$ is convex for each $x \in G$, then (Φ3) holds with $A_2 = 1$; namely $\phi(x,\cdot)$ is non-decreasing for each $x \in G$.

Let $\bar{\phi}(x,t) = \sup_{0\leq s\leq t} \phi(x,s)$ and

$$\overline{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)\,dr$$

for $x \in G$ and $t \geq 0$. Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \leq \overline{\Phi}(x,t) \leq A_2\Phi(x,t)$$

for all $x \in G$ and $t \geq 0$. In fact, the first inequality is seen as follows:

$$\overline{\Phi}(x,t) \geq \int_{t/2}^t \bar{\phi}(x,r)\,dr \geq \frac{t}{2}\phi(x,t/2) \geq \frac{1}{2A_3}\Phi(x,t).$$

We shall also consider the following condition:

(Φ5) for every $\gamma > 0$, there exists a constant $B_\gamma \geq 1$ such that

$$\phi(x,t) \leq B_\gamma\phi(y,t)$$

whenever $|x-y| \leq \gamma t^{-1/N}$ and $t \geq 1$.

Example 2.1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1, \ldots, k$, be measurable functions on G such that

(P1) $1 \le p^- := \inf_{x\in G} p(x) \le \sup_{x\in G} p(x) =: p^+ < \infty$

and

(Q1) $-\infty < q_j^- := \inf_{x\in G} q_j(x) \le \sup_{x\in G} q_j(x) =: q_j^+ < \infty$

for all $j = 1, \ldots, k$.

Set $L_c(t) = \log(c+t)$ for $c \ge e$ and $t \ge 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$ and

$$\Phi(x,t) = t^{p(x)} \prod_{j=1}^{k} (L_c^{(j)}(t))^{q_j(x)}.$$

Then, $\Phi(x,t)$ satisfies (Φ1), (Φ2) and (Φ4). It satisfies (Φ3) if there is a constant $K \ge 0$ such that $K(p(x)-1) + q_j(x) \ge 0$ for all $x \in G$ and $j = 1, \ldots, k$; in particular if $p^- > 1$ or $q_j^- \ge 0$ for all $j = 1, \ldots, k$.

$\Phi(x,t)$ satisfies (Φ5) if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{C_p}{L_e(1/|x-y|)}$$

with a constant $C_p \ge 0$ and

(Q2) $q_j(\cdot)$ is j-log-Hölder continuous, namely

$$|q_j(x) - q_j(y)| \le \frac{C_{q_j}}{L_e^{(j)}(1/|x-y|)}$$

with constants $C_{q_j} \ge 0$, $j = 1, \ldots k$.

Given $\Phi(x,t)$ as above, the associated Musielak-Orlicz space

$$L^\Phi(G) = \left\{ f \in L^1_{loc}(G)\,;\ \int_G \Phi\big(y, |f(y)|\big)\, dy < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(G)} = \inf\left\{ \lambda > 0\,;\ \int_G \overline{\Phi}\big(y, |f(y)|/\lambda\big)\, dy \le 1 \right\}$$

(cf. [18]).

3 Lemmas

Throughout this paper, let C denote various constants independent of the variables in question and $C(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$.

We denote by $B(x, r)$ the open ball centered at x of radius r. For a measurable set E, we denote by $|E|$ the Lebesgue measure of E.

For a locally integrable function f on G, the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap G} |f(y)|\, dy.$$

We know the following boundedness of maximal operator on $L^\Phi(G)$.

LEMMA 3.1 ([12, Corollary 4.4]). *Suppose that $\Phi(x,t)$ satisfies ($\Phi5$) and further assume:*

($\Phi3^*$) $t \mapsto t^{-\varepsilon_0}\phi(x,t)$ *is uniformly almost increasing on* $(0,\infty)$ *for some* $\varepsilon_0 > 0$.

Then the maximal operator M is bounded from $L^\Phi(G)$ into itself, namely, there is a constant $C > 0$ such that

$$\|Mf\|_{L^\Phi(G)} \le C\|f\|_{L^\Phi(G)}$$

for all $f \in L^\Phi(G)$.

We consider the function

$$\gamma(x,t) : G \times (0, d_G) \to [0, \infty)$$

satisfying the following conditions ($\gamma1$) and ($\gamma2$):

($\gamma1$) $\gamma(\,\cdot\,, t)$ is measurable on G for each $0 < t < d_G$ and $\gamma(x, \,\cdot\,)$ is continuous on $(0, d_G)$ for each $x \in G$;

($\gamma2$) there exists a constant $B_0 \ge 1$ such that

$$B_0^{-1} \le \gamma(x,t) \le B_0 t^{-N} \quad \text{for all } x \in G \quad \text{whenever } 0 < t < d_G.$$

Further we consider the function

$$\Gamma_\alpha(x,t) : G \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions ($\Gamma1$) and ($\Gamma2$):

($\Gamma1$) $\Gamma_\alpha(\,\cdot\,, t)$ is measurable on G for each $t \ge 0$ and $\Gamma_\alpha(x, \,\cdot\,)$ is continuous on $[0,\infty)$ for each $x \in G$;

($\Gamma2$) $\Gamma_\alpha(x, \cdot)$ is uniformly almost increasing, namely there exists a constant $B_1 \ge 1$ such that

$$\Gamma_\alpha(x,t) \le B_1 \Gamma_\alpha(x,s) \quad \text{for all } x \in G \quad \text{whenever } 0 \le t < s;$$

(Γ3) there exist constants $\alpha_0 > 0, B_2 \geq 1$ and $B_3 \geq 1$ such that

$$t^{\alpha-N}\phi(x,\gamma(x,t))^{-1} \leq B_2\Gamma_\alpha(x,1/t)$$

for all $x \in G$ and $\alpha \geq \alpha_0$ whenever $0 < t < d_G$ and

$$\int_t^{d_G} \rho^\alpha \gamma(x,\rho)\,\frac{d\rho}{\rho} \leq B_3\Gamma_\alpha(x,1/t)$$

for all $x \in G, 0 < t \leq d_G/2$ and $\alpha \geq \alpha_0$.

LEMMA 3.2. *Suppose that $\Phi(x,t)$ satisfies (Φ5) and $\alpha_0 \leq \alpha < N$. Then there exists a constant $C > 0$ such that*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-N} f(y)\,dy \leq C\Gamma_\alpha\left(x, \frac{1}{\delta}\right)$$

for all $x \in G$, $0 < \delta \leq d_G/2$ and nonnegative $f \in L^\Phi(G)$ with $\|f\|_{L^\Phi(G)} \leq 1$.

Proof. Let f be a nonnegative measurable function with $\|f\|_{L^\Phi(G)} \leq 1$. Since

$$\phi(y,\gamma(x,|x-y|))^{-1} \leq B'\phi(x,\gamma(x,|x-y|))^{-1}$$

with some constant $B' > 0$ by (γ2), (Φ3), (Φ4) and (Φ5), we have by (Φ3), (Γ2) and (Γ3)

$$\begin{aligned}
&\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-N} f(y)\,dy \\
\leq\ & \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-N}\gamma(x,|x-y|)\,dy \\
&+A_2\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-N} f(y)\frac{\phi(y,f(y))}{\phi(y,\gamma(x,|x-y|))}\,dy \\
\leq\ & C\int_\delta^{d_G} \rho^\alpha\gamma(x,\rho)\,\frac{d\rho}{\rho} + A_2B'\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-N}\phi(x,\gamma(x,|x-y|))^{-1}\Phi(y,f(y))\,dy \\
\leq\ & CB_3\Gamma_\alpha(x,1/\delta) + A_2B_1B_2B'\Gamma_\alpha(x,1/\delta)\int_{G\setminus B(x,\delta)} \Phi(y,f(y))\,dy \\
\leq\ & (CB_3 + A_2B_1B_2B')\Gamma_\alpha(x,1/\delta).
\end{aligned}$$

Thus we obtain the required results. □

LEMMA 3.3. *Let $\alpha \geq \alpha_0$. Then there exists a constant $C' > 0$ such that $\Gamma_\alpha(x,2/d_G) \geq C'$ for all $x \in G$.*

Proof. By (Γ3) and (γ2),

$$\begin{aligned}
\Gamma_\alpha(x,2/d_G) &\geq B_3^{-1}\int_{d_G/2}^{d_G} \rho^\alpha\gamma(x,\rho)\,\frac{d\rho}{\rho} \geq B_0^{-1}B_3^{-1}\int_{d_G/2}^{d_G}\rho^\alpha\,\frac{d\rho}{\rho} \\
&= B_0^{-1}B_3^{-1}\alpha^{-1}d_G^\alpha(1-2^{-\alpha}) = C'
\end{aligned}$$

for all $x \in G$, as required. □

Lemma 3.4 (cf. [15, Lemma 2.1]). *Suppose $\Gamma_\alpha(x,t)$ satisfies the uniform log-type condition:*

($\Gamma_{\log}$) *there exists a constant $c_\Gamma > 0$ such that*

$$c_\Gamma^{-1}\Gamma_\alpha(x,s) \le \Gamma_\alpha(x,s^2) \le c_\Gamma \Gamma_\alpha(x,s)$$

for all $x \in G$ and $s > 0$.

Then, for every $c > 1$, there exists $C > 0$ such that $\Gamma_\alpha(x,cs) \le C\Gamma_\alpha(x,s)$ for all $x \in G$ and $s > 0$.

4 Trudinger's inequality

For $0 < \alpha < N$, we define the Riesz potential of order α for a locally integrable function f on G by

$$I_\alpha f(x) = \int_G |x-y|^{\alpha-N} f(y)\, dy.$$

Theorem 4.1. *Assume that $\Phi(x,t)$ satisfies (Φ5) and ($\Phi3^*$). Suppose that $\Gamma_\alpha(x,t)$ satisfies ($\Gamma_{\log}$). For each $x \in G$, let $\gamma_\alpha(x) = \sup_{s>0} \Gamma_\alpha(x,s)$. Suppose $\Psi_\alpha(x,t) : G \times [0,\infty) \to [0,\infty]$ satisfies the following conditions:*

(Ψ_α1) *$\Psi_\alpha(\cdot,t)$ is measurable on G for each $t \in [0,\infty)$; $\Psi_\alpha(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in G$;*

(Ψ_α2) *there is a constant $B_4 \ge 1$ such that $\Psi_\alpha(x,t) \le \Psi_\alpha(x,B_4 s)$ for all $x \in G$ whenever $0 < t < s$;*

(Ψ_α3) *there are constants $B_5,\ B_6 \ge 1$ and $t_0 > 0$ such that $\Psi_\alpha(x,\Gamma_\alpha(x,t)/B_5) \le B_6 t$ for all $x \in G$ and $t \ge t_0$.*

Then there exist constants $c_1, c_2 > 0$ such that $I_\alpha f(x)/c_1 < \gamma_\alpha(x)$ for a.e. $x \in G$ and

$$\int_G \Psi_\alpha\left(x, \frac{I_\alpha f(x)}{c_1}\right) dx \le c_2$$

for all $\alpha_0 \le \alpha < N$ and $f \ge 0$ satisfying $\|f\|_{L^\Phi(G)} \le 1$.

Proof. Let $f \ge 0$ and $\|f\|_{L^\Phi(G)} \le 1$. Note from Lemma 3.1 that

$$\int_G Mf(x)\, dx \le |G| + A_1 A_2 \int_G \Phi(x, Mf(x))\, dx \le C_M. \tag{4.1}$$

Fix $x \in G$. For $0 < \delta \le d_G/2$, Lemma 3.2 implies

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-N} f(y)\, dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-N} f(y)\, dy \\ &\le C\left\{Mf(x) + \Gamma_\alpha\left(x, \frac{1}{\delta}\right)\right\} \end{aligned}$$

with constants $C > 0$ independent of x.

If $Mf(x) \le 2/d_G$, then we take $\delta = d_G/2$. Then, by Lemma 3.3

$$I_\alpha f(x) \le C\Gamma_\alpha\left(x, \frac{2}{d_G}\right).$$

By Lemma 3.4, there exists $C_1^* > 0$ independent of x such that

$$I_\alpha f(x) \le C_1^* \Gamma_\alpha(x, t_0) \qquad \text{if } Mf(x) \le 2/d_G. \tag{4.2}$$

Next, suppose $2/d_G < Mf(x) < \infty$. Let $m = \sup_{s \ge 2/d_G, x\in G} \Gamma_\alpha(x,s)/s$. By $(\Gamma_{\log})$, $m < \infty$. Define δ by

$$\delta^\alpha = \frac{(d_G/2)^\alpha}{m}\Gamma_\alpha(x, Mf(x))(Mf(x))^{-1}.$$

Since $\Gamma_\alpha(x, Mf(x))(Mf(x))^{-1} \le m$, $0 < \delta \le d_G/2$. Then by Lemma 3.3

$$\begin{aligned}\frac{1}{\delta} &\le C\Gamma_\alpha(x, Mf(x))^{-1/\alpha}(Mf(x))^{1/\alpha} \\ &\le C\Gamma_\alpha(x, 2/d_G)^{-1/\alpha}(Mf(x))^{1/\alpha} \le C(Mf(x))^{1/\alpha}.\end{aligned}$$

Hence, using $(\Gamma_{\log})$ and Lemma 3.4, we obtain

$$\Gamma_\alpha\left(x, \frac{1}{\delta}\right) \le C\Gamma_\alpha\big(x, C(Mf(x))^{1/\alpha}\big) \le C\Gamma_\alpha(x, Mf(x)).$$

By Lemma 3.4 again, we see that there exists a constant $C_2^* > 0$ independent of x such that

$$I_\alpha f(x) \le C_2^*\Gamma_\alpha\left(x, \frac{t_0 d_G}{2}Mf(x)\right) \qquad \text{if } 2/d_G < Mf(x) < \infty. \tag{4.3}$$

Now, let $c_1 = B_4B_5\max(C_1^*, C_2^*)$. Then, by (4.2) and (4.3),

$$\frac{I_\alpha f(x)}{c_1} \le \frac{1}{B_4B_5}\max\left\{\Gamma_\alpha(x, t_0), \Gamma_\alpha\left(x, \frac{t_0 d_G}{2}Mf(x)\right)\right\}$$

whenever $Mf(x) < \infty$. Since $Mf(x) < \infty$ for a.e. $x \in G$ by Lemma 3.1, $I_\alpha f(x)/c_1 < \gamma_\alpha(x)$ a.e. $x \in G$, and by $(\Psi_\alpha 2)$ and $(\Psi_\alpha 3)$, we have

$$\begin{aligned}&\Psi_\alpha\left(x, \frac{I_\alpha f(x)}{c_1}\right) \\ &\le \max\left\{\Psi_\alpha(x, \Gamma_\alpha(x, t_0)/B_5), \Psi_\alpha\left(x, \Gamma_\alpha\left(x, \frac{t_0 d_G}{2}Mf(x)\right)/B_5\right)\right\} \\ &\le B_6 t_0 + \frac{B_6 t_0 d_G}{2}Mf(x)\end{aligned}$$

for a.e. $x \in G$. Thus, we have by (4.1)

$$\begin{aligned}\int_G \Psi_\alpha\left(x, \frac{I_\alpha f(x)}{c_1}\right)dx &\le B_6 t_0|G| + \frac{B_6 t_0 d_G}{2}\int_G Mf(x)\,dx \\ &\le B_6 t_0|G| + \frac{B_6 t_0 d_G C_M}{2} = c_2.\end{aligned}$$

□

Applying Theorem 4.1 to special Φ given in Example 2.1, we obtain the following corollary.

COROLLARY 4.2. *Let Φ be as in Example 2.1.*

(1) *Suppose there exists an integer $1 \le j_0 \le k$ such that*

$$\inf_{x\in G}(p(x) - q_{j_0}(x) - 1) > 0 \tag{4.4}$$

and

$$\sup_{x\in G}(p(x) - q_j(x) - 1) \le 0 \tag{4.5}$$

for all $j \le j_0 - 1$ in case $j_0 \ge 2$. Then there exist constants $c_1, c_2 > 0$ such that

$$\int_G E_+^{(j_0)}\left(\left(\frac{I_\alpha f(x)}{c_1}\right)^{p(x)/(p(x)-q_{j_0}(x)-1)} \right.$$
$$\left. \times \prod_{j=1}^{k-j_0}\left(L_e^{(j)}\left(\frac{I_\alpha f(x)}{c_1}\right)\right)^{q_{j_0+j}(x)/(p(x)-q_{j_0}(x)-1)}\right) dx \le c_2$$

for all $N/p^- \le \alpha < N$ and $f \ge 0$ satisfying $\|f\|_{L^\Phi(G)} \le 1$, where $E^{(1)}(t) = e^t - e$, $E^{(j+1)}(t) = \exp(E^j(t)) - e$ and $E_+^{(j)}(t) = \max(E^{(j)}(t), 0)$.

(2) *If*

$$\sup_{x\in G}(p(x) - q_j(x) - 1) \le 0$$

for all $j = 1, \dots, k$, then there exist constants $c_1, c_2 > 0$ such that

$$\int_G E^{(k+1)}\left(\left(\frac{I_\alpha f(x)}{c_1}\right)^{p(x)/(p(x)-1)}\right) dx \le c_2$$

for all $N/p^- \le \alpha < N$ and $f \ge 0$ satisfying $\|f\|_{L^\Phi(G)} \le 1$.

Proof. First we show the case (1). In this case, set

$$\gamma(x,t) = t^{-N/p(x)}\left(\prod_{j=1}^{j_0-1}[L_e^{(j)}(1/t)]^{-1}\right)[L_e^{(j_0)}(1/t)]^{-(q_{j_0}(x)+1)/p(x)}\left(\prod_{j=j_0+1}^{k}[L_e^{(j)}(1/t)]^{-q_j(x)/p(x)}\right)$$

and

$$\Gamma_\alpha(x,t) = [L_e^{(j_0)}(t)]^{(p(x)-q_{j_0}(x)-1)/p(x)}\left(\prod_{j=j_0+1}^{k}[L_e^{(j)}(t)]^{-q_j(x)/p(x)}\right).$$

Here note that $\gamma(x,t)$ satisfies ($\gamma 2$) and $\Gamma_\alpha(x,t)$ is uniformly almost increasing on

t and satisfies $(\Gamma_{\log})$ by (4.4). We have by $N/p^- \le \alpha$ and (4.5)

$$\begin{aligned}
& t^{\alpha-N}\phi(x,\gamma(x,t))^{-1} \\
\le\ & Ct^{\alpha-N/p(x)}\left(\prod_{j=1}^{j_0-1}[L_e^{(j)}(1/t)]^{p(x)-q_j(x)-1}\right) \\
& \times[L_e^{(j_0)}(1/t)]^{(p(x)-q_{j_0}(x)-1)/p(x)}\left(\prod_{j=j_0+1}^{k}[L_e^{(j)}(1/t)]^{-q_j(x)/p(x)}\right) \\
\le\ & C[L_e^{(j_0)}(1/t)]^{(p(x)-q_{j_0}(x)-1)/p(x)}\left(\prod_{j=j_0+1}^{k}[L_e^{(j)}(1/t)]^{-q_j(x)/p(x)}\right) \\
=\ & C\Gamma_\alpha(x,1/t)
\end{aligned}$$

for all $x \in G$ and $\alpha_0 = N/p^- \le \alpha < N$ whenever $0 < t < d_G$. By (4.4), we find $\varepsilon_0 > 0$ such that $\inf_{x\in G}\{1-(q_{j_0}(x)+1)/p(x)\} > \varepsilon_0$. We see from $N/p^- \le \alpha$, (4.4) and (4.5) that

$$\begin{aligned}
& \int_t^{d_G}\rho^\alpha\gamma(x,\rho)\,\frac{d\rho}{\rho} \\
\le\ & C\int_t^{d_G}\left(\prod_{j=1}^{j_0-1}[L_e^{(j)}(1/\rho)]^{-1}\right)[L_e^{(j_0)}(1/\rho)]^{-(q_{j_0}(x)+1)/p(x)}\left(\prod_{j=j_0+1}^{k}[L_e^{(j)}(1/\rho)]^{-q_j(x)/p(x)}\right)\frac{d\rho}{\rho} \\
\le\ & C[L_e^{(j_0)}(1/t)]^{1-(q_{j_0}(x)+1)/p(x)-\varepsilon_0}\left(\prod_{j=j_0+1}^{k}[L_e^{(j)}(1/t)]^{-q_j(x)/p(x)}\right) \\
& \times\int_t^{d_G}\left(\prod_{j=1}^{j_0-1}[L_e^{(j)}(1/\rho)]^{-1}\right)[L_e^{(j_0)}(1/\rho)]^{-1+\varepsilon_0}\,\frac{d\rho}{\rho} \\
\le\ & C\Gamma_\alpha(x,1/t)
\end{aligned}$$

for all $0 < t \le d_G/2$ and $N/p^- \le \alpha < N$. Hence, $\Gamma_\alpha(x,t)$ satisfies $(\Gamma3)$.

Now, set

$$\psi(x,t) = t^{p(x)/(p(x)-q_{j_0}(x)-1)}\prod_{i=1}^{k-j_0}[L_e^{(i)}(t)]^{q_{j_0+i}(x)/(p(x)-q_{j_0}(x)-1)}$$

for $x \in G$ and $t > 0$. Then

$$\psi(x,\Gamma_\alpha(x,s)) \le C_1L_e^{(j_0)}(s)$$

for $s > 0$.

Since $\inf_{x\in G} p(x)/(p(x)-q_{j_0}(x)-1) > 0$, there are constants $0 < \theta \le 1$ and $C_2 \ge 1$ such that

$$\psi(x,ct) \le C_2c^\theta\psi(x,t) \tag{4.6}$$

for all $x \in G$, $t > 0$ and $0 < c \le 1$. Hence, choosing $B \ge 1$ such that $C_1C_2B^{-\theta} \le 1$, we have

$$\psi(x,\Gamma_\alpha(x,s)/B) \le C_2B^{-\theta}\psi(x,\Gamma_\alpha(x,s)) \le C_2B^{-\theta}C_1L_e^{(j_0)}(s) \le L_e^{(j_0)}(s)$$

for $s > 0$. Thus,

$$E^{(j_0)}(\psi(x, \Gamma_\alpha(x,s)/B)) \le s \quad \text{for } s > 0. \tag{4.7}$$

Let $u_0 > 0$ be the unique solution of the equation $e^u - e = u$. Then $E^{(1)}(u) \ge u_0$ if and only if $u \ge u_0$. Choose $t_0 > 0$ such that $\psi(x,t) \ge u_0$ for $t \ge t_0$ and define

$$\Psi(x,t) = \begin{cases} E^{(j_0)}(\psi(x,t)) & \text{for } t \ge t_0, \\ \Psi(x,t_0)\dfrac{t}{t_0} & \text{for } 0 < t < t_0. \end{cases}$$

Noting that

$$\psi(x,t) = \psi\left(x, \frac{t}{C_2^{1/\theta}s} C_2^{1/\theta} s\right) \le \psi(x, C_2^{1/\theta} s)$$

for $0 < t \le s$ by (4.6), $\Psi(x,t)$ satisfies $(\Psi_\alpha 1)$, $(\Psi_\alpha 2)$ (with $B_4 = C_2^{1/\theta}$, say) and $(\Psi_\alpha 3)$, in view of (4.6) and (4.7).

Thus Theorem 4.1 implies the existence of constants $c_1, C_3 > 0$ such that

$$\int_G \Psi\left(x, \frac{I_\alpha f(x)}{c_1}\right) dx \le C_3$$

for all $N/p^- \le \alpha < N$ and $f \ge 0$ satisfying $\|f\|_{L^\Phi(G)} \le 1$. Let $S_f = \{x \in G : I_\alpha f(x) \ge c_1 t_0\}$. Then

$$\begin{aligned} \int_G E_+^{(j_0)}\left(\psi\left(x, \frac{I_\alpha f(x)}{c_1}\right)\right) dx &\le C_4 \int_{G \setminus S_f} dx + \int_{G \cap S_f} \Psi\left(x, \frac{I_\alpha f(x)}{c_1}\right) dx \\ &\le C_4 |G| + C_3 = c_2 \end{aligned}$$

for all $N/p^- \le \alpha < N$ and $f \ge 0$ satisfying $\|f\|_{L^\Phi(G)} \le 1$, which shows the assertion of (1).

In the case (2), setting

$$\begin{aligned} \gamma(x,t) &= t^{-N/p(x)} \left(\prod_{j=1}^{k} [L_e^{(j)}(1/t)]^{-1}\right) [L_e^{(k+1)}(1/t)]^{-1/p(x)}, \\ \Gamma_\alpha(x,t) &= [L_e^{(k+1)}(1/t)]^{1-1/p(x)} \end{aligned}$$

and

$$\psi(x,t) = t^{p(x)/(p(x)-1)},$$

the above discussion yields the required result. □

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