Riesz decomposition for super-polyharmonic functions in the punctured unit ball

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January 6, 2011

Abstract

We consider a Riesz decomposition theorem for super-polyharmonic functions satisfying certain growth condition on surface integrals in the punctured unit ball. We give a condition that super-polyharmonic functions u have the bound

$$u(x) = O(\mathcal{R}_2(x)),$$

where $\mathcal{R}_2$ denotes the fundamental solution for $-\Delta u$ in $\mathbf{R}^n$.

1 Introduction

Let $B(x,r)$ denote the open ball centered at x with radius r, whose boundary is written as $S(x,r) = \partial B(x,r)$. If $x = 0$, then we simply write $B(r) = B(0,r)$ and $S(r) = S(0,r)$. Fix r_0, $0 < r_0 < 1$. For $0 < r < r_0$, we set

$$A(r) = \{x \in \mathbf{R}^n : r < |x| < r_0\}.$$

For a Borel measurable function u on $S(r)$, let us define the spherical mean over $S(r)$ by

$$M(u,r) = \fint_{S(r)} u(x)\ dS(x) = \frac{1}{\omega_n r^{n-1}} \int_{S(r)} u(x)\ dS(x),$$

where ω_n denotes the surface area of the unit sphere $S(1)$.

A real-valued function u on an open set $\Omega \subset \mathbf{R}^n$ is called polyharmonic of order m on Ω if $u \in C^{2m}(\Omega)$ and $\Delta^m u = 0$ on Ω, where m is a positive integer and Δ^m denotes the Laplacian iterated m times. We denote by $\mathcal{H}^m(\Omega)$ the space of polyharmonic functions of order m on Ω; for fundamental properties of polyharmonic

2000 Mathematics Subject Classification : Primary 31B30, 31B05, 31B15

Key words and phrases : polyharmonic functions, super-polyharmonic functions, spherical means, Riesz decomposition

functions, we refer the reader to the book by N. Aronszajn, T. M. Creese and L. J. Lipkin [2].

We say that a locally integrable Borel function u on Ω is super-polyharmonic of order m in Ω if

(1) $(-\Delta)^m u$ is a nonnegative measure on Ω, that is,

$$\int_\Omega u(x)(-\Delta)^m\varphi(x)\,dx \geq 0 \qquad \text{for all nonnegative } \varphi \in C_0^\infty(\Omega);$$

(2) u is lower semicontinuous in Ω;

(3) every point of Ω is a Lebesgue point of u

(see [5]); $(-\Delta)^m u$ is referred to as the Riesz measure of u and denoted by μ_u.

Consider the Riesz kernel of order $2m$ defined by

$$\mathcal{R}_{2m}(x) = \begin{cases} \alpha_{n,m}(-1)^{\frac{2m-n}{2}}|x|^{2m-n}\log(1/|x|) & \text{if } 2m-n \text{ is an even nonnegative integer,} \\ \alpha_{n,m}(-1)^{\max\{0,\frac{(2m-n+1)}{2}\}}|x|^{2m-n} & \text{otherwise,} \end{cases}$$

where $\alpha_{n,m}$ is a positive constant chosen such that $(-\Delta)^m\mathcal{R}_{2m}$ is the Dirac measure at the origin. Note here that if $2m \leq n$, then

$$\lim_{x\to 0}\mathcal{R}_{2m}(x) = \begin{cases} \infty & \text{if } 2m \leq n, \\ 0 & \text{if } 2m > n \end{cases} \tag{1.1}$$

Following the book by K. Hayman and P. B. Kennedy [8], we consider the remainder term in the Taylor expansion of $\mathcal{R}_{2m}(\cdot - y)$ given by

$$\mathcal{R}_{2m,L}(x,y) = \mathcal{R}_{2m}(x-y) - \sum_{|\lambda|\leq L}\frac{(-y)^\lambda}{\lambda!}D^\lambda\mathcal{R}_{2m}(x),$$

where L is a real number; in case $L < 0$, set $\mathcal{R}_{2m,L}(x,y) = \mathcal{R}_{2m}(x-y)$.

For a nonnegative measure μ on $A(0)$, we define

$$\mathcal{R}_{2m,L}\mu(x) = \int_{A(0)}\mathcal{R}_{2m,L}(x,y)d\mu(y).$$

Our first aim in this paper is to establish the following result.

THEOREM 1.1. *The following are equivalent:*

(1) *there is a super-polyharmonic function u on $B(1)\setminus\{0\}$ with $\mu_u = (-\Delta)^m u \geq 0$ such that*

   (1-1) $M((-1)^m u, r) \leq \mathcal{R}_2(r)$ *for all* $0 < r < r_0$*;*

   (1-2) $\liminf_{r\to 0}\ \mathcal{R}_2(r)^{-1}M(((-1)^m u)^+, r) < \infty$*;*

(1-3) $\liminf_{x\to 0} \; \mathcal{R}_2(x)^{-1}(-1)^m u(x) = -\infty$,

where $v^+ = \max\{v, 0\}$.

(2) *m is odd and $2m \le n$.*

This extends a recent result by M. Ghergu, A. Moradifam and S. D. Taliaferro [7, Theorem 1.1]; our result is stated as follows:

Corollary 1.2. *The following are equivalent:*

(1) *there is a super-polyharmonic function u on $B(1)\backslash\{0\}$ with $\mu_u = (-\Delta)^m u \ge 0$ such that*

(1-4) $(-1)^m u \le 0$,

(1-5) $\liminf_{x\to 0} \; \mathcal{R}_2(x)^{-1}(-1)^m u(x) = -\infty$;

(2) *m is odd and $2m \le n$.*

It is easy to see from (1.1) that (2) is equivalent to

(3) $\lim_{x\to 0} (-1)^m \mathcal{R}_{2m}(x) = -\infty$.

To show Theorem 1.1, we apply the following Riesz decomposition theorem for super-polyharmonic functions in the punctured unit ball (see also M. Ghergu, A. Moradifam and S. D. Taliaferro [7, Theorem 3.1], where they treated the case when $(-1)^m u \le 0$).

Theorem 1.3. *Let u be a super-polyharmonic function on $B(1) \setminus \{0\}$ with $\mu_u = (-\Delta)^m u \ge 0$. Suppose that there exist constants $\alpha \ge n-2$ and $C > 0$ with*

$$M((-1)^m u, r) \le C \begin{cases} r^{-\alpha} & \text{when } \alpha > 0, \\ \log(1/r) & \text{when } \alpha = 0, \end{cases}$$

whenever $0 < r < r_0$. Suppose further that

$$\liminf_{r\to 0} \; r^{\alpha-(n-2)} \mathcal{R}_2(r)^{-1} M(((-1)^m u)^+, r) < \infty.$$

Then there exist a function $h \in \mathcal{H}^m(B(r_0))$ and constants c_λ such that if $\alpha > n-2$, then

$$u(x) = \mathcal{R}_{2m,L}\mu_u(x) + h(x) + \sum_{|\lambda|\le L} c_\lambda D^\lambda \mathcal{R}_{2m}(x)$$

for all $x \in A(0)$, where L is the integer such that $L \le 2m - n + \alpha < L+1$; and if $\alpha = n-2$, then

$$u(x) = \mathcal{R}_{2m,2m-3}\mu_u(x) + h(x) + \sum_{|\lambda|\le 2m-2} c_\lambda D^\lambda \mathcal{R}_{2m}(x)$$

for all $x \in A(0)$.

The case $\alpha > n-2$ was treated in [6, Theorems 1.3 and 1.4]. For further related results, we refer the reader to [1, 3, 4, 5, 9, 10].

Throughout this paper, let C denote various positive constants independent of the variables in question and let $C(a, b, \cdots)$ be a positive constant which may depend on $a, b, \ldots$.

2 Preliminaries and fundamental lemmas

Since $\Delta^k \mathcal{R}_{2m}(x)$ is of rotation free, we write

$$\Delta^k \mathcal{R}_{2m}(r) = \Delta^k \mathcal{R}_{2m}(x)$$

when $r = |x|$.

LEMMA 2.1. *For $r > 0$ and $y \in \mathbf{R}^n$,*

$$M(\mathcal{R}_{2m}(\cdot - y), r) = \begin{cases} \displaystyle\sum_{j=0}^{m-1} a_j r^{2j} \Delta^j \mathcal{R}_{2m}(y) & \text{if } |y| > r, \\ \displaystyle\sum_{j=0}^{m-1} a_j |y|^{2j} \Delta^j \mathcal{R}_{2m}(r) & \text{if } |y| \le r, \end{cases}$$

where $a_0 = 1$ and

$$a_j = \frac{1}{2^j j! n(n+2)\cdots(n+2j-2)}$$

for positive integers j.

Proof. Since $\Delta^m \mathcal{R}_{2m}(\cdot - y) = 0$ in $B(0, |y|)$, this equality holds for $r < |y|$ by Pizetti's formula [12].

If $|y| < r$, then we have

$$\begin{aligned} M(\mathcal{R}_{2m}(\cdot - y), r) &= \frac{1}{\omega_n r^{n-1}} \int_{S(r)} \mathcal{R}_{2m}\left(\frac{|y|}{r} x - \frac{r}{|y|} y\right) dS(x) \\ &= \frac{1}{\omega_n |y|^{n-1}} \int_{S(|y|)} \mathcal{R}_{2m}\left(x' - \frac{r}{|y|} y\right) dS(x') \\ &= \sum_{j=0}^{m-1} a_j |y|^{2j} \Delta^j \mathcal{R}_{2m}\left(\frac{r}{|y|} y\right). \end{aligned}$$

Since $M(\mathcal{R}_{2m}(\cdot - y), r)$ is a continuous function of r, the present lemma follows. □

LEMMA 2.2. *Let u be a super-polyharmonic function on $B(1) \setminus \{0\}$ with $\mu_u = (-\Delta)^m u \ge 0$. Then*

$$\begin{aligned} M(u, r) &= \int_{A(r)} \left(\sum_{j=0}^{m-1} a_j r^{2j} \Delta^j \mathcal{R}_{2m}(y) - \sum_{j=0}^{m-1} a_j |y|^{2j} \Delta^j \mathcal{R}_{2m}(r) \right) d\mu_u(y) \\ &\quad + \sum_{j=0}^{m-1} \left(b_j r^{2j} + c_j r^{2m-n-2j} + d_j r^{2m-n-2j} \log(1/r) \right) \end{aligned}$$

for $0 < r < r_0$, *where* b_j, c_j *and* d_j *are constants satisfying* $d_j = 0$ *when* $j > (2m-n)/2$ *or* $2m-n$ *is odd.*

Proof. For $0 < R < r < r_0$, u is represented as

$$\begin{aligned} u(x) &= \int_{A(R)} \mathcal{R}_{2m}(x-y)\; d\mu_u(y) + h_{1,R}(x) \\ &= \int_{A(R)} \left(\mathcal{R}_{2m}(x-y) - \sum_{j=0}^{m-1} a_j |y|^{2j} \Delta^j \mathcal{R}_{2m}(x) \right)\; d\mu_u(y) + h_{2,R}(x) \end{aligned}$$

when $x \in A(R)$, where $h_{1,R},\ h_{2,R} \in \mathcal{H}^m(A(R))$. Hence we have by Lemma 2.1

$$\begin{aligned} M(u,r) &= \int_{A(R)} \left(M(\mathcal{R}_{2m}(\cdot - y), r) - \sum_{j=0}^{m-1} a_j |y|^{2j} \Delta^j \mathcal{R}_{2m}(r) \right)\; d\mu_u(y) + M(h_{2,R}, r) \\ &= \int_{A(r)} \left(\sum_{j=0}^{m-1} a_j r^{2j} \Delta^j \mathcal{R}_{2m}(y) - \sum_{j=0}^{m-1} a_j |y|^{2j} \Delta^j \mathcal{R}_{2m}(r) \right)\; d\mu_u(y) + M(h_{2,R}, r). \end{aligned}$$

This equality implies that $M(h_{2,R}, r)$ does not depend on R, so that, by [3, Lemma 1], we can find a constants $b_j, c_j, d_j (j = 0, 1, \cdots m-1)$ independent of R such that $d_j = 0$ when $j > (2m-n)/2$ or $2m-n$ is odd, and

$$M(h_{2,R}, r) = \sum_{j=0}^{m-1} \left(b_j r^{2j} + c_j r^{2m-n-2j} + d_j r^{2m-n-2j} \log(1/r) \right),$$

as required. □

Set

$$g_m(t,r) = \sum_{j=0}^{m-1} a_j r^{2j} \Delta^j \mathcal{R}_{2m}(t) - \sum_{j=0}^{m-1} a_j t^{2j} \Delta^j \mathcal{R}_{2m}(r).$$

Lemma 2.3. *The following hold:*

(1) $(-1)^m g_m(t,r) \geq 0$ *for* $r < t$;

(2) $(-1)^m g_m(t,r) \geq C(a) t^{2m-2} \varphi(t,r)$ *for* $t > ar$ *and* $a > 1$, *where* $C(a)$ *is a positive constant and*

$$\varphi(t,r) = \begin{cases} \log(t/r) & \text{if } n = 2, \\ r^{2-n} & \text{if } n \geq 3. \end{cases}$$

Proof. For fixed r, set $g_m(t) = g_m(t,r)$. We prove this lemma by induction on m. In case $m = 1$, we have

$$g_1(t) = \begin{cases} \alpha_{2,1} \log(r/t) & \text{if } n = 2, \\ \alpha_{n,1}(t^{2-n} - r^{2-n}) & \text{if } n \geq 3. \end{cases}$$

Hence (1) and (2) hold for $m = 1$.

Suppose that (1) and (2) hold for $m-1$ where $m \geq 2$. Note that

$$\Delta g_m(t) = g_m''(t) + \frac{n-1}{t} g_m'(t) = -g_{m-1}(t).$$

Since $M(\mathcal{R}_{2m}(\cdot - y), r) \in C^{2m-2}(\mathbf{R}^n)$, we see that $g_m(r) = g_m'(r) = 0$, so that

$$\begin{aligned}(-1)^m g_m(t) &= (-1)^m \int_r^t s^{1-n} \left(\int_r^s \left(\xi^{n-1} g_m'(\xi)\right)' d\xi \right) ds \\ &= \int_r^t s^{1-n} \left(\int_r^s \xi^{n-1} (-1)^{m-1} g_{m-1}(\xi) d\xi \right) ds, \qquad (2.1)\end{aligned}$$

which gives $(-1)^m g_m(t) \geq 0$ for $t > r$.

First we consider the case $n \geq 3$. For $a > 1$, take a_1 and a_2 such that $1 < a_2 < a_1 < a$. Then we have for $t \geq ar$

$$\begin{aligned}(-1)^m g_m(t) &\geq \int_{a_1 r}^t s^{1-n} \left(\int_{a_2 r}^s \xi^{n-1} (-1)^{m-1} g_{m-1}(\xi) d\xi \right) ds \\ &\geq C(a) r^{2-n} \int_{a_1 r}^t s^{1-n} \left(\int_{a_2 r}^s \xi^{2m+n-5} d\xi \right) ds \\ &\geq C(a) t^{2m-2} r^{2-n}.\end{aligned}$$

Thus (2) holds for m.

Next we deal with the case $n = 2$. Since

$$r^{2m-2}(-1)^m g_m(a,1) \leq (-1)^m g_m(t,r) \leq r^{2m-2}(-1)^m g_m(b,1)$$

for $ar < t < br$, it is sufficient to find constants $c(m) > 1$ such that $c(m)$ is increasing for m and

$$(-1)^m g_m(t,r) \geq C t^{2m-2} \log(t/r)$$

for $t \geq c(m) r$. By (2.1), we have

$$\begin{aligned}(-1)^m g_m(t) &= \int_r^t \xi (-1)^{m-1} g_{m-1}(\xi) \log \frac{t}{\xi}\, d\xi \\ &\geq C \int_{c(m-1)r}^t \xi^{2m-3} \left(\log \frac{\xi}{r} \right) \log \frac{t}{\xi}\, d\xi \\ &= C t^{2m-2} \int_{c(m-1)r/t}^1 s^{2m-3} \left(\log \frac{ts}{r} \right) \log \frac{1}{s}\, ds \\ &\geq C t^{2m-2} \left\{ \log \frac{t}{r} \int_{c(m-1)/c(m)}^1 s^{2m-3} \log \frac{1}{s}\, ds - C(m) \right\} \\ &\geq C t^{2m-2} \log(t/r).\end{aligned}$$

The induction is completed. □

LEMMA 2.4. *Let u be a super-polyharmonic function on $B(1) \setminus \{0\}$ with $\mu_u = (-\Delta)^m u \geq 0$. Suppose that there exist constants $\alpha \geq n-2$ and $C > 0$ such that*

$$M((-1)^m u, r) \leq C \begin{cases} r^{-\alpha} & \text{when } \alpha > 0, \\ \log(1/r) & \text{when } \alpha = 0 \end{cases}$$

for $0 < r < r_0$.

(1) *If $\alpha > n-2$, then*

$$\limsup_{r \to +0} r^{\alpha+2-n} \int_{A(r)} |y|^{2m-2} \, d\mu_u(y) < \infty.$$

(2) *If $\alpha = n-2$, then*

$$\int_{A(0)} |y|^{2m-2} \, d\mu_u(y) < \infty.$$

Proof. First we show the case $n \geq 3$. For $0 < r < r_0$, we have by Lemmas 2.2 and 2.3,

$$\begin{aligned} M((-1)^m u, r) &\geq \int_{A(2r)} (-1)^m g_m(|y|, r) \, d\mu_u(y) - Cr^{2-n} \\ &\geq C(2) r^{2-n} \int_{A(2r)} |y|^{2m-2} \, d\mu_u(y) - Cr^{2-n}. \end{aligned}$$

Then we see that

$$r^{2-n} \int_{A(2r)} |y|^{2m-2} \, d\mu_u(y) \leq C \left(r^{-\alpha} + r^{2-n} \right),$$

so that we have the required result.

Next we prove the case $n = 2$. For $0 < r < r_0$, we have by Lemmas 2.2 and 2.3,

$$\begin{aligned} M((-1)^m u, r) &\geq \int_{A(2r)} (-1)^m g_m(|y|, r) \, d\mu_u(y) - C \log(1/r) \\ &\geq C(2) \int_{A(2r)} |y|^{2m-2} \log(|y|/r) \, d\mu_u(y) - C \log(1/r), \end{aligned}$$

as required. □

LEMMA 2.5. *Let u be a super-polyharmonic function on $B(1) \setminus \{0\}$ with $\mu_u = (-\Delta)^m u \geq 0$. Suppose that there exist constants $\alpha \geq n-2$ and $C > 0$ such that*

$$M((-1)^m u, r) \leq C \begin{cases} r^{-\alpha} & \text{when } \alpha > 0, \\ \log(1/r) & \text{when } \alpha = 0, \end{cases}$$

whenever $0 < r < r_0$. If $\alpha > n-2$, then

$$\int_{A(0)} |y|^{2m-n+\alpha+\varepsilon} \, d\mu_u(y) < \infty$$

for $\varepsilon > 0$; if $\alpha = n-2$, then one can take $\varepsilon = 0$.

Proof. By Lemma 2.4 (1), we have

$$\int_{B(r)\setminus B(r/2)} |y|^{2m-2}\, d\mu_u(y) \le C r^{-\alpha} \mathcal{R}_2(r)^{-1}$$

for $0 < r < r_0$, so that

$$\begin{aligned}
&\int_{A(0)} |y|^{2m-n+\alpha+\varepsilon}\, d\mu_u(y) \\
= &\sum_{j=0}^{\infty} \int_{B(2^{-j}r_0)\setminus B(2^{-j-1}r_0)} |y|^{2m-n+\alpha+\varepsilon}\, d\mu_u(y) \\
\le &\sum_{j=0}^{\infty} (2^{-j}r_0)^{2-n+\alpha+\varepsilon} \int_{B(2^{-j}r_0)\setminus B(2^{-j-1}r_0)} |y|^{2m-2}\, d\mu_u(y) \\
\le & C \sum_{j=0}^{\infty} 2^{-j\varepsilon} < \infty,
\end{aligned}$$

as required. □

Finally we discuss a fine limit property for Riesz potentials (see also [7, Proposition 4.2]). For this purpose, set

$$\mathcal{R}_{2m}\mu(x) = \int_{A(0)} |\mathcal{R}_{2m}(x-y)|\, d\mu(y),$$

where μ is a nonnegative measure on $A(0)$.

LEMMA 2.6. *Let $2m \le n$. Then for every $\alpha, \beta > 0$ there exists a nonnegative measure μ such that*

(1) $\displaystyle\int_{A(0)} |y|^{-\alpha}\, d\mu(y) < \infty$;

(2) $\displaystyle\limsup_{x\to 0} |x|^{\beta} \mathcal{R}_{2m}\mu(x) = \infty$.

Proof. Take a sequence $\{x_j\}$ such that $|x_j| = 1/j$. For sequences $\{a_j\}$ and $\{r_j\}$ of positive numbers, set

$$\mu = \sum_j a_j r_j^{-n} \chi_{B(x_j, r_j)},$$

where χ_E denotes the characteristic function of a measurable set E. Now it suffices to choose $\{a_j\}$ and $\{r_j\}$ such that

(1) $\displaystyle\int_{A(0)} |y|^{-\alpha}\, d\mu(y) \le C \sum_j j^{\alpha} a_j < \infty$;

(2) $\mathcal{R}_{2m}\mu(x_j) \ge C\mathcal{R}_{2m}(r_j) a_j \ge C j^{2\beta}$ for each j;

this is possible since $\lim_{x\to 0}\mathcal{R}_{2m}(x) = \infty$ when $2m \le n$. In fact, we choose $\{a_j\}$ such that $0 < a_j < j^{-\alpha}2^{-j}$ and $\{r_j\}$ such that $\{B(x_j, r_j)\}$ is a disjoint family and $\mathcal{R}_{2m}(r_j) > a_j^{-1} j^{2\beta}$. $\square$

Note here that (1) gives

(3) $M(\mathcal{R}_{2m}\mu, r)$ is bounded when $2m \le n$.

3 Representation formula

In the same manner as [11, Lemmas 6, 8 and 9] (see also [7, (3.12), (4.5)]), we have the following results.

Lemma 3.1. *Let $2m-3 \le L \le 2m-2$. Then there exists a constant $C > 0$ such that*

$$|\mathcal{R}_{2m,L}(x,y)| \le C \begin{cases} |y|^{L+1}\min\{\log(1/|y|), \log(1/|x|)\} & \text{if } L = 2m-3 \text{ and } n = 2, \\ |y|^L |x|^{2m-L-3}\mathcal{R}_2(x)\min\{|y|, |x|\} & \text{if } L = 2m-2 \text{ or } n \ge 3 \end{cases}$$

for all $x, y \in B(1)$ and $|x-y| > |x|/2$; if $2m > n$, then

$$|\mathcal{R}_{2m,L}(x,y)| \le C \begin{cases} |y|^{L+1}\min\{\log(1/|y|), \log(1/|x|)\} & \text{if } L = 2m-3 \text{ and } n = 2, \\ |y|^L |x|^{2m-L-3}\mathcal{R}_2(x)\min\{|y|, |x|\} & \text{if } L = 2m-2 \text{ or } n \ge 3 \end{cases}$$

for all $x, y \in B(1)$.

Lemma 3.2. *If $L > 2m-2$, then there exists a constant $C > 0$ such that*

$$|\mathcal{R}_{2m,L}(x,y)| \le C|y|^L |x|^{2m-n-L-1}\min\{|y|, |x|\}$$

for all $x, y \in B(1)$ and $|x-y| > |x|/2$; if $2m > n$, then

$$|\mathcal{R}_{2m,L}(x,y)| \le C|y|^L |x|^{2m-n-L-1}\min\{|y|, |x|\}$$

for all $x, y \in B(1)$.

Lemma 3.3. *Let u be a super-polyharmonic function on $B(1) \setminus \{0\}$ with $\mu_u = (-\Delta)^m u \ge 0$. Suppose that there exist constants $\alpha \ge n-2$ and $C > 0$ such that*

$$M((-1)^m u, r) \le C \begin{cases} r^{-\alpha} & \text{when } \alpha > 0, \\ \log(1/r) & \text{when } \alpha = 0, \end{cases}$$

whenever $0 < r < r_0$. Then

$$\lim_{r\to 0} r^q M(|\mathcal{R}_{2m,L}\mu_u|, r) = 0$$

for all $q > \alpha$, where L is the integer such that $2m-n+\alpha-1 < L \le 2m-n+\alpha$ when $\alpha > n-2$ and $L = 2m-3$ when $\alpha = n-2$.

Proof. First we consider the case $\alpha > n-2$, and take $\varepsilon > 0$ such that $2m-n+\alpha+\varepsilon-1 \le L < 2m-n+\alpha+\varepsilon$ and $\varepsilon < q-\alpha$. If $2m > n$, then, since $L \ge 2m-2$ in case $\alpha > n-2$, we have by Lemmas 2.5, 3.1 and 3.2

$$\begin{aligned} |\mathcal{R}_{2m,L}\mu_u(x)| &\le C|x|^{2m-L-3}\mathcal{R}_2(x)\int_{A(0)}|y|^L\min\{|y|,|x|\}d\mu_u(y) \\ &\le C|x|^{-\alpha-\varepsilon+n-2}\mathcal{R}_2(x)\int_{A(0)}|y|^{2m-n+\alpha+\varepsilon}d\mu_u(y) \\ &\le C|x|^{-\alpha-\varepsilon+n-2}\mathcal{R}_2(x), \end{aligned}$$

which gives

$$\lim_{r\to 0} r^q \fint_{S(r)} |\mathcal{R}_{2m,L}\mu_u(x)|dS(x) = 0 \tag{3.1}$$

for $q > \alpha$. If $2m-n \le 0$, then

$$\begin{aligned} |\mathcal{R}_{2m,L}(x,y)| &\le C\mathcal{R}_{2m}(x-y)\chi_{\{y:|x-y|<|x|/2\}}(y) \\ &\quad + C|y|^L|x|^{2m-L-3}\mathcal{R}_2(x)\min\{|y|,|x|\}. \end{aligned}$$

Noting from Lemma 2.1 that

$$\fint_{S(r)} \mathcal{R}_{2m}(x-y)\ dS(x) \le C\mathcal{R}_{2m}(r)$$

when $r/2 < |y| < 3r/2$, we obtain (3.1) for $q > \alpha$, as above.

Similarly, if $\alpha = n-2$ and $L = 2m-3$, then

$$\fint_{S(r)} |\mathcal{R}_{2m,L}\mu_u(x)|dS(x) \le Cr^{2-n}\log(1/r)\int_{A(0)}|y|^{2m-2}d\mu_u(y) \le Cr^{-\alpha}\log(1/r),$$

which implies (3.1) for $q > \alpha$, as required. □

THEOREM 3.4. *Let u be a super-polyharmonic function on $B(1)\setminus\{0\}$ with $\mu_u = (-\Delta)^m u \ge 0$. Suppose that there exist constants $\alpha \ge n-2$ and $C > 0$ such that*

$$M((-1)^m u, r) \le C \begin{cases} r^{-\alpha} & \text{when } \alpha > 0, \\ \log(1/r) & \text{when } \alpha = 0, \end{cases}$$

whenever $0 < r < r_0$. Then there exists a function $h_0 \in \mathcal{H}^m(A(0))$ such that

$$u(x) = \mathcal{R}_{2m,L}\mu_u(x) + h_0(x)$$

for $x \in A(0)$, where L is the integer such that $2m-n+\alpha-1 < L \le 2m-n+\alpha$ when $\alpha > n-2$ and $L = 2m-3$ when $\alpha = n-2$.

Proof. Since $\mathcal{R}_{2m,L}\mu_u(x)$ is super-polyharmonic in $A(0)$ by Lemma 3.3, we see that

$$u(x) - \mathcal{R}_{2m,L}\mu_u(x)$$

is polyharmonic of order m in $A(0)$ (in the sense of distributions). By Weyl's lemma, there exists a function $h_0 \in \mathcal{H}^m(A(0))$ such that

$$h_0(x) = u(x) - \mathcal{R}_{2m,L}\mu_u(x)$$

for $x \in A(0)$. $\square$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. By Theorem 3.4, there exists a function $h_0 \in \mathcal{H}^m(A(0))$ such that

$$u(x) = \mathcal{R}_{2m,L}\mu_u(x) + h_0(x)$$

for $x \in A(0)$. Since

$$\begin{aligned} ((-1)^m h_0(x))^+ &\leq ((-1)^m u(x))^+ + \left((-1)^{m+1}\mathcal{R}_{2m,L}\mu_u(x)\right)^+ \\ &\leq ((-1)^m u(x))^+ + |\mathcal{R}_{2m,L}\mu_u(x)|, \end{aligned}$$

we have

$$\liminf_{r\to 0}\ r^q M(((-1)^m h_0)^+, r) = 0$$

for $q > \alpha$, by Lemma 3.3. Hence, in case $\alpha > n-2$, we find a function $h \in \mathcal{H}^m(B(r_0))$ and constants $\{c_\lambda\}$ such that

$$\begin{aligned} h_0(x) &= \sum_{|\lambda|<q+2m-n} c_\lambda D^\lambda \mathcal{R}_{2m}(x) + h(x) \\ &= \sum_{|\lambda|\leq L} c_\lambda D^\lambda \mathcal{R}_{2m}(x) + h(x) \end{aligned}$$

for $x \in A(0)$ with the aid of [6, Theorem 1.2]; in case $\alpha = n-2$,

$$h_0(x) = \sum_{|\lambda|\leq 2m-2} c_\lambda D^\lambda \mathcal{R}_{2m}(x) + h(x).$$

Thus the theorem is obtained. $\square$

4 Proof of Theorem 1.1

For a proof of Theorem 1.1 we prepare the following lemma (see also [7, Proposition 4.1]).

LEMMA 4.1. *Let m be even. Suppose that u is a super-polyharmonic function on $B(1) \setminus \{0\}$ with $\mu_u = (-\Delta)^m u \geq 0$ and*

$$M((-1)^m u, r) \leq C\mathcal{R}_2(r) \qquad \text{whenever } 0 < r < r_0$$

for some constant $C > 0$. Then

$$\liminf_{x\to 0}\ \mathcal{R}_2(x)^{-1}(-1)^m \mathcal{R}_{2m,2m-3}\mu_u(x) \geq 0.$$

Proof. By Lemma 2.4 (2) we have

$$\int_{A(0)} |y|^{2m-2} d\mu_u(y) < \infty.$$

First we show the case $n \geq 3$. Then Lemma 3.1 gives

$$(-1)^m \mathcal{R}_{2m,2m-3}(x,y) \geq -C|y|^{2m-3}|x|^{2-n} \min\{|y|, |x|\},$$

since $(-1)^m \mathcal{R}_{2m}(x-y) \geq 0$ when m is even and $2m \leq n$. Hence, if m is even, then

$$(-1)^m \mathcal{R}_{2m,2m-3}\mu_u(x) \geq -C|x|^{2-n} \int_{A(0)} |y|^{2m-3} \min\{|y|, |x|\} d\mu_u(y),$$

which together with Lebesgue's dominated convergence theorem gives

$$\liminf_{x \to 0} \ |x|^{n-2}(-1)^m \mathcal{R}_{2m,2m-3}\mu_u(x) \geq 0.$$

Next we consider the case $n = 2$. The above discussions yield

$$(-1)^m \mathcal{R}_{2m,2m-3}\mu_u(x) \geq -C \int_{A(0)} |y|^{2m-2} \min\{\log(1/|y|), \log(1/|x|)\} d\mu_u(y),$$

so that

$$\liminf_{x \to 0} \ (\log(1/|x|))^{-1}(-1)^m \mathcal{R}_{2m,2m-3}\mu_u(x) \geq 0,$$

as required. □

Proof of Theorem 1.1. First we show that (1) implies (2). For this purpose, we suppose that m is even or $2m > n$, and take a super-polyharmonic function u on $B(1) \setminus \{0\}$ with $\mu_u = -\Delta^m u \geq 0$ satisfying (1-1) – (1-3). By Theorem 1.3, there exist a function $h \in \mathcal{H}^m(B(r_0))$ and constants c_λ such that

$$u(x) = \mathcal{R}_{2m,2m-3}\mu_u(x) + h(x) + \sum_{|\lambda| \leq 2m-2} c_\lambda D^\lambda \mathcal{R}_{2m}(x)$$

for $x \in A(0)$. Then, by Lemma 4.1, we have

$$\liminf_{x \to 0} \ \mathcal{R}_2(x)^{-1}(-1)^m u(x) \geq -C > -\infty,$$

which contradicts with (1-3).

The implication (2) ⇒ (1) is obtained by Lemma 2.6. □

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