Trudinger's exponential integrability for Riesz potentials of functions in generalized grand Morrey spaces

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Abstract

Our aim in this paper is to discuss Trudinger's exponential integrability for Riesz potentials of functions in generalized grand Morrey spaces. Our result will imply the boundedness of the Riesz potential operator from a grand Morrey space to a Morrey space.

1 Introduction

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. We denote by $B(x,r)$ the open ball centered at x of radius r and denote by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbf{R}^n$. In this paper, let G be a bounded open set in $\mathbf{R}^n$. We denote by d_G the diameter of G

In 1938, Morrey [8] considered the integral growth condition on derivatives over balls, in order to study the existence and regularity for partial differential equations. A family of functions with the integral growth condition is then called a Morrey space after his name. A systematical study for Morrey spaces was done by Peetre [10] in 1969, where the Morrey space $L^{p,\nu}(G)$ is a family of $f \in L^1_{loc}(G)$ satisfying the Morrey condition

$$\sup_{x\in G, 0<r<d_G} \frac{r^\nu}{|B(x,r)|} \int_{G\cap B(x,r)} |f(y)|^p dy < \infty$$

for $p \geq 1$ and $\nu > 0$ (see also Nakai [9]). Grand Lebesgue spaces were introduced in [2] for the sake of study of the integrability of the Jacobian (see also [3, 4, 11, 12]).

For $0 < \alpha < n$ and a locally integrable function f on G, we define the Riesz potential $U_\alpha f$ of order α by

$$U_\alpha f(x) = \int_G |x-y|^{\alpha-n} f(y)\, dy;$$

for fundamental properties of Riesz potentials, we refer the reader to the book by the first author [6].

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Meskhi [5] investigated the boundedness for several integral operators, including the Riesz potential operator, in the grand Morrey spaces $L^{p),\nu,\theta}(G)$ which consists of all functions $f \in L^1_{loc}(G)$ satisfying the grand Morrey condition

$$\sup_{x \in G, 0<r<d_G, 0<\varepsilon<p-1} \varepsilon^\theta \frac{r^\nu}{|B(x,r)|} \int_{G \cap B(x,r)} |f(y)|^{p-\varepsilon} \, dy < \infty$$

for $p > 1, \nu > 0$ and $\theta > 0$; in what follows, let $f = 0$ outside G. Our main aim in this paper is to establish Trudinger's exponential integrability for Riesz potentials of functions in generalized grand Morrey spaces which will be mentioned below.

In view of Fusco, Lions and Sbordone [1], we see that if f is a measurable function on G satisfying the grand Lebesgue condition

$$\lim_{\varepsilon \to 0+} \varepsilon^\theta \int_G |f(y)|^{n-\varepsilon} \, dy = 0,$$

then

$$\int_G \exp(|U_1 f(x)|^{n/(n-1+\theta)}) \, dx < \infty.$$

We also obtain Trudinger's exponential integrability for Riesz potentials of functions in grand Lebesgue spaces.

Throughout this paper, let C denote various constants independent of the variables in question, and $C(a, b, \cdots)$ a constant that depends on $a, b, \cdots$. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 0$.

2 Grand Morrey spaces

Let φ be a positive nondecreasing function on $(0, \infty)$ satisfying the following condition:

(φ1) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} r^n \le \varphi(r) \le A_1 \quad \text{for } 0 < r < 1;$$

(φ2) φ is doubling on $(0, \infty)$, namely there exists a constant $A_2 \ge 1$ such that

$$\varphi(2t) \le A_2 \varphi(t) \quad \text{for } t > 0.$$

For $\beta > 0$, set

$$\psi_\beta(r) = \int_{1/r}^{2d_G} t^\beta \varphi(t)^{-1/p} (\log(2d_G/t))^{\theta/p} \frac{dt}{t}$$

when $r \ge 1/d_G$; and set

$$\psi_\beta(r) = d_G \psi_\beta(1/d_G) r$$

when $0 < r < 1/d_G$.

Let us begin with the following result, which is easily proved by (φ2).

Lemma 2.1. *For $\beta > 0$, ψ_β is increasing and doubling on $[0, \infty)$.*

Now, for $p > 1$ and $\theta > 0$, we introduce the generalized grand Morrey space $L^{p),\varphi,\theta}(G)$ which consists of all measurable functions f on G such that

$$\begin{aligned}\|f\|_{L^{p),\varphi,\theta}(G)} &= \inf\left\{\lambda > 0 : \sup_{x\in G, 0<r<d_G, 0<\varepsilon<p-1} \varepsilon^{\theta}\frac{\varphi(r)}{|B(x,r)|}\right.\\ &\qquad\qquad\left.\times \int_{B(x,r)} |f(y)/\lambda|^{p-\varepsilon}\, dy \le 1\right\} < \infty;\end{aligned}$$

recall here that $f = 0$ outside G.

In case $\varphi(r) = r^{\nu}$, $L^{p),\varphi,\theta}(G)$ is denoted by $L^{p),\nu,\theta}(G)$ for simplicity; in particular, $L^{p),n,\theta}(G)$ is usually written as $L^{p),\theta}(G)$.

Our main aim in this paper is to establish the following exponential integrability for Riesz potentials of functions in $L^{p),\varphi,\theta}(G)$.

THEOREM 2.2. *For $0 < \beta < \alpha$ there exist constants $c_1, c_2 > 0$ such that*

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} \left\{\psi_\alpha^{-1}(c_1 U_\alpha f(x))\right\}^{\alpha-\beta}\, dx \le c_2\psi_\beta(1/r)$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ with $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$.

REMARK 2.3. In Theorem 2.2, set

$$\Phi_\alpha(r) = \left\{\psi_\alpha^{-1}(c_1 r)\right\}^{\alpha-\beta}$$

and

$$\varphi_\beta(r) = \psi_\beta(1/r)^{-1}.$$

Then the theorem insists that $U_\alpha f$ is an element of the Morrey space $L^{\Phi_\alpha,\varphi_\beta}(G)$ (which is defined in a natural way) when $f \in L^{p),\varphi,\theta}(G)$.

EXAMPLE 2.4. Let $\varphi(r) = r^{\alpha p}(\log(c_0 + r^{-1}))^{\tau_1}\{\log(\log(c_0 + r^{-1}))\}^{\tau_2}$, where τ_1, τ_2 are constants and $c_0 > 1$ is chosen so that φ is decreasing on $(0, \infty)$.

(1) If $\tau_1 < p + \theta$, then

$$\psi_\alpha(r) \sim (\log(c_0 + r))^{(p+\theta-\tau_1)/p}\{\log(\log(c_0 + r))\}^{-\tau_2/p}$$

and

$$(\psi_\alpha)^{-1}(r) \sim \exp\left(r^{p/(p+\theta-\tau_1)}(\log(c_0 + r))^{\tau_2/(p+\theta-\tau_1)}\right);$$

(2) if $\tau_1 = p + \theta$ and $\tau_2 < p$, then

$$\psi_\alpha(r) \sim \{\log(\log(c_0 + r))\}^{1-\tau_2/p}$$

and

$$(\psi_\alpha)^{-1}(r) \sim \exp(\exp(r^{p/(p-\tau_2)}));$$

(3) if $\tau_1 = p + \theta$ and $\tau_2 = p$, then

$$\psi_\alpha(r) \sim \log(\log(\log(c_0 + r)))$$

and

$$(\psi_\alpha)^{-1}(r) \sim \exp(\exp(\exp r));$$

(4) if $\tau_1 = p + \theta$ and $\tau_2 > p$, then $\psi_\alpha(\infty) < \infty$, so that

$$\psi_\alpha(r) \sim 1$$

for large $r > 0$.

Corollary 2.5. *Let $\varphi(r) = r^{\alpha p}(\log(c_0 + r^{-1}))^{\tau_1}\{\log(\log(c_0 + r^{-1}))\}^{\tau_2}$ as above. If $0 < \eta < \alpha$, then there exist constants $c_1, c_2 > 0$ (depending on η) such that*

(1) *in case $\tau_1 < p + \theta$,*

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} \exp\left[c_1 U_\alpha f(x)^{1/(1+(\theta-\tau_1)/p)} \times (\log(c_0 + U_\alpha f(x)))^{\tau_2/(p+\theta-\tau_1)}\right] dx \le c_2 r^{-\eta};$$

(2) *in case $\tau_1 = p + \theta$ and $\tau_2 < p$,*

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} \exp\left[\exp\left(c_1 U_\alpha f(x)^{1/(1-\tau_2/p)}\right)\right] dx \le c_2 r^{-\eta}$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ with $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$.

In fact, to prove (1), letting $0 < \alpha - \beta < \eta < \alpha$, we see from Theorem 2.2 and Example 2.4 (1) that

$$\begin{aligned} &\frac{1}{|B(z,r)|}\int_{B(z,r)} \exp\left[c_1(\alpha-\beta) U_\alpha f(x)^{1/(1+(\theta-\tau_1)/p)} \times (\log(c_0 + U_\alpha f(x)))^{\tau_2/(p+\theta-\tau_1)}\right] dx \\ \le\ & c_2 r^{-(\alpha-\beta)}(\log(c_0 + r))^{(\theta-\tau_1)/p}\{\log(\log(c_0 + r))\}^{-\tau_2/p} \end{aligned}$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ with $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$. Hence it suffices to note that

$$c_2 r^{-(\alpha-\beta)}(\log(c_0 + r))^{(\theta-\tau_1)/p}\{\log(\log(c_0 + r))\}^{-\tau_2/p} \le C(\eta) r^{-\eta}$$

when $0 < r < d_G$. Assetion (2) can be proved similarly.

For a proof of Theorem 2.2, we prepare some lemmas.

Lemma 2.6. *There exists a constant $C > 1$ such that*

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\, dy \le C\varphi(r)^{-1/p}(\log(2d_G/r))^{\theta/p} \tag{2.1}$$

for all $x \in G$, $0 < r < d_G$ and $f \ge 0$ with $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$.

Proof. Let f be a nonnegative measurable function on G such that $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$. Then note that

$$\varepsilon^\theta \frac{\varphi(r)}{|B(x,r)|}\int_{B(x,r)} f(y)^{p-\varepsilon}\, dy \le 1$$

for all $x \in G, 0 < r < d_G$ and $0 < \varepsilon < p-1$. We have by Jensen's inequality

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy \le \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p-\varepsilon}\,dy\right)^{1/(p-\varepsilon)}$$
$$\le \varepsilon^{-\theta/(p-\varepsilon)}\varphi(r)^{-1/(p-\varepsilon)}.$$

Here, taking $\varepsilon = \min\{(p-1)/2, (\log(2d_G/r))^{-1}\}$, we find by ($\varphi$1)

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy \le C\varphi(r)^{-1/p}(\log(2d_G/r))^{\theta/p},$$

since $r^{-1/(\log(2d_G/r))}$ is bounded above when $0 < r < d_G$. This proves the lemma. $\square$

LEMMA 2.7. *Let $0 < \beta < \alpha$. Then there exists a constant $C > 0$ such that*

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} U_\beta f(x)\,dx \le C\beta^{-1}\psi_\beta(1/r)$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying (2.1).

Proof. Let $z \in G, 0 < r < d_G$ and $0 < \beta < \alpha$. For $f \ge 0$ satisfying (2.1) , write

$$\begin{aligned} U_\beta f(x) &= \int_{B(z,2r)} |x-y|^{\beta-n} f(y)\,dy + \int_{G\setminus B(z,2r)} |x-y|^{\beta-n} f(y)\,dy \\ &= U_1(x) + U_2(x). \end{aligned}$$

By Fubini's theorem, (2.1) and (φ2), we have

$$\begin{aligned} \frac{1}{|B(z,r)|}\int_{B(z,r)} U_1(x)\,dx &= \frac{1}{|B(z,r)|}\int_{B(z,2r)}\left(\int_{B(z,r)} |x-y|^{\beta-n}\,dx\right) f(y)\,dy \\ &\le C\beta^{-1}r^\beta \frac{1}{|B(z,r)|}\int_{B(z,2r)} f(y)\,dy \\ &\le C\beta^{-1}r^\beta\varphi(2r)^{-1/p}(\log(2d_G/r))^{\theta/p} \\ &\le C\beta^{-1}\psi_\beta(1/r) \end{aligned}$$

since

$$\psi_\beta(1/r) \ge \int_r^{3r/2} t^\beta\varphi(t)^{-1/p}(\log(2d_G/t))^{\theta/p}\,\frac{dt}{t} \ge r^\beta\varphi(r)^{-1/p}(\log(2d_G/r))^{\theta/p}.$$

For U_2, note that

$$U_2(x) \le C\int_{G\setminus B(z,2r)} |z-y|^{\beta-n} f(y)\,dy$$

for $x \in B(z,r)$. Here we have only to consider the case $0 < r < d_G/2$ since $U_2(x) = 0$ for $r \ge d_G/2$. Hence we obtain

$$\begin{aligned} U_2(x) &\le C\int_{2r}^{2d_G} t^{\beta-n}\left(\int_{B(z,t)} f(y)\,dy\right)\frac{dt}{t} \\ &\le C\int_{2r}^{2d_G} t^\beta\varphi(t)^{-1/p}(\log(2d_G/t))^{\theta/p}\,\frac{dt}{t} \\ &\le C\psi_\beta(1/r) \end{aligned}$$

by Lemma 2.1, which proves the present lemma. $\square$

Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let f be a nonnegative measurable function on G satisfying $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$. Then we have by Lemma 2.6

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy \le C\varphi(r)^{-1/p}(\log(2d_G/r))^{\theta/p}$$

for all $x \in G$ and $0 < r < d_G$.

For $x \in G$ and $0 < \delta \le d_G$, write

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)\,dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)\,dy \\ &= U_1(x) + U_2(x). \end{aligned}$$

For $0 < \beta < \alpha$ we have

$$U_1(x) \le \delta^{\alpha-\beta}\int_{B(x,\delta)} |x-y|^{\beta-n} f(y)\,dy \le \delta^{\alpha-\beta} U_\beta f(x)$$

and

$$\begin{aligned} U_2(x) &\le C\int_\delta^{2d_G} t^{\alpha-N}\left(\int_{B(x,t)} f(y)\,dy\right)\frac{dt}{t} \\ &\le C\int_\delta^{2d_G} t^{\alpha}\varphi(t)^{-1/p}(\log(2d_G/t))^{\theta/p}\frac{dt}{t} \\ &\le C\psi_\alpha(\delta^{-1}) \end{aligned}$$

since

$$\int_{d_G}^{2d_G} t^{\alpha-N}\left(\int_{B(x,t)} f(y)\,dy\right)\frac{dt}{t} \le C \le C\int_{d_G}^{2d_G} t^{\alpha}\varphi(t)^{-1/p}(\log(2d_G/t))^{\theta/p}\frac{dt}{t}.$$

Hence it follows that

$$U_\alpha f(x) \le C\left\{\delta^{\alpha-\beta}U_\beta f(x) + \psi_\alpha(\delta^{-1})\right\}.$$

If $\{U_\beta f(x)\}^{-1/(\alpha-\beta)} \le d_G$, then we take $\delta = \{U_\beta f(x)\}^{-1/(\alpha-\beta)}$ to obtain

$$U_\alpha f(x) \le C\psi_\alpha(\{U_\beta f(x)\}^{1/(\alpha-\beta)});$$

if $\{U_\beta f(x)\}^{-1/(\alpha-\beta)} \ge d_G$, then we take $\delta = d_G$ to obtain

$$U_\alpha f(x) \le C.$$

Hence

$$U_\alpha f(x) \le C_1\psi_\alpha(1 + \{U_\beta f(x)\}^{1/(\alpha-\beta)}),$$

which together with Lemma 2.7 gives

$$\begin{aligned} \frac{1}{|B(z,r)|}\int_{B(z,r)} \left\{\psi_\alpha^{-1}(U_\alpha f(x)/C_1)\right\}^{\alpha-\beta} dx &\le C\frac{1}{|B(z,r)|}\int_{B(z,r)} \{1 + U_\beta f(x)\}\,dx \\ &\le C\left\{1 + \beta^{-1}\psi_\beta(1/r)\right\} \\ &\le C(\beta)\psi_\beta(1/r) \end{aligned}$$

for $z \in G$ and $0 < r < d_G$. The proof is now completed. □

Remark 2.8. If $f \in L^{p,\nu}(G)$ with $\nu = \alpha p < n$, then, in view of the proof of Theorem 2.2, we find constants $c_1, c_2 > 0$ such that

$$\frac{r^\eta}{|B(z,r)|} \int_{B(z,r)} \exp\left(c_1 \eta |U_\alpha f(x)|\right) dx \le c_2 \eta^{-1}$$

for all $z \in G$, $0 < r < d_G$ and $0 < \eta < \alpha$; for this, see also [7]. Here we can not add an exponent $q > 1$ such that

$$\frac{r^\eta}{|B(z,r)|} \int_{B(z,r)} \exp(c_1 \eta |U_\alpha f(x)|^q)\, dx \le c_2 \eta^{-1}.$$

But, in case $\nu = \alpha p = n$, this is not the case, as will be seen in Theorem 3.1.

For this, consider the function $f(y) = |y|^{-\alpha} \chi_{\mathbf{B}}(y)$, where $\mathbf{B} = B(0,1)$. If $n - \alpha p > 0$, then

$$\frac{r^\nu}{|B(z,r)|} \int_{B(x,r)} f(y)^p\, dy \le \frac{r^\nu}{|B(z,r)|} \int_{B(x,r)} |x-y|^{-\alpha p}\, dy \le C$$

for all $x \in \mathbf{B}$ and $0 < r < 2$, so that $f \in L^{p,\nu}(\mathbf{B})$.

On the other hand, we see that

$$\begin{aligned} U_\alpha f(x) &\ge \int_{\mathbf{B} \setminus B(0,2|x|)} |x-y|^{\alpha-n} f(y)\, dy \\ &\ge 2^{\alpha-n} \int_{\mathbf{B} \setminus B(0,2|x|)} |y|^{-n}\, dy \\ &\ge C \log(1/|x|) \end{aligned}$$

for $x \in B(0,1/3)$. Hence

$$\int_{B(0,r)} \exp\left(c\{U_\alpha f(x)\}^q\right) dx = \infty$$

for all $r > 0$, $c > 0$ and $q > 1$.

Remark 2.9. If $(\alpha - 1)p < \nu < \alpha p$, then there exists a constant $C > 0$ such that

$$|U_\alpha f(x) - U_\alpha f(z)| \le C|x-z|^{\alpha - \nu/p} (\log(e + 1/|x-z|))^{\theta/p}$$

for all $x, z \in G$ and $f \ge 0$ satisfying $\|f\|_{L^{p),\nu,\theta}(G)} \le 1$ (for instance, see [7]).

3 Grand Lebesgue spaces

In view of Fusco, Lions and Sbordone [1], we see that if

$$\lim_{\varepsilon \to 0+} \varepsilon^\theta \int_G |f(y)|^{n-\varepsilon}\, dy = 0,$$

then

$$\int_G \exp\left(|U_1 f(x)|^{n/(n-1+\theta)}\right) dx < \infty.$$

In connection with their result, we can prove the following result.

THEOREM 3.1. *Let $\alpha p = n$. Then for $0 < \eta < \alpha$ there exist constants $c_1, c_2 > 0$ such that*

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} \exp\left(c_1\{U_\alpha f(x)\}^{1/(1+(\theta-1)/p)}\right) dx \le c_2 r^{-\eta}$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{p),\theta}(G)} \le 1$.

REMARK 3.2. In Corollary 2.5, letting $0 < \eta < \alpha$ and $\varphi(r) = r^n(\log(c_0 + r^{-1}))^\tau$ with $\tau < p + \theta$, we can find constants $c_1, c_2 > 0$ such that

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} \exp\left(c_1\{U_\alpha f(x)\}^{1/(1+(\theta-\tau)/p)}\right) dx \le c_2 r^{-\eta}$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$. When $\tau < 1$, Theorem 3.1 gives a result better than this.

For a proof of Theorem 3.1, we have only to give the next result.

PROPOSITION 3.3. *Let $\alpha p = n$. Then for $0 < \beta < \alpha$ there exist constants $c_1, c_2 > 0$ such that*

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} \exp\left(c_1\{U_\alpha f(x)\}^{1/(1+(\theta-1)/p)}\right)\, dx \le c_2 \psi_\beta(1/r)$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$.

To prove Proposition 3.3, we prepare the next lemma.

LEMMA 3.4. *Let $\alpha p = n$. Then there exists a constant $C > 0$ such that*

$$\int_{G\setminus B(x,r)} |x-y|^{\alpha-n} f(y)\, dy \le C(\log(2d_G/r))^{1-(1-\theta)/p}$$

for all $x \in G$, $0 < r \le d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$.

Proof. Let $p = n/\alpha$ and f be a nonnegative measurable function on G satisfying $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$. Then note that

$$\int_G f(y)^{p-\varepsilon}\, dy \le \varepsilon^{-\theta}$$

for all $0 < \varepsilon < p - 1$. For $x \in G$, $0 < r \le d_G$ and $0 < \varepsilon < p - 1$, we have by

Hölder's inequality

$$\begin{aligned}
& \int_{G\setminus B(x,r)} |x-y|^{\alpha-n} f(y)\,dy \\
\leq\ & \left(\int_{G\setminus B(x,r)} |x-y|^{(\alpha-n)(p-\varepsilon)'}\,dy \right)^{1/(p-\varepsilon)'} \left(\int_{G\setminus B(x,r)} f(y)^{p-\varepsilon}\,dy \right)^{1/(p-\varepsilon)} \\
\leq\ & C \left(\int_r^{d_G} t^{(\alpha-n)(p-\varepsilon)'+n-1}\,dt \right)^{1/(p-\varepsilon)'} \varepsilon^{-\theta/(p-\varepsilon)} \\
\leq\ & C \left(\int_r^{d_G} t^{-\alpha\varepsilon/(p-\varepsilon-1)-1}\,dt \right)^{1/(p-\varepsilon)'} \varepsilon^{-\theta/(p-\varepsilon)} \\
\leq\ & C \left(\frac{r^{-\alpha\varepsilon/(p-\varepsilon-1)}}{\alpha\varepsilon/(p-\varepsilon-1)} \right)^{1/(p-\varepsilon)'} \varepsilon^{-\theta/(p-\varepsilon)} \\
\leq\ & C r^{-\alpha\varepsilon/(p-\varepsilon)} \varepsilon^{-1/(p-\varepsilon)'-\theta/(p-\varepsilon)} \\
\leq\ & C r^{-\alpha\varepsilon/(p-\varepsilon)} \varepsilon^{-1/p'-\theta/p}.
\end{aligned}$$

Now, taking $\varepsilon = \min\{(p-1)/2, (\log(2d_G/r))^{-1}\}$, we find

$$\int_{G\setminus B(x,r)} |x-y|^{\alpha-n} f(y)\,dy \leq C(\log(2d_G/r))^{1/p'+\theta/p},$$

which gives the result. □

Proof of Proposition 3.3. Let f be a nonnegative measurable function on G satisfying $\|f\|_{L^{p),\varphi,\theta}(G)} \leq 1$. Then for $0<\beta<\alpha$ we have by Lemma 3.4

$$\begin{aligned}
U_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)\,dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)\,dy \\
&\leq \delta^{\alpha-\beta} U_\beta f(x) + C(\log(2d_G/r))^{1-(1-\theta)/p}.
\end{aligned}$$

Here, as in the proof of Theorem 2.2, we have the inequality

$$U_\alpha f(x) \leq C_1(\log(e+U_\beta f(x)))^{1-(1-\theta)/p}.$$

Hence we find

$$\begin{aligned}
& \frac{1}{|B(z,r)|} \int_{B(z,r)} \exp\left(\{U_\alpha f(x)/C_1\}^{1/(1-(1-\theta)/p)} \right) dx \\
\leq\ & C \frac{1}{|B(z,r)|} \int_{B(z,r)} \{1+U_\beta f(x)\}\,dx \\
\leq\ & C\psi_\beta(1/r)
\end{aligned}$$

for all $z \in G$ and $0<r<d_G$, in view of Lemma 2.7 . Now we obtain the present result. □

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