Capacity for potentials of functions in Musielak-Orlicz spaces

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Abstract

We define a capacity for potentials of functions in Musielak-Orlicz spaces. Basic properties of such capacity are studied. We also estimate the capacity of balls and give some applications of the estimates.

1 Introduction

The notion of classical Newton capacity has been generalized to various forms. Among others, Meyers [11] introduced a general notion of L^p-capacity, which is defined by general potentials of functions in the Lebesgue space L^p and such notion of capacity has been proved to provide rich results in the nonlinear potential theory as well as in the study of various function spaces and partial differential equations; see e.g., [1]. Most useful L^p-capacities are Riesz capacity and Bessel capacity, and we can estimate the capacities of balls $B(x,r)$ for these special cases, which are used to obtain relations between these capacities and Hausdorff measures (cf. the references cited above as well as [18] and [12]).

In [3], the notion of L^p-capacity was generalized by replacing L^p by Orlicz space. Recently, there appeared several papers dealing with capacities for special type of Orlicz spaces and estimates of the capacity of balls: [2], [9], [4], [14].

In the mean time, variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth condition, and Sobolev capacity for variable exponent Sobolev space has been studied in connection with the related nonlinear potential theory: [7] and [6]. The Riesz capacity for the variable exponent Lebesgue space $L^{p(\cdot)}$ was considered in [5], and then that for the space $L^{p(\cdot)}(\log L)^{q(\cdot)}$ in [13].

The spaces $L^{p(\cdot)}$ and $L^{p(\cdot)}(\log L)^{q(\cdot)}$ are special cases of the Musielak-Orlicz spaces (or, generalized Orlicz spaces); see, [8], [15]. The purpose of the present paper is to extend the notion of capacity to that defined by general potentials of functions in fairy general Musielak-Orlicz spaces and to show that the capacity

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thus defined still satisfies fundamental properties shared by those capacities stated above. We also give estimates of balls for our capacity and apply the estimates to obtain local behavior of functions in the space and to relate the capacity with a generalized harmonic measure.

2 Preliminaries

We consider a function $\Phi(x,t) : \mathbf{R}^N \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Φ1) $\Phi(\cdot,t)$ is measurable on $\mathbf{R}^N$ for each $t \ge 0$;

(Φ2) $\Phi(x,0) = 0$ and $\Phi(x,t) > 0$ for $t > 0$;

(Φ3) $\Phi(x,\cdot)$ is convex for each $x \in \mathbf{R}^N$;

(Φ4) $\Phi(x,1)$ and $1/\Phi(x,1)$ are bounded;

(Φ5) There exists a constant $A_1 \ge 1$ such that

$$\Phi(x,2t) \le A_1 \Phi(x,t) \quad \text{for all } x \in \mathbf{R}^N \text{ and } t > 0.$$

Remark 2.1. By (Φ2) and (Φ3), we see that $\Phi(x,t)/t$ is nondecreasing in t and $\Phi(x,t)$ is strictly increasing in t for each $x \in \mathbf{R}^N$. In fact,

$$\Phi(x,s) = \Phi\left(x, \frac{s}{t}t\right) \le \frac{s}{t}\Phi(x,t) < \Phi(x,t)$$

for $0 < s < t$.

For an open set $G \subset \mathbf{R}^N$, the Musielak-Orlicz space

$$L^\Phi(G) = \left\{ f \in L^1_{loc}(G); \int_G \Phi(y, |f(y)|/\lambda)dy < \infty \text{ for some } \lambda > 0 \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(G)} = \inf\left\{ \lambda > 0; \int_G \Phi(y, |f(y)|/\lambda)dy \le 1 \right\}$$

(cf. [15]).

Example 2.2. Let $p_j(x)$ $(j = 1,\dots,m)$ and $q_j(x)$ $(j = 1,\dots,m)$ be measurable functions on $\mathbf{R}^N$ such that

(P) $1 \le p_j^- := \operatorname{ess\,inf}_{x\in\mathbf{R}^N} p_j(x) \le \operatorname{ess\,sup}_{x\in\mathbf{R}^N} p_j(x) := p_j^+ < \infty$ for all $j = 1,\dots,m$;

(Q) $-\infty < q_j^- := \operatorname{ess\,inf}_{x\in\mathbf{R}^N} q_j(x) \le \operatorname{ess\,sup}_{x\in\mathbf{R}^N} q_j(x) := q_j^+ < \infty$ for all $j = 1, \ldots, m$.

Let $a_j \ge e$ satisfy $(1+\log a_j)(p_j(x)-1)+q_j(x) \ge 0$ for all $x \in \mathbf{R}^N$ and $j = 1, \ldots, m$. For positive numbers b_j, set

$$\Phi_{\{p_j(\cdot)\},\{q_j(\cdot)\},\{a_j\},\{b_j\}}(x,t) = \sum_{j=1}^{m} b_j t^{p_j(x)}(\log(a_j+t))^{q_j(x)}.$$

Then this function satisfies (Φi) for $i = 1,2,3,4,5$ (cf. [10]).

As a kernel function on $\mathbf{R}^N$, we consider $k(x) = k(|x|)$ (with the abuse of notation) with a positive nonincreasing lower semicontinuous function $k(r)$ on $(0,\infty)$ such that

(k1) $\int_0^1 k(r)r^{N-1}dr < \infty$.

By (k1), $k(\cdot) \in L^1_{loc}(\mathbf{R}^N)$. Further we see that $\lim_{r\to 0+} r^N k(r) = 0$. We set $k(0) = \lim_{r\to 0+} k(r)$. We also consider the condition:

(k2) There exists a constant $A_2 \ge 1$ such that $k(r) \le A_2 k(r+1)$ for all $r \ge 1$.

For $0 < \alpha < N$, the Riesz kernel $I_\alpha(x) = 1/|x|^{N-\alpha}$ and the Bessel kernel g_α of order α are typical examples of $k(x)$ satisfying (k1) and (k2).

We define the k-potential for a locally integrable function f on $\mathbf{R}^N$ by

$$k * f(x) = \int_{\mathbf{R}^N} k(x-y)f(y)dy.$$

Here it is natural to assume that

$$\int_{\mathbf{R}^N} k(1+|y|)|f(y)|dy < \infty, \tag{2.1}$$

which is equivalent to the condition that $k * |f| \not\equiv \infty$ by the conditions (k1) and (k2) (see [12, Theorem 1.1, Chapter 2]). Note that $k * f \in L^1_{loc}(\mathbf{R}^N)$ under this assumption.

3 (k, Φ)-capacity

Throughout this paper, let A denote various positive constants independent of the variables in question and let $A(a, b, \cdots)$ be a constant which may depend on $a, b, \ldots$. For a measurable subset E of $\mathbf{R}^N$, we denote by $|E|$ the Lebesgue measure of E and by χ_E the characteristic function of E.

We introduce a notion of capacity as an extension of Meyers [11] and Mizuta [12]. For a set $E \subset \mathbf{R}^N$ and an open set $G \subset \mathbf{R}^N$, we define the (k,Φ)-capacity of E relative to G by

$$C_{k,\Phi}(E;G) = \inf_{f\in S_k(E;G)} \int_G \Phi\left(y, f(y)\right) dy,$$

where $S_k(E;G)$ is the family of all nonnegative measurable functions f on $\mathbf{R}^N$ such that f vanishes outside G and $k * f(x) \ge 1$ for every $x \in E$ (cf. Futamura-Mizuta-Shimomura [5]). Here, note that $E \subset G$ is not required.

PROPOSITION 3.1 (cf. [12, Theorem 1.1, Chapter 5]). *$C_{k,\Phi}(\,\cdot\,;G)$ is a countably subadditive, nondecreasing and outer capacity.*

Proof. Clearly,

$$C_{k,\Phi}(E_1;G) \le C_{k,\Phi}(E_2;G) \quad \text{whenever } E_1 \subset E_2.$$

Let $f \in S_k(E;G)$. Since $k * f$ is lower semicontinuous, $\omega(a) = \{x \in \mathbf{R}^N : k * f(x) > a\}$ is an open set for every $a > 0$. Note that

$$C_{k,\Phi}(E;G) \le C_{k,\Phi}(\omega(a);G) \le \int_G \Phi\left(y, \frac{f(y)}{a}\right) dy$$

whenever $0 < a < 1$. Since the condition ($\Phi5$) implies

$$\Phi\left(y, \frac{f(y)}{a}\right) \le A_1 \Phi\left(y, f(y)\right)$$

for $1/2 < a < 1$, Lebesgue's dominated convergence theorem implies that

$$C_{k,\Phi}(E;G) \le \inf_{\omega \supset E,\, \omega:\text{open}} C_{k,\Phi}(\omega;G) \le \int_G \Phi\left(y, f(y)\right) dy,$$

which gives

$$C_{k,\Phi}(E;G) = \inf_{\omega \supset E,\, \omega:\text{open}} C_{k,\Phi}(\omega;G).$$

Finally, let $\{E_j\}$ be a countable family of sets in $\mathbf{R}^N$ and set $E = \bigcup_j E_j$. We may assume that $C_{k,\Phi}(E;G) < \infty$. For $\varepsilon > 0$, take $f_j \in S_k(E_j;G)$ such that

$$\int_G \Phi\left(y, f_j(y)\right) dy \le C_{k,\Phi}(E_j;G) + \varepsilon 2^{-j}.$$

Consider the function $f(y) = \sup_j f_j(y)$. Then it is easy to see that $k * f(x) \ge k * f_j(x) \ge 1$ for all $x \in E_j$. Hence

$$\begin{aligned} C_{k,\Phi}(E;G) &\le \int_G \Phi\left(y, f(y)\right) dy \\ &\le \sum_{j=1}^{\infty} \int_G \Phi\left(y, f_j(y)\right) dy \le \sum_{j=1}^{\infty} C_{k,\Phi}(E_j;G) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, we have

$$C_{k,\Phi}(E;G) \le \sum_{j=1}^{\infty} C_{k,\Phi}(E_j;G).$$

□

Proposition 3.2 (cf. [12, Proposition 1.1, Chapter 5]). *$C_{k,\Phi}(E;G)=0$ if and only if there exists a nonnegative function $f \in L^{\Phi}(\mathbf{R}^N)$ such that $f(x)=0$ for $x \in \mathbf{R}^N \setminus G$ and*

$$k * f(x) = \infty \quad \text{whenever } x \in E.$$

Proof. Suppose $C_{k,\Phi}(E;G)=0$. Then, for each positive integer j, there exists a function $f_j \in S_k(E;G)$ with

$$\int_G \Phi(y, f_j(y))dy < A_1^{-j}$$

for A_1 in (Φ5). Then $\|f_j\|_{L^{\Phi}(\mathbf{R}^N)} \le 2^{-j}$, so that setting $f=\sum_j f_j$, we have $f \in L^{\Phi}(\mathbf{R}^N)$. Since

$$k * f(x) = \sum_{j=1}^{\infty} k * f_j(x) = \infty$$

for $x \in E$, we obtain the required result.

Conversely, if there exists a nonnegative function $f \in L^{\Phi}(\mathbf{R}^N)$ such that $f(x)=0$ for $x \in \mathbf{R}^N \setminus G$ and $k*f(x)=\infty$ whenever $x \in E$, then we see that $f/a \in S_k(E;G)$ for all $a>1$. Hence we have by (Φ2) and (Φ3)

$$C_{k,\Phi}(E;G) \le \frac{1}{a}\int_G \Phi(y, f(y))dy \to 0$$

as $a \to \infty$, as required. □

We say that E is of (k,Φ)-capacity zero, written as $C_{k,\Phi}(E)=0$, if

$$C_{k,\Phi}(E \cap G; G) = 0 \qquad \text{for every bounded open set } G.$$

Proposition 3.3 (cf. [12, Theorem 1.2, Chapter 5], [5, Lemma 4.1]). *Suppose k satisfies* (k2). *For $E \subset \mathbf{R}^N$, $C_{k,\Phi}(E)=0$ if and only if there exists a nonnegative function $f \in L^{\Phi}(\mathbf{R}^N)$ such that $k*f \not\equiv \infty$ and*

$$k * f(x) = \infty \quad \text{whenever } x \in E.$$

Proof. First, suppose there exists a nonnegative function $f \in L^{\Phi}(\mathbf{R}^N)$ such that $k*f \not\equiv \infty$ and $k*f(x)=\infty$ whenever $x \in E$. Let G be a bounded open set. If $x \in G$, then by (k2) there exists $A(x)>0$ such that $k(x-y) \le A(x)k(1+|y|)$ for all $y \in \mathbf{R}^N \setminus G$. Hence $k*f \not\equiv \infty$ implies

$$\int_{\mathbf{R}^N \setminus G} k(x-y)f(y)dy < \infty$$

for all $x \in G$. Hence we have

$$\int_G k(x-y)f(y)dy = \infty \quad \text{whenever } x \in E \cap G,$$

which implies from Proposition 3.2 that $C_{k,\Phi}(E \cap G; G) = 0$. Thus $C_{k,\Phi}(E) = 0$ follows.

Conversely, suppose $C_{k,\Phi}(E) = 0$. Then, since $C_{k,\Phi}(E \cap B(0,j); B(0,j)) = 0$ for any $j \geq 1$, we can find a function $f_j \in S_k(E \cap B(0,j); B(0,j))$ such that $\int_{B(0,j)} \Phi(y, f_j(y))dy \leq A_1^{-j}$. Then, $\|f_j\|_{L^\Phi(B(0,j))} \leq 2^{-j}$ and from Remark 2.1, (Φ4) and (Φ5) we see that

$$
\begin{aligned}
& \int_{B(0,j)} k(1+|y|) f_j(y)dy \\
\leq\ & k(1) \int_{B(0,j)} f_j(y)dy \\
=\ & k(1) \left\{ \int_{B(0,j)\cap\{y: f_j(y) \leq 2^{-j}\}} f_j(y)dy + \int_{B(0,j)\cap\{y: f_j(y) > 2^{-j}\}} f_j(y)dy \right\} \\
\leq\ & k(1) \left\{ \sigma_N j^N 2^{-j} + \int_{B(0,j)} f_j(y) \frac{\Phi(y, f_j(y))}{f_j(y)} \frac{2^{-j}}{\Phi(y, 2^{-j})} dy \right\} \\
\leq\ & k(1) \left\{ \sigma_N j^N 2^{-j} + \frac{2^{-j} A_1^j}{\inf_{y \in \mathbf{R}^N} \Phi(y,1)} \int_{B(0,j)} \Phi(y, f_j(y))dy \right\} \\
\leq\ & A 2^{-j} j^N,
\end{aligned}
$$

where σ_N denotes the Lebesgue measure of the unit ball $B(0,1)$. Hence setting $f = \sum_{j=1}^\infty f_j$, we have $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq 1$ and

$$
\int_{\mathbf{R}^N} k(1+|y|) f(y)dy \leq \sum_{j=1}^\infty \int_{B(0,j)} k(1+|y|) f_j(y)dy \leq A \sum_{j=1}^\infty 2^{-j} j^N < \infty.
$$

Thus we have $f \in L^\Phi(\mathbf{R}^N)$ such that $k * f \not\equiv \infty$ and $k * f(x) = \infty$ whenever $x \in E$. □

Corollary 3.4 (cf. [12, Corollary 1.2, Chapter 5]). *Suppose k satisfies* (k2). *If $C_{k,\Phi}(E; G) = 0$ for some bounded open set G, then $C_{k,\Phi}(E) = 0$.*

Lemma 3.5 (cf. [12, Theorem 1.3, Chapter 5]). *Suppose G is an open set and k satisfies* (k2) . *Let $\{f_j\}$ be a sequence in $L^\Phi(G)$ which converges to f in $L^\Phi(G)$. Then there exist a subsequence $\{f_{j_n}\}$ and a set $F \subset G$ such that*

$$
\lim_{n\to\infty} k * f_{j_n}(x) = k * f(x) \quad \text{for } x \in G \setminus F,
$$

and $C_{k,\Phi}(F; G) = 0$.

Proof. We may assume that $C_{k,\Phi}(G; G) > 0$, so that every $f \in L^\Phi(G)$ satisfies (2.1). For each positive integers i and j, consider the set

$$
E_{i,j} = \{x \in G : |k * f_j(x) - k * f(x)| > 2^{-i}\}.
$$

Then we have by (Φ5)

$$\begin{aligned} C_{k,\Phi}(E_{i,j};G) &\le \int_G \Phi\left(y, 2^i|f_j(y)-f(y)|\right) dy \\ &\le A_1^i \int_G \Phi\left(y, |f_j(y)-f(y)|\right) dy. \end{aligned}$$

Since $\{f_j\}$ converges to f in $L^\Phi(G)$, we can find a subsequence $\{f_{j_n}\}$ such that

$$\int_G \Phi(y, |f_{j_n}(y)-f(y)|)dy \le (A_1 2)^{-n}.$$

Set $F = \bigcap_{\ell=1}^\infty \bigcup_{n=\ell}^\infty E_{n,j_n}$. Then it follows from Proposition 3.1 that

$$C_{k,\Phi}(F;G) \le \sum_{n=\ell}^\infty C_{k,\Phi}(E_{n,j_n};G) \le \sum_{n=\ell}^\infty 2^{-n} \to 0,$$

as $\ell \to \infty$, from which $C_{k,\Phi}(F;G) = 0$ follows. Since

$$\lim_{n\to\infty} |k * f_{j_n}(x) - k * f(x)| = 0$$

for $x \in G \setminus F$, we obtain the required result. □

We say that a property holds (k,Φ)-q.e. on a set E if it holds for all $x \in E$ except those in a set F with $C_{k,\Phi}(F) = 0$. By Corollary 3.4 and Lemma 3.5, we have the following result.

COROLLARY 3.6. *Suppose G is a bounded open set and k satisfies* (k2) . *Let $\{f_j\}$ be a sequence in $L^\Phi(G)$ which converges to f in $L^\Phi(G)$. Then there exists a subsequence $\{f_{j_n}\}$ such that*

$$\lim_{n\to\infty} k * f_{j_n}(x) = k * f(x) \quad \text{for } (k,\Phi)\text{-q.e. } x \in G.$$

PROPOSITION 3.7 (cf. [12, Theorem 1.4, Chapter 5]). *Suppose k satisfies* (k2). *Suppose further that $L^\Phi(\mathbf{R}^N)$ is reflexive. If $E_j \subset E_{j+1}$ and $E = \bigcup_{j=1}^\infty E_j$, then*

$$\lim_{j\to\infty} C_{k,\Phi}(E_j;G) = C_{k,\Phi}(E;G).$$

Proof. We may assume that there exists a constant $M > 1$ such that

$$C_{k,\Phi}(E_j;G) \le M.$$

For each positive integer j, we take $f_j \in S_k(E_j;G)$ such that

$$\int_G \Phi(y, f_j(y))dy \le C_{k,\Phi}(E_j;G) + 2^{-j}.$$

Then $\|f_j\|_{L^\Phi(\mathbf{R}^N)} \le M+1$. By the reflexivity of $L^\Phi(\mathbf{R}^N)$, there exists a subsequence $\{f_{j_k}\}$ which converges weakly to f in $L^\Phi(\mathbf{R}^N)$. Then Mazur's lemma implies that there exists a sequence $\{\lambda_{\ell,k}\}$ of nonnegative numbers such that $\sum_{k\ge\ell}\lambda_{\ell,k} = 1$ and

$$g_\ell = \sum_{k\ge\ell} \lambda_{\ell,k} f_{j_k}$$

converges to f in $L^\Phi(\mathbf{R}^N)$. Then we see that $g_\ell(x) = 0$ for $x \in \mathbf{R}^N \setminus G$ and

$$k * g_\ell(x) \ge 1 \quad \text{for } x \in E_{j_\ell}.$$

We apply Lemma 3.5 to obtain a subsequence $\{g_{\ell_k}\}$ and a set $F \subset G$ such that $k * g_{\ell_k}(x)$ converges to $k * f(x)$ for every $x \in E \setminus F$ and $C_{k,\Phi}(F; G) = 0$. Thus

$$k * f(x) \ge 1 \quad \text{whenever } x \in E \setminus F.$$

Hence, Fatou's lemma and (Φ3) yield

$$\begin{aligned}
C_{k,\Phi}(E;G) &\le C_{k,\Phi}(F;G) + C_{k,\Phi}(E\setminus F;G) \\
&\le \int_G \Phi(y, f(y))dy \\
&\le \liminf_{\ell\to\infty} \int_G \Phi(y, g_\ell(y))dy \\
&\le \lim_{\ell\to\infty}\left(\lim_{j\to\infty} C_{k,\Phi}(E_j;G) + 2^{-\ell}\right) \\
&\le \lim_{j\to\infty} C_{k,\Phi}(E_j;G).
\end{aligned}$$

Thus the required equality holds. □

REMARK 3.8. For reflexivity of $L^\Phi(\mathbf{R}^N)$, see Appendix. In particular, for $\Phi_{p(\cdot),q(\cdot),a}(x,t) = t^{p(x)}(\log(a+t))^{q(x)}$, $L^{\Phi_{p(\cdot),q(\cdot),a}}(\mathbf{R}^N)$ $(= L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^N)$; cf [10]) is reflexive if $1 < p^- \le p^+ < \infty$, $-\infty < q^- \le q^+ < \infty$ and $(1+\log a)(p(x)-1) + q(x) \ge 0$ for all $x \in \mathbf{R}^N$ (see Corollary 5.4; also cf. Ohno [16]).

We say that a function u is (k,Φ)-quasicontinuous on $\mathbf{R}^n$ if, for any $\varepsilon > 0$ and $R > 0$, there exists an open set ω such that $C_{k,\Phi}(\omega; B(0,R)) < \varepsilon$ and u is continuous on $B(0,R) \setminus \omega$.

PROPOSITION 3.9 (cf. [12, Theorem 7.1, Chapter 6]). *Suppose k is continuous on $\mathbf{R}^N \setminus \{0\}$ and satisfies* (k2). *If $f \in L^\Phi(\mathbf{R}^N)$ satisfies* (2.1), *then $k * f$ is (k,Φ)-quasicontinuous on $\mathbf{R}^N$.*

Proof. For $R > 0$, write

$$\begin{aligned}
k * f(x) &= \int_{B(0,R)} k(x-y)f(y)dy + \int_{\mathbf{R}^N\setminus B(0,R)} k(x-y)f(y)dy \\
&= u_1(x) + u_2(x).
\end{aligned}$$

For $L > 0$, define

$$u_{1,L}(x) = \int_{B(0,R)} k(x-y) f_L(y) dy,$$

where $f_L = \max\{\min\{f, L\}, -L\}$. Then note that u_2 and $u_{1,L}$ are continuous on $B(0,R)$. Consider the open sets

$$E_{j,L} = \left\{ x \in \mathbf{R}^N : \int_{B(0,R)} k(x-y)|f(y) - f_L(y)| dy > 2^{-j} \right\}.$$

For each j, there exists $L_j > 0$ such that

$$\int_{\mathbf{R}^N} \Phi(y, |f(y) - f_{L_j}(y)|) dy < (A_1 2)^{-j}.$$

Since $2^j |f - f_{L_j}| \chi_{B(0,R)} \in S_k(E_{j,L_j}; B(0,R))$, it follows from ($\Phi 5$) that

$$C_{k,\Phi}(E_{j,L_j}; B(0,R)) \le A_1^j \int_{B(0,R)} \Phi(y, |f(y) - f_{L_j}(y)|) dy \le 2^{-j}.$$

Now, letting

$$\omega_\ell = \bigcup_{j=\ell}^{\infty} E_{j,L_j},$$

we see that

$$\begin{aligned} C_{k,\Phi}(\omega_\ell; B(0,R)) &\le \sum_{j=\ell}^{\infty} C_{k,\Phi}(E_{j,L_j}; B(0,R)) \\ &\le \sum_{j=\ell}^{\infty} 2^{-j} \to 0 \end{aligned}$$

as $\ell \to \infty$ and u_{1,L_j} converges to u_1 uniformly on $B(0,R) \setminus \omega_\ell$. Thus $k * f$ is continuous on $B(0,R) \setminus \omega_\ell$ and the present proposition is obtained. $\square$

4 Estimates of (k, Φ)-capacity of balls and a generalized Hausdorff measure

For $r > 0$ and $x \in \mathbf{R}^N$, define

$$h_{k,\Phi}(r; x) = r^N \sup_{y \in B(x,r)} \Phi(y, r^{-N} \bar{k}(r)^{-1})$$

and

$$\tilde{h}_{k,\Phi}(r; x) = r^N \inf_{y \in B(x,r)} \Phi(y, r^{-N} \bar{k}(r)^{-1}),$$

where

$$\bar{k}(r) = \frac{1}{|B(0,r)|}\int_{B(0,r)} k(x)dx = \frac{N}{r^N}\int_0^r k(\rho)\rho^{N-1}d\rho.$$

Note that $\bar{k}(r)$ is a nonincreasing positive continuous function satisfying doubling condition, $\bar{k}(r) \ge k(r)$ and $\lim_{r\to 0+} r^N \bar{k}(r) = 0$.

Example 4.1. (1) If $k(r) = r^{\alpha-N}$ with $0 < \alpha < N$ and $\Phi(x,t) = t^{p(x)}$ with $1 \le p^- \le p^+ < \infty$, then

$$h_{k,\Phi}(r;x) \approx r^{N-\alpha p^+_{B(x,r)}}, \quad p^+_{B(x,r)} = \sup_{y\in B(x,r)} p(y)$$

and

$$\tilde{h}_{k,\Phi}(r;x) \approx r^{N-\alpha p^-_{B(x,r)}}, \quad p^-_{B(x,r)} = \inf_{y\in B(x,r)} p(y)$$

for $0 < r \le 1$. (Here $h_1(r) \approx h_2(r)$ means that $A^{-1}h_2(r) \le h_1(r) \le Ah_2(r)$ for a constant $A > 0$.)

If $p(\cdot)$ is log-Hölder continuous, then

$$h_{k,\Phi}(r;x) \approx \tilde{h}_{k,\Phi}(r;x) \approx r^{N-\alpha p(x)}$$

for $0 < r \le 1$.

(2) If $k(r)$ is as above and $\Phi(x,t) = t^{p(x)}(\log(a+t))^{q(x)}$ with $p(\cdot)$ as above, $-\infty < q^- \le q^+ < \infty$, $a \ge e$ and $(1+\log a)(p(x)-1) + q(x) \ge 0$ for all $x \in \mathbf{R}^N$, then

$$h_{k,\Phi}(r;x) \le Ar^{N-\alpha p^+_{B(x,r)}}(\log(e+1/r))^{q^+_{B(x,r)}}$$

and

$$\tilde{h}_{k,\Phi}(r;x) \ge A'r^{N-\alpha p^-_{B(x,r)}}(\log(e+1/r))^{q^-_{B(x,r)}}$$

for $0 < r \le 1$. If $p(\cdot)$ is log-Hölder continuous and $q(\cdot)$ is log-log-Hölder continuous, then

$$h_{k,\Phi}(r;x) \approx \tilde{h}_{k,\Phi}(r;x) \approx r^{N-\alpha p(x)}(\log(e+1/r))^{q(x)}$$

for $0 < r \le 1$.

Proposition 4.2 (cf. [13, Lemma 4.1]). *There exists a constant $A > 0$ such that*

$$A^{-1}\tilde{h}_{k,\Phi}(r;x) \le C_{k,\Phi}(B(x,r);B(x,r)) \le Ah_{k,\Phi}(r;x)$$

for all $0 < r \le 1$ and $x \in \mathbf{R}^N$.

Proof. For $0 < r \le 1$, consider the function

$$f_r(y) = \chi_{B(x,r)}(y).$$

If $z \in B(x,r)$, then

$$k * f_r(z) = \int_{B(x,r)} k(z-y)\,dy \ge \theta_N \int_{B(0,r)} k(y)\,dy = \theta'_N r^N \bar{k}(r)$$

with positive constants θ_N and θ'_N depending only on N. It follows from the definition of capacity that

$$
\begin{aligned}
C_{k,\Phi}(B(x,r);B(x,r)) &\le \int_{B(x,r)} \Phi(y, {\theta'_N}^{-1} r^{-N}\bar{k}(r)^{-1})dy \\
&\le A\int_{B(x,r)} \Phi(y, r^{-N}\bar{k}(r)^{-1})dy \\
&\le Ah_{k,\Phi}(r;x),
\end{aligned}
$$

which proves the second inequality of the proposition.

Let $f \in S_k(B(x,r);B(x,r))$. Then

$$
\begin{aligned}
1 &\le \frac{1}{|B(x,r)|}\int_{B(x,r)} k * f(z)\,dz \\
&= \int_{B(x,r)} \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} k(z-y)\,dz\right) f(y)dy \\
&\le \int_{B(x,r)} \left(\frac{1}{|B(0,r)|}\int_{B(0,r)} k(z)dz\right) f(y)\,dy \\
&= \bar{k}(r)\int_{B(x,r)} f(y)dy.
\end{aligned}
$$

For $\varepsilon > 0$, we see from $(\Phi 3)$ and $(\Phi 5)$ that

$$
\begin{aligned}
\int_{B(x,r)} f(y)dy &\le |B(0,1)|\varepsilon\bar{k}(r)^{-1} + \int_{B(x,r)} \Phi(y,f(y))\frac{\varepsilon r^{-N}\bar{k}(r)^{-1}}{\Phi(y,\varepsilon r^{-N}\bar{k}(r)^{-1})}dy \\
&\le |B(0,1)|\varepsilon\bar{k}(r)^{-1} \\
&\quad + A(\varepsilon)r^{-N}\bar{k}(r)^{-1}\big(\inf_{y\in B(x,r)} \Phi(y, r^{-N}\bar{k}(r)^{-1})\big)^{-1}\int_{B(x,r)} \Phi(y,f(y))dy \\
&= |B(0,1)|\varepsilon\bar{k}(r)^{-1} + A(\varepsilon)\bar{k}(r)^{-1}\tilde{h}_{k,\Phi}(r;x)^{-1}\int_{B(x,r)} \Phi(y,f(y))dy,
\end{aligned}
$$

so that

$$1 \le |B(0,1)|\varepsilon + A(\varepsilon)\tilde{h}_{k,\Phi}(r;x)^{-1}C_{k,\Phi}(B(x,r);B(x,r)).$$

Hence, letting $|B(0,1)|\varepsilon = 1/2$ we have

$$C_{k,\Phi}(B(x,r);B(x,r)) \ge A\tilde{h}_{k,\Phi}(r;x).$$

□

REMARK 4.3. Proposition 4.2 implies that

$$C_{k,\Phi}(B(x,r);B(x,1)) \le Ah_{k,\Phi}(r;x)$$

whenever $0 < r \le 1$. This estimate is trivial if

$$\liminf_{r\to 0+} h_{k,\Phi}(r;x) > 0. \tag{4.1}$$

As the case where $k(r) = r^{\alpha-N}$ $(0 < \alpha < N)$ and $\Phi(x,t) = t^{N/\alpha}$ shows, it can happen that

$$\lim_{r\to 0+} C_{k,\Phi}(B(x,r);B(x,1)) = 0 \tag{4.2}$$

even if (4.1) holds (cf., e.g., [12, Chapter 5, Theorem 2.1]).

In view of this remark, it is desirable to obtain non-trivial estimates in case both (4.1) and (4.2) occur. Here, we consider the special case where $\Phi(x,t)$ is of the form

$$\Phi_{p(\cdot),\varphi}(x,t) = t^{p(x)}\varphi(x,t)$$

with a variable exponent $p(\cdot)$ such that $1 < p^- \le p^+ < \infty$ and $\varphi(x,t)$ is uniformly of log-type in $t \ge 1$: there exists a constant $A_0 \ge 1$ such that

$$A_0^{-1}\varphi(x,t) \le \varphi(x,t^2) \le A_0\varphi(x,t) \quad \text{whenever} \quad x \in \mathbf{R}^N \text{ and } t \ge 1. \tag{4.3}$$

We naturally assume that $\Phi_{p(\cdot),\varphi}(x,t)$ satisfies $(\Phi1)$, $(\Phi2)$, $(\Phi3)$, $(\Phi4)$ and $(\Phi5)$.

Proposition 4.4. *Set*

$$\psi(x,s) = \sup_{|x-y|=s^{-1},\ s\le t\le s^2} \varphi(y,t) \quad \textit{for } x \in \mathbf{R}^N,\ s \ge 1$$

and

$$h(x,r) = \left(\int_r^1 k(2\rho)^{p(x)'}\psi(x,\rho^{-1})^{1-p(x)'}\rho^{N-1}d\rho\right)^{1-p(x)}$$

for $x \in \mathbf{R}^N$ and $0 < r \le 1/2$, where $p(x)' = p(x)/(p(x)-1)$.

Assume that $p(\cdot)$ is log-Hölder continuous at $x_0 \in \mathbf{R}^N$ and

(A) $r \mapsto r^N k(r)^{p(x_0)'}$ *is of log-type for $0 < r \le 1$.*

Then, there exists a constant $A = A(x_0) > 0$ such that

$$C_{k,\Phi_{p(\cdot),\varphi}}(B(x_0,r);B(x_0,1)) \le Ah(x_0,r)$$

for $0 < r \le 1/2$.

Remark 4.5. Assumption (A) is nearly a necessary condition for (4.1) and (4.2).

Proof of Proposition 4.4. We assume that $x_0 = 0$. For simplicity, let $p = p(0)$, $\psi(s) = \psi(0,s)$, and $h(r) = h(0,r)$. We may assume that $h(r) \to 0$ as $r \to 0+$. Set

$$g(r) = h(r)^{1/(1-p)} = \int_r^1 k(2\rho)^{p'}\psi(\rho^{-1})^{1-p'}\rho^{N-1}d\rho.$$

First note that (4.3), $(\Phi4)$ and $(\Phi5)$ imply

$$A_2^{-1}[\log(e+t)]^{-\sigma} \le \varphi(x,t) \le A_2[\log(e+t)]^{\sigma}$$

for $x \in \mathbf{R}^N$ and $t \geq 1$ with some constants $\sigma > 0$ and $A_2 \geq 1$, so that,

$$A_2^{-1}[\log(e+s)]^{-\sigma} \leq \psi(s) \leq A_3[\log(e+s)]^{\sigma} \tag{4.4}$$

for $s \geq 1$. Hence, together with our assumption (A), we see that

$$\lim_{r \to 0+} r^{\varepsilon} g(r) = 0 \tag{4.5}$$

for any $\varepsilon > 0$.

Now, let $a = 5p/N$. By (4.5), there is $r_1 : 0 < r_1 \leq 1/2$ such that

$$g(r) \leq \frac{1}{2} r^{-1/a} \quad \text{for} \quad 0 < r \leq r_1. \tag{4.6}$$

Then

$$g(r)^{-a} \geq 2^a r > r \quad \text{for} \quad 0 < r \leq r_1.$$

Choose $r_2 : 0 < r_2 \leq r_1$ such that $g(r_2) \geq r_1^{-1/a}$. Then, $g(r)^{-a} \leq r_1$ for $0 < r \leq r_2$ and (4.6) implies

$$g(g(r)^{-a}) \leq \frac{1}{2} g(r) \quad \text{for} \quad 0 < r \leq r_2. \tag{4.7}$$

For $0 < r \leq r_2$, consider the function

$$f_r(y) = 2\omega_N^{-1} g(r)^{-1} k(2|y|)^{p'-1} \psi(|y|^{-1})^{1-p'} \chi_{B(0,g(r)^{-a}) \setminus B(0,r)}(y),$$

where ω_N is the surface area of the unit sphere in $\mathbf{R}^N$. If $x \in B(0,r)$, then $|x-y| \leq 2|y|$ for $|y| \geq r$, so that

$$\begin{aligned} k * f_r(x) &\geq \int_{B(0,g(r)^{-a}) \setminus B(0,r)} k(2|y|) f_r(y)\, dy \\ &= 2g(r)^{-1} \int_r^{g(r)^{-a}} k(2\rho)^{p'} \psi(\rho^{-1})^{1-p'} \rho^{N-1}\, d\rho \\ &= 2g(r)^{-1}\big(g(r) - g(g(r)^{-a})\big) \geq 1, \end{aligned}$$

where we used (4.7) to deduce the last inequality. Hence $f_r \in S_k(B(0,r); B(0,1))$, so that

$$C_{k,\Phi_{p(\cdot),\varphi}}(B(0,r); B(0,1)) \leq \int_{B(0,1)} f_r(y)^{p(y)} \varphi(y, f_r(y))\, dy. \tag{4.8}$$

Let $b = (3/5)(N/p) = 3/a$. We shall show that there exists $r_0 : 0 < r_0 \leq r_2$ such that

$$|y|^{-b} \leq f_r(y) \leq |y|^{-2b} \quad \text{for } r \leq |y| < g(r)^{-a} \tag{4.9}$$

whenever $0 < r \leq r_0$.

Let $\eta(\rho) = 2\omega_N^{-1} k(2\rho)^{p'-1} \psi(\rho^{-1})^{1-p'}$ $(0 < \rho \leq 1)$. Since

$$k(2\rho)^{p'-1} \psi(\rho^{-1})^{1-p'} = \rho^{-N/p} [\rho^N k(2\rho)^{p'}]^{1/p} \psi(\rho^{-1})^{1-p'}$$

and $-N/p = -5/a$, by (A) and (4.4) there exists $r_0 : 0 < r_0 \le r_2$ such that

$$\rho^{-4/a} \le \eta(\rho) \le \rho^{-6/a} \quad \text{for} \quad 0 < \rho \le g(r_0)^{-a}. \tag{4.10}$$

Let $0 < r \le r_0$. Since $f_r(y) = g(r)^{-1}\eta(|y|)$ for $r \le |y| < g(r)^{-a}$ ($\le 1/2$), (4.10) implies

$$f_r(y) \ge g(r)^{-1}|y|^{-4/a} > |y|^{1/a}|y|^{-4/a} = |y|^{-3/a} = |y|^{-b}$$

for $r \le |y| < g(r)^{-a}$, and

$$f_r(y) \le g(r)^{-1}|y|^{-6/a} \le |y|^{-6/a} = |y|^{-2b}$$

for $r \le |y| < g(r)^{-a}$. Thus (4.9) holds.

In view of (4.9), the log-Hölder continuity of $p(\cdot)$ at 0 implies

$$f_r(y)^{p(y)} \le A_4 f_r(y)^p$$

for $r \le |y| < g(r)^{-a}$.

To estimate $\varphi(y, f_r(y))$, consider an auxiliary function

$$\tilde{\psi}(y,s) = \sup_{s \le t \le s^2} \varphi(y,t)$$

for $s \ge 1$. Choose $m \in \mathbf{N}$ such that $2^{m-1} \le b < 2^m$ in case $b \ge 1$, $2^{-m} \le b < 2^{-m+1}$ in case $b < 1$. Then, by (4.3) we have

$$\tilde{\psi}(y, s^b) \le A_0^m \tilde{\psi}(y,s)$$

for $s \ge 1$. Hence, by (4.9)

$$\varphi(y, f_r(y)) \le \tilde{\psi}(y, |y|^{-b}) \le A_0^m \tilde{\psi}(y, |y|^{-1}) \le A_0^m \psi(|y|^{-1})$$

for $r \le |y| < g(r)^{-a}$. Hence we have

$$f_r(y)^{p(y)}\varphi(y, f_r(y)) \le A g(r)^{-p} k(2|y|)^{p'} \psi(|y|^{-1})^{1-p'}$$

for $y \in B(0, g(r)^{-a}) \setminus B(0,r)$, so that by (4.8)

$$\begin{aligned} &C_{k,\Phi_{p(\cdot),\varphi}}(B(0,r); B(0,1)) \\ &\qquad \le A g(r)^{-p} \int_r^{g(r)^{-a}} k(2\rho)^{p'} \psi(\rho^{-1})^{1-p'} \rho^{N-1}\, d\rho \le A g(r)^{1-p} = A h(r) \end{aligned}$$

for $0 < r \le r_0$. The required estimate is trivial for $r_0 < r \le 1/2$. □

Example 4.6. Let $k(r) = r^{\alpha - N}$ $(0 < \alpha < N)$ and

$$\Phi(x,t) = \Phi_{p(\cdot),q(\cdot),a}(x,t) = t^{p(x)}(\log(a+t))^{q(x)},$$

namely, $\varphi(x,t) = (\log(a+t))^{q(x)}$. For these k and Φ, $C_{k,\Phi}$ is denoted by $C_{\alpha,p(\cdot),q(\cdot)}$. Suppose $\alpha p(x_0) = N$, $p(\cdot)$ is log-Hölder continuous and $q(\cdot)$ is log-log-Hölder continuous at x_0. Then $\psi(x_0, s) \approx \log(e+s)^{q(x_0)}$, so that

$$h(x_0, r) \approx \left(\int_r^1 \log(e+1/\rho)^{q(x_0)(1-p(x_0)')} \frac{d\rho}{\rho} \right)^{1-p(x_0)}.$$

Thus, applying the above proposition, we have (cf. [2]):

(1) if $q(x_0) < p(x_0) - 1$, then

$$C_{\alpha,p(\cdot),q(\cdot)}(B(x_0,r); B(x_0,1)) \le A \log(e+1/r)^{q(x_0)+1-p(x_0)}$$

for $0 < r \le 1/2$; and

(2) if $q(x_0) = p(x_0) - 1$, then

$$C_{\alpha,p(\cdot),q(\cdot)}(B(x_0,r); B(x_0,1)) \le A \log\bigl(\log(e+1/r)\bigr)^{1-p(x_0)}$$

for $0 < r \le 1/2$.

If $q(x_0) > p(x_0) - 1$, then $\lim_{r\to 0+} h(x_0, r) > 0$.

Theorem 4.7 (cf. [5, Lemma 4.4],[13, Lemma 4.3]). *If $f \in L^{\Phi}(\mathbf{R}^N)$, then there exists a set $E \subset \mathbf{R}^N$ with $C_{k,\Phi}(E) = 0$ such that*

$$\lim_{r\to 0+} \frac{1}{C_{k,\Phi}(B(x,5r); B(x,5r))} \int_{B(x,r)} \Phi(y, |f(y)|)dy = 0$$

for all $x \in \mathbf{R}^N \setminus E$.

Proof. For $\delta > 0$, consider the set

$$E_\delta = \left\{ x \in \mathbf{R}^N : \limsup_{r\to 0+} C_{k,\Phi}(B(x,5r); B(x,5r))^{-1} \int_{B(x,r)} \Phi(y, f(y))dy > \delta \right\}.$$

By subadditivity and Corollary 3.4, it suffices to show that $C_{k,\Phi}(E_\delta \cap B(0,R); B(0,2R)) = 0$ for all $R > 1$. Let $0 < \varepsilon < 1$. For each $x \in E_\delta \cap B(0,R)$, we find $0 < r(x) < \varepsilon/5$ such that

$$\int_{B(x,r(x))} \Phi(y, |f(y)|)dy > \delta C_{k,\Phi}(B(x,5r(x)); B(x,5r(x))). \tag{4.11}$$

By the Vitali covering lemma (see [17, Lemma, p. 9]), there exists a disjoint family $\{B_j\}$ of balls $B_j = B(x_j, r(x_j))$ such that $\bigcup_j B(x_j, 5r(x_j)) \supset E_\delta \cap B(0,R)$. Then we have by (4.11)

$$\begin{aligned} C_{k,\Phi}(E_\delta \cap B(0,R); B(0,2R)) &\le \sum_j C_{k,\Phi}(B(x_j,5r(x_j)); B(x_j,5r(x_j))) \\ &\le \delta^{-1} \int_{\bigcup_j B_j} \Phi(y, |f(y)|)dy. \end{aligned} \tag{4.12}$$

Set $L(\varepsilon) = \inf_{0<r<\varepsilon} r^{-N}\bar{k}(r)^{-1}$. Then $L(\varepsilon) \to \infty$ as $\varepsilon \to 0+$ and we may assume that $L(\varepsilon) > 1$. By Proposition 4.2, (4.11), (Φ3) and (Φ4),

$$\begin{aligned}
|B_j| &= A\tilde{h}_{k,\Phi}(5r(x_j); x_j) \left\{ \inf_{y \in B(x_j, 5r(x_j))} \Phi\big(y, (5r(x_j))^{-N}\bar{k}(5r(x_j))^{-1}\big) \right\}^{-1} \\
&\leq AC_{k,\Phi}(B(x_j, 5r(x_j)); B(x_j, 5r(x_j))) \left\{ \inf_{y \in B(x_j, 5r(x_j))} \Phi\big(y, L(\varepsilon)\big) \right\}^{-1} \\
&\leq A\delta^{-1} L(\varepsilon)^{-1} \left\{ \inf_{y \in B(x_j, 5r(x_j))} \Phi(y, 1) \right\}^{-1} \int_{B_j} \Phi(y, |f(y)|) dy \\
&\leq A\delta^{-1} L(\varepsilon)^{-1} \int_{B_j} \Phi(y, |f(y)|) dy.
\end{aligned}$$

Hence we have

$$\left| \bigcup_j B_j \right| \leq A\delta^{-1} L(\varepsilon)^{-1} \int_{\mathbf{R}^N} \Phi(y, |f(y)|) dy \to 0$$

as $\varepsilon \to 0+$. Thus, from (4.12) and the absolute continuity of integral we obtain

$$C_{k,\Phi}(E_\delta \cap B(0, R); B(0, 2R)) = 0,$$

as required. $\square$

Corollary 4.8. *Suppose*

$$\limsup_{r \to 0+} \frac{\sup_{y \in B(x,r)} \Phi(y, r^{-N}\bar{k}(r)^{-1})}{\inf_{y \in B(x,r)} \Phi(y, r^{-N}\bar{k}(r)^{-1})} < \infty \tag{4.13}$$

for every $x \in \mathbf{R}^N$. *If* $f \in L^\Phi(\mathbf{R}^N)$, *then there exists a set* $E \subset \mathbf{R}^N$ *with* $C_{k,\Phi}(E) = 0$ *such that*

$$\lim_{r \to 0+} \frac{1}{r^N \Phi(x, r^{-N}\bar{k}(r)^{-1})} \int_{B(x,r)} \Phi(y, |f(y)|) dy = 0$$

for all $x \in \mathbf{R}^N \setminus E$.

Remark 4.9. Condition (4.13) is a kind of continuity of $\Phi(x, t)$ in x. For example, in the case $k(r) = r^{\alpha - N}$ $(0 < \alpha < N)$ and $\Phi(x, t) = t^{p(x)}(\log(a + t))^{q(x)}$ as in Example 4.1 (2), (4.13) is satisfied if $p(x)$ is log-Hölder continuous and $q(x)$ is log-log-Hölder continuous.

A function $h(r; x) : (0, \tilde{r}) \times \mathbf{R}^N \to (0, \infty)$ $(\tilde{r} > 0)$ may be called a variable measure function if $\lim_{r \to 0+} h(r; x) = 0$ for every $x \in \mathbf{R}^N$. Given such a function, the generalized Hausdorff measure H_h can be defined as in the case of ordinary measure function.

Let $\bar{h}(r;x) = \sup_{0<\rho\le r} h(\rho;x)$. For $E \subset \mathbf{R}^N$ and $0 < \delta < \tilde{r}$, let

$$H_h^{(\delta)}(E) = \inf\left\{\sum_j \bar{h}(r_j;x_j); E \subset \bigcup_j B(x_j,r_j), 0 < r_j \le \delta\right\}.$$

Since $H_h^{(\delta)}(E)$ increases as δ decreases, we define the generalized Hausdorff measure with respect to h by

$$H_h(E) = \lim_{\delta\to 0+} H_h^{(\delta)}(E).$$

Clearly, $H_h^{(\delta)}(E)$ and $H_h(E)$ are measures on $\mathbf{R}^N$.

THEOREM 4.10. *Suppose that* $\lim_{r\to 0+} h_{k,\Phi}(r;x) = 0$ *for every* $x \in \mathbf{R}^N$. *If* $H_{h_{k,\Phi}}(E) = 0$, *then* $C_{k,\Phi}(E) = 0$.

Proof. Suppose $H_{h_{k,\Phi}}(E) = 0$. By subadditivity and Corollary 3.4, it suffices to show that $C_{k,\Phi}(E\cap B(0,R); B(0,2R)) = 0$ for all $R > 1$. Since $H_{h_{k,\Phi}}(E\cap B(0,R)) = 0$, for $\varepsilon > 0$, there exists a family $\{B(x_j,r_j)\}$ such that $0 < r_j < 1$, $E \subset \bigcup_j B(x_j,r_j)$ and

$$\sum_j h_{k,\Phi}(r_j;x_j) < \varepsilon.$$

Then Proposition 4.2 yields

$$\begin{aligned} C_{k,\Phi}(E\cap B(0,R); B(0,2R)) &\le \sum_j C_{k,\Phi}(B(x_j,r_j); B(x_j,r_j)) \\ &\le A\sum_j h_{k,\Phi}(r_j;x_j) < A\varepsilon. \end{aligned}$$

Hence, we have $C_{k,\Phi}(E\cap B(0,R); B(0,2R)) = 0$. □

5 Appendix

In this section, we consider two functions $\Phi(x,t)$ and $\Psi(x,t)$ satisfying conditions (Φi), $i = 1,2,3,4,5$, and discuss the duality between $L^\Phi(\mathbf{R}^N)$ and $L^\Psi(\mathbf{R}^N)$:

PROPOSITION 5.1. *Let* $\varphi(x,t) = \Phi(x,t)/t$ *and* $\psi(x,t) = \Psi(x,t)/t$ *for* $x \in \mathbf{R}^N$ *and* $t > 0$. *Suppose there is a constant* $\widetilde{A} \ge 1$ *such that*

$$\widetilde{A}^{-1}t \le \psi\big(x,\varphi(x,t)\big) \le \widetilde{A}t \tag{5.1}$$

for all $x \in \mathbf{R}^N$ *and* $t > 0$. *Then* $L^\Psi(\mathbf{R}^N)$ *is the dual space of* $L^\Phi(\mathbf{R}^N)$.

It is known that if Φ^* is the complementary function of Φ, then $L^{\Phi^*}(\mathbf{R}^N)$ is the dual space of $L^\Phi(\mathbf{R}^N)$ provided that

$$\lim_{t\to 0+}\varphi(x,t) = 0 \quad \text{and} \quad \lim_{t\to\infty}\varphi(x,t) = \infty \tag{5.2}$$

(see, e.g., [15, Theorem 13.15 and Theorem 13.17]). Note that assumption (5.1) implies (5.2) as well as

$$\lim_{t\to 0+} \psi(x,t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \psi(x,t) = \infty.$$

Therefore, we show the above proposition by verifying that Ψ is comparable to the complementary function Φ^*, namely we show

LEMMA 5.2. *Under the assumptions of Proposition 5.1, there exists a constant $A \ge 1$ such that*

$$A^{-1}\Phi^*(x,t) \le \Psi(x,t) \le A\Phi^*(x,t)$$

for all $x \in \mathbf{R}^N$ and $t > 0$.

Proof. First, note that $\varphi(x,\cdot)$ and $\psi(x,\cdot)$ are non-decreasing continuous on $(0,\infty)$ by condition (Φ3).

Recall that

$$\Phi^*(x,t) = \sup_{s>0}\big(st - \Phi(x,s)\big) = \sup_{s>0} s\big(t - \varphi(x,s)\big)$$

for $t > 0$. By the continuity of $\varphi(x,\cdot)$, there exists $\sigma = \sigma(x,t) > 0$ such that

$$\Phi^*(x,t) = \sigma\big(t - \varphi(x,\sigma)\big).$$

Note that $0 < \varphi(x,\sigma) < t$. Hence, by (5.1),

$$\widetilde{A}^{-1}\sigma \le \psi\big(x,\varphi(x,\sigma)\big) \le \psi(x,t).$$

Therefore,

$$\Phi^*(x,t) \le t\sigma \le \widetilde{A}t\psi(x,t) = \widetilde{A}\Psi(x,t).$$

Conversely, given $t > 0$, choose $s > 0$ such that $\varphi(x,s) = t/2$. Then, by (5.1)

$$\Phi^*(x,t) \ge \frac{t}{2}\,s \ge \frac{t}{2}\widetilde{A}^{-1}\psi(x,\varphi(x,s)) = \widetilde{A}^{-1}\Psi(x,t/2) \ge A^{-1}\Psi(x,t),$$

where we used (Φ5) for Ψ to derive the last inequality. □

EXAMPLE 5.3. Let $p(\cdot)$ and $q(\cdot)$ be measurable variable exponents on $\mathbf{R}^N$ satisfying

(p1) $1 < p^- \le p^+ < \infty$;

(q1) $-\infty < q^- \le q^+ < \infty$.

If we choose $a \ge e$ such that $(1+\log a)(p(x)-1) + q(x) \ge 0$ for all $x \in \mathbf{R}^N$, then

$$\Phi_{p(\cdot),q(\cdot),a}(x,t) = t^{p(x)}(\log(a+t))^{q(x)}$$

satisfies conditions (Φi), $i = 1,2,3,4,5$.

Set $p^* = p(x)/(p(x)-1)$, $q^*(x) = -q(x)/(p(x)-1)$ and $\log a^* = \max(1, q^+ - 1)$. Then we see that $\Phi_{p^*(\cdot),q^*(\cdot),a^*}(x,t)$ also satisfies the conditions (Φi), $i = 1,2,3,4,5$.

By elementary calculus, we see that (5.1) holds for $\Phi = \Phi_{p(\cdot),q(\cdot),a}$ and $\Psi = \Phi_{p^*(\cdot),q^*(\cdot),a^*}$. Thus, by Proposition 5.1, we have

COROLLARY 5.4. *The dual space of $L^{\Phi_{p(\cdot),q(\cdot),a}}(\mathbf{R}^N)$ is $L^{\Phi_{p^*(\cdot),q^*(\cdot),a^*}}(\mathbf{R}^N)$, and $L^{\Phi_{p(\cdot),q(\cdot),a}}(\mathbf{R}^N)$ is reflexive.*

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