Trudinger's inequality and continuity for Riesz potentials of functions in Musielak-Orlicz-Morrey spaces on metric measure spaces

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Abstract

In this paper we are concerned with Trudinger's inequality and continuity for Riesz potentials of functions in Musielak-Orlicz-Morrey spaces on metric measure spaces.

1 Introduction

A famous Trudinger inequality ([42]) insists that Sobolev functions in $W^{1,N}(G)$ satisfy finite exponential integrability, where G is an open bounded set in $\mathbf{R}^N$ (see also [2], [5], [36], [43]). For $0 < \alpha < N$, we define the Riesz potential of order α for a locally integrable function f on $\mathbf{R}^N$ by

$$U_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\, dy.$$

Great progress on Trudinger type inequalities has been made for Riesz potentials of order α in the limiting case $\alpha p = N$ (see e.g. [8], [9], [10], [11], [41]). Trudinger type exponential integrability was studied on Orlicz spaces in [3], [28] and [32], on generalized Morrey spaces $L^{1,\varphi}$ in [23] and [24], and on Orlicz-Morrey spaces in [33] and [38]. For Morrey spaces, which were introduced to estimate solutions of partial differential equations, we refer to [35] and [40].

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth condition. For a survey, see [6] and [7]. Trudinger type exponential integrability was investigated on variable exponent Lebesgue spaces $L^{p(\cdot)}$ in [12], [13] and [14] and on two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$ in [27]. See also [26] for two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$.

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For $x \in \mathbf{R}^N$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x with radius r and $d_\Omega = \sup\{d(x,y) : x, y \in \Omega\}$ for a set $\Omega \subset \mathbf{R}^N$. For bounded measurable functions $\nu(\cdot) : \mathbf{R}^N \to (0, N]$ and $\beta(\cdot) : \mathbf{R}^N \to \mathbf{R}$, let $L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)$ be the set of all measurable functions f on G such that $\|f\|_{L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)} < \infty$, where

$$\|f\|_{L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)} = \inf\left\{\lambda > 0 : \sup_{x\in G, 0<r\le d_G} \frac{r^{\nu(x)}(\log(e+1/r))^{\beta(x)}}{|B(x,r)|} \times \int_{B(x,r)} \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} \left(\log\left(e + \frac{|f(y)|}{\lambda}\right)\right)^{q(y)} dy \le 1\right\};$$

we set $f = 0$ outside G. As an extension of Trudinger [42] and [24, Corollaries 4.6 and 4.8], Mizuta, Nakai and the authors [25] proved Trudinger type exponential integrability for two variable exponent Morrey spaces $L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)$ when $p(\cdot)$ and $q(\cdot)$ are variable exponents satisfying the log-Hölder and loglog-Hölder conditions on G, respectively. The result is an improvement of [31, Theorems 4.4 and 4.5]. In fact we proved the following:

THEOREM A. *Suppose* $\inf_{x\in\mathbf{R}^N} \nu(x) > 0$ *and* $\inf_{x\in\mathbf{R}^N}(\alpha - \nu(x)/p(x)) \ge 0$ *hold. Let* ε *be a constant such that*

$$\inf_{x\in\mathbf{R}^n} (\nu(x)/p(x) - \varepsilon) > 0 \text{ and } 0 < \varepsilon < \alpha.$$

Then there exist constants $C_1, C_2 > 0$ *such that*

(1) *in case* $\sup_{x\in\mathbf{R}^N} (q(x)+\beta(x))/p(x) < 1$,

$$\frac{r^{\nu/p(z)-\varepsilon}}{|B(z,r)|} \int_{B(z,r)} \exp\left(\frac{|U_\alpha f(x)|^{p(x)/(p(x)-q(x)-\beta(x))}}{C_1}\right) dx \le C_2;$$

(2) *in case* $\inf_{x\in\mathbf{R}^N} (q(x)+\beta(x))/p(x) \ge 1$,

$$\frac{r^{\nu/p(z)-\varepsilon}}{|B(z,r)|} \int_{B(z,r)} \exp\left(\exp\left(\frac{|U_\alpha f(x)|}{C_1}\right)\right) dx \le C_2$$

for all $z \in G$, $0 < r < d_G$ *and* f *satisfying* $\|f\|_{L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)} \le 1$.

Recently, Theorem A was extended to Musielak-Orlicz-Morrey spaces in [20]. Our main aim in this paper is to give a general version of Trudinger type exponential integrability for Riesz potentials $I_\alpha f$ of functions in Musielak-Orlicz-Morrey spaces $L^{\Phi,\kappa}(X)$ on metric measure spaces X (e.g., Corollary 4.6) as an extension of the above results (see Section 2 for the definitions of Φ and κ and Section 3 for the definition of $I_\alpha f$). Since we discuss the Morrey version, our strategy is to find an estimate of Riesz potentials by use of Riesz potentials of order ε, which plays a role of the maximal functions (see Section 3). What is new about this paper is that we can pass our results to the metric measure setting; the technique in [20] still works.

Beginning with Sobolev's embedding theorem (see e.g. [1], [2]), continuity properties of Riesz potentials or Sobolev functions have been studied by many authors. Continuity of Riesz potentials of functions in Orlicz spaces was studied in [11], [21], [22], [29] and [32] (cf. also [30]). Then such continuity was investigated on generalized Morrey spaces $L^{1,\varphi}$ in [23] and [24], on Orlicz-Morrey spaces in [34], on variable exponent Lebesgue spaces in [12], [13] and [16] and on variable exponent Morrey spaces in [34]. In [25], Mizuta, Nakai and the authors also proved continuity for Riesz potentials of functions in two variable exponent Morrey spaces $L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)$.

In [20], these results have been extended to Musielak-Orlicz-Morrey spaces. Our second aim in this paper is to give a general version of continuity for Riesz potentials $I_\alpha f$ of functions in Musielak-Orlicz-Morrey spaces $L^{\Phi,\kappa}(X)$ on metric measure spaces (e.g., Corollary 5.6) as an extension of the above results.

In [39], we established Trudinger type exponential integrability for Musielak-Orlicz spaces in the Euclidean setting by use of the maximal functions, which are a crucial tool as in Hedberg [18]. Our third aim in this paper is to give a general version of Trudinger type exponential integrability for Riesz potentials $I_\alpha f$ of functions in Musielak-Orlicz spaces $L^{\Phi}(X)$ on metric measure spaces (e.g., Corollary 7.2) as an extension of [13], [17] and [39]. To obtain our results, we need the boundedness of maximal operator on $L^{\Phi}(X)$ (see Lemma 6.1).

In the final section, we show the continuity for Riesz potentials $I_\alpha f$ of functions in Musielak-Orlicz spaces $L^{\Phi}(X)$ on metric measure spaces (see Corollary 8.2).

2 Preliminaries

Throughout this paper, let C denote various constants independent of the variables in question.

We denote by (X, d, μ) a metric measure space, where X is a set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x with radius r and $d_\Omega = \sup\{d(x, y) : x, y \in \Omega\}$ for a set $\Omega \subset X$.

We say that the measure μ is a doubling measure if there exists a constant $c_0 > 0$ such that $\mu(B(x, 2r)) \le c_0\mu(B(x, r))$ for every $x \in X$ and $0 < r < d_X$. We say that X is a doubling space if μ is a doubling measure.

In this paper, we assume that X is a bounded set and a doubling space, that is $d_X < \infty$. This implies that $\mu(X) < \infty$.

We consider a function

$$\Phi(x, t) = t\phi(x, t) : X \times [0, \infty) \to [0, \infty)$$

satisfying the following conditions $(\Phi1)$ – $(\Phi4)$:

$(\Phi1)$ $\phi(\,\cdot\,, t)$ is measurable on X for each $t \ge 0$ and $\phi(x, \,\cdot\,)$ is continuous on $[0, \infty)$ for each $x \in X$;

$(\Phi2)$ there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \phi(x, 1) \le A_1 \quad \text{for all } x \in X;$$

(Φ3) $\phi(x,\cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \ge 1$ such that

$$\phi(x,t) \le A_2\phi(x,s) \quad \text{for all } x \in X \quad \text{whenever } 0 \le t < s;$$

(Φ4) there exists a constant $A_3 \ge 1$ such that

$$\phi(x,2t) \le A_3\phi(x,t) \quad \text{for all } x \in X \text{ and } t > 0.$$

Note that (Φ2), (Φ3) and (Φ4) imply

$$0 < \inf_{x\in X} \phi(x,t) \le \sup_{x\in X} \phi(x,t) < \infty$$

for each $t > 0$.

If $\Phi(x,\cdot)$ is convex for each $x \in X$, then (Φ3) holds with $A_2 = 1$; namely $\phi(x,\cdot)$ is non-decreasing for each $x \in X$.

Let $\bar{\phi}(x,t) = \sup_{0\le s\le t} \phi(x,s)$ and

$$\overline{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)\,dr \tag{2.1}$$

for $x \in X$ and $t \ge 0$. Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \le \overline{\Phi}(x,t) \le A_2\Phi(x,t) \tag{2.2}$$

for all $x \in X$ and $t \ge 0$.

We shall also consider the following condition:

(Φ5) for every $\gamma_1, \gamma_2 > 0$, there exists a constant $B_{\gamma_1,\gamma_2} \ge 1$ such that

$$\phi(x,t) \le B_{\gamma_1,\gamma_2}\phi(y,t)$$

whenever $d(x,y) \le \gamma_1 t^{-1/\gamma_2}$ and $t \ge 1$.

Example 2.1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1,\dots,k$, be measurable functions on X such that

(P1) $1 < p^- := \inf_{x\in X} p(x) \le \sup_{x\in X} p(x) =: p^+ < \infty$

and

(Q1) $-\infty < q_j^- := \inf_{x\in X} q_j(x) \le \sup_{x\in X} q_j(x) =: q_j^+ < \infty$

for all $j = 1,\dots,k$.

Set $L_c(t) = \log(c+t)$ for $c \ge e$ and $t \ge 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$ and

$$\Phi(x,t) = t^{p(x)} \prod_{j=1}^{k} (L_c^{(j)}(t))^{q_j(x)}.$$

Then, $\Phi(x,t)$ satisfies (Φ1), (Φ2), (Φ3) and (Φ4).

Moreover, we see that $\Phi(x,t)$ satisfies (Φ5) if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{C_p}{L_e(1/d(x,y))}$$

with a constant $C_p \ge 0$ and

(Q2) $q_j(\cdot)$ is $j+1$-log-Hölder continuous, namely

$$|q_j(x) - q_j(y)| \le \frac{C_{q_j}}{L_e^{(j+1)}(1/d(x,y))}$$

with constants $C_{q_j} \ge 0$, $j = 1, \dots k$.

EXAMPLE 2.2. Let $p(\cdot)$ be a measurable function on X satisfying (P1) and (P2). Let $q_1(\cdot)$ be a measurable function on X satisfying (Q1) and (Q2) and let $q_2(\cdot)$ be a measurable function on X satisfying (Q1). Then

$$\Phi(x,t) = t^{p(x)} (\log(e+t))^{q_1(x)} (\log(e+1/t))^{q_2(x)}$$

satisfies (Φ1), (Φ2), (Φ3), (Φ4) and (Φ5).

In view of (2.2), given $\Phi(x,t)$ as above, the associated Musielak-Orlicz space

$$L^{\Phi}(X) = \left\{ f \in L^1_{loc}(X) \,;\, \int_X \Phi\big(y, |f(y)|\big)\, d\mu(y) < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(X)} = \inf \left\{ \lambda > 0 \,;\, \int_X \overline{\Phi}\big(y, |f(y)|/\lambda\big)\, d\mu(y) \le 1 \right\}$$

(cf. [37]).

We also consider a function $\kappa(x,r) : X \times (0, d_X] \to (0, \infty)$ satisfying the following conditions:

(κ1) $\kappa(x,\cdot)$ is measurable for each $x \in X$;

(κ2) $\kappa(x,\cdot)$ is uniformly almost increasing on $(0, d_X]$, namely there exists a constant $Q_1 \ge 1$ such that

$$\kappa(x,r) \le Q_1 \kappa(x,s)$$

for all $x \in X$ whenever $0 < r < s \le d_X$;

(κ3) there are constants $Q > 0$ and $Q_2 \ge 1$ such that

$$Q_2^{-1} \min(1, r^Q) \le \kappa(x,r) \le Q_2$$

for all $x \in X$ and $0 < r \le d_X$.

Example 2.3. For $Q > 0$, let $\nu(\cdot)$ and $\beta_j(\cdot)$, $j = 1, \dots k$ be measurable functions on X such that $\inf_{x \in X} \nu(x) > 0$, $\sup_{x \in X} \nu(x) \le Q$ and $-c(Q - \nu(x)) \le \beta_j(x) \le c$ for all $x \in X$, $j = 1, \dots, k$ and some constant $c > 0$. Then

$$\kappa(x, r) = r^{\nu(x)} \prod_{j=1}^{k} (L_e^{(j)}(1/r))^{\beta_j(x)}$$

satisfies $(\kappa 1)$, $(\kappa 2)$ and $(\kappa 3)$.

For a locally integrable function f on X, define the $L^{\Phi,\kappa}$ norm

$$\|f\|_{L^{\Phi,\kappa}(X)} = \inf\left\{\lambda > 0 : \sup_{x \in X, 0 < r \le d_X} \frac{\kappa(x, r)}{\mu(B(x, r))} \int_{X \cap B(x,r)} \overline{\Phi}(y, |f(y)|/\lambda)\, d\mu(y) \le 1\right\}.$$

See (2.1) for the definition of $\overline{\Phi}$. Let $L^{\Phi,\kappa}(X)$ denote the set of all functions f such that $\|f\|_{L^{\Phi,\kappa}(X)} < \infty$ (cf. [38]), which we call a Musielak-Orlicz-Morrey space. Note that $L^{\Phi,\kappa}(X) = L^{\Phi}(X)$ if $\mu(B(x, r)) \sim \kappa(x, r)$ for all $x \in X$ and $0 < r \le d_X$. (Here $h_1(x, s) \sim h_2(x, s)$ means that $C^{-1} h_2(x, s) \le h_1(x, s) \le C h_2(x, s)$ for a constant $C > 0$.)

3 Lemmas for Musielak-Orlicz-Morrey spaces

Set

$$\Phi^{-1}(x, s) = \sup\{t > 0\, ;\, \Phi(x, t) < s\}$$

for $x \in X$ and $s > 0$.

Lemma 3.1 ([19, Lemma 5.1]). *$\Phi^{-1}(x, \cdot)$ is non-decreasing;*

$$\Phi^{-1}(x, \lambda s) \le A_2 \lambda \Phi^{-1}(x, s) \tag{3.1}$$

for all $x \in X$, $s > 0$ and $\lambda \ge 1$ and

$$\min\left\{1, \frac{s}{A_1 A_2}\right\} \le \Phi^{-1}(x, s) \le \max\{1, A_1 A_2 s\} \tag{3.2}$$

for all $x \in X$ and $s > 0$, where A_1 and A_2 are the constants appearing in $(\Phi 2)$ and $(\Phi 3)$.

Lemma 3.2. *There exists a constant $C > 0$ such that*

$$C^{-1} \le \Phi^{-1}(x, \kappa(x, r)^{-1}) \le C r^{-Q} \tag{3.3}$$

for all $x \in X$ and $0 < r \le d_X$.

Proof. By $(\kappa 3)$,

$$Q_2^{-1} \le \kappa(x, r)^{-1} \le Q_2 \max(1, r^{-Q})$$

for $x \in X$ and $0 < r \le d_X$. Hence, by (3.2), we obtain (3.3). □

As in [19, Lemma 5.3], we can prove the following result.

Lemma 3.3 (cf. [19, Lemma 5.3]). *Assume that* $\Phi(x,t)$ *satisfies* ($\Phi5$). *Then there exists a constant* $C>0$ *such that*

$$\int_{X\cap B(x,r)} f(y)\,d\mu(y) \le C\mu(B(x,r))\Phi^{-1}(x,\kappa(x,r)^{-1})$$

for all $x\in X$, $0<r\le d_X$ *and* $f\ge 0$ *satisfying* $\|f\|_{L^{\Phi,\kappa}(X)}\le 1$.

For $\alpha>0$, we define the Riesz potential of order α for a locally integrable function f on X by

$$I_\alpha f(x) = \int_X \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,d\mu(y)$$

(e.g. see [15]).

Set

$$\Gamma(x,s) = \int_{1/s}^{d_X} \rho^\alpha \Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\,\frac{d\rho}{\rho}$$

for $s\ge 2/d_X$ and $x\in X$. For $0\le s<2/d_X$ and $x\in X$, we set $\Gamma(x,s)=\Gamma(x,2/d_X)(d_X/2)s$. Then note that $\Gamma(x,\cdot)$ is strictly increasing and continuous for each $x\in X$.

Lemma 3.4 (cf. [20, Lemma 3.5]). *There exists a positive constant* C' *such that* $\Gamma(x,2/d_X)\ge C'>0$ *for all* $x\in X$.

Lemma 3.5. *Assume that* $\Phi(x,t)$ *satisfies* ($\Phi5$). *Then there exists a constant* $C>0$ *such that*

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,d\mu(y) \le C\Gamma\left(x,\frac{1}{\delta}\right)$$

for all $x\in X$, $0<\delta\le d_X/2$ *and nonnegative* $f\in L^{\Phi,\kappa}(X)$ *with* $\|f\|_{L^{\Phi,\kappa}(X)}\le 1$.

Proof. Let j_0 be the smallest positive integer such that $2^{j_0}\delta\ge d_X$. By Lemma 3.3, we have

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,d\mu(y)\\
=\ &\sum_{j=1}^{j_0}\int_{X\cap(B(x,2^j\delta)\setminus B(x,2^{j-1}\delta))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,d\mu(y)\\
\le\ &\sum_{j=1}^{j_0}(2^j\delta)^\alpha \frac{1}{\mu(B(x,2^{j-1}\delta))}\int_{X\cap B(x,2^j\delta)} f(y)\,d\mu(y)\\
\le\ &c_0\sum_{j=1}^{j_0}(2^j\delta)^\alpha \frac{1}{\mu(B(x,2^j\delta))}\int_{X\cap B(x,2^j\delta)} f(y)\,d\mu(y)\\
\le\ &C\left(\sum_{j=1}^{j_0-1}(2^j\delta)^\alpha\Phi^{-1}(x,\kappa(x,2^j\delta)^{-1}) + d_X^\alpha\Phi^{-1}(x,\kappa(x,d_X)^{-1})\right).
\end{aligned}$$

By (κ2) and (3.1), we have

$$\int_{2^{j-1}\delta}^{2^j\delta} t^\alpha \Phi^{-1}(x,\kappa(x,t)^{-1})\,\frac{dt}{t} \geq (2^{j-1}\delta)^\alpha \Phi^{-1}(x, Q_1^{-1}\kappa(x,2^j\delta)^{-1})\log 2$$
$$\geq \frac{(2^j\delta)^\alpha \log 2}{2^\alpha A_2 Q_1}\Phi^{-1}(x,\kappa(x,2^j\delta)^{-1}) = C(2^j\delta)^\alpha \Phi^{-1}(x,\kappa(x,2^j\delta)^{-1})$$

and

$$\begin{aligned}\int_{d_X/2}^{d_X} t^\alpha \Phi^{-1}(x,\kappa(x,t)^{-1})\,\frac{dt}{t} &\geq \frac{d_X^\alpha \log 2}{2^\alpha A_2 Q_1}\Phi^{-1}(x,\kappa(x,d_X)^{-1}) \\ &= C d_X^\alpha \Phi^{-1}(x,\kappa(x,d_X)^{-1}).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,d\mu(y) \\ \leq\ & C\left(\sum_{j=1}^{j_0-1}\int_{2^{j-1}\delta}^{2^j\delta} t^\alpha \Phi^{-1}(x,\kappa(x,t)^{-1})\,\frac{dt}{t} + \int_{d_X/2}^{d_X} t^\alpha \Phi^{-1}(x,\kappa(x,t)^{-1})\,\frac{dt}{t}\right) \\ \leq\ & C\Gamma\left(x,\frac{1}{\delta}\right),\end{aligned}$$

as required □

LEMMA 3.6. *Assume that $\Phi(x,t)$ satisfies (Φ5). Let $\varepsilon > 0$ and define*

$$\lambda_\varepsilon(z,r) = \frac{1}{1+\displaystyle\int_r^{d_X} \rho^\varepsilon \Phi^{-1}(z,\kappa(z,\rho)^{-1})\,\frac{d\rho}{\rho}}$$

for $z \in X$. Then there exists a constant $C_{I,\varepsilon} > 0$ such that

$$\frac{\lambda_\varepsilon(z,r)}{\mu(B(z,r))}\int_{X\cap B(z,r)} I_\varepsilon f(x)\,d\mu(x) \leq C_{I,\varepsilon}$$

for all $z \in X$, $0 < r \leq d_X$ and $f \geq 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(X)} \leq 1$.

Proof. Let $z \in X$. Write

$$\begin{aligned}I_\varepsilon f(x) &= \int_{X\cap B(z,2r)} \frac{d(x,y)^\varepsilon f(y)}{\mu(B(x,d(x,y)))}\,d\mu(y) + \int_{X\setminus B(z,2r)} \frac{d(x,y)^\varepsilon f(y)}{\mu(B(x,d(x,y)))}\,d\mu(y) \\ &= I_1(x) + I_2(x)\end{aligned}$$

for $x \in X$. By Fubini's theorem,

$$\begin{aligned}
&\int_{X\cap B(z,r)} I_1(x)\,d\mu(x)\\
=\ &\int_{X\cap B(z,2r)}\left(\int_{X\cap B(z,r)}\frac{d(x,y)^\varepsilon}{\mu(B(x,d(x,y)))}\,d\mu(x)\right)f(y)\,d\mu(y)\\
\le\ &\int_{X\cap B(z,2r)}\left(\int_{X\cap B(y,3r)}\frac{d(x,y)^\varepsilon}{\mu(B(x,d(x,y)))}\,d\mu(x)\right)f(y)\,d\mu(y)\\
\le\ &\int_{X\cap B(z,2r)}\left(\sum_{j=0}^{\infty}\int_{X\cap(B(y,2^{-j+2}r)\setminus B(y,2^{-j+1}r))}\frac{d(x,y)^\varepsilon}{\mu(B(x,d(x,y)))}\,d\mu(x)\right)f(y)\,d\mu(y)\\
\le\ &\int_{X\cap B(z,2r)}\left(\sum_{j=0}^{\infty}\int_{X\cap(B(y,2^{-j+2}r)\setminus B(y,2^{-j+1}r))}\frac{(2^{-j+2}r)^\varepsilon}{\mu(B(x,2^{-j+1}r))}\,d\mu(x)\right)f(y)\,d\mu(y).
\end{aligned}$$

Since μ is a doubling measure, we have

$$\begin{aligned}
&\int_{X\cap B(z,r)} I_1(x)\,d\mu(x)\\
\le\ &c_0^2\int_{X\cap B(z,2r)}\left(\sum_{j=0}^{\infty}\int_{X\cap(B(y,2^{-j+2}r)\setminus B(y,2^{-j+1}r))}\frac{(2^{-j+2}r)^\varepsilon}{\mu(B(x,2^{-j+3}r))}\,d\mu(x)\right)f(y)\,d\mu(y)\\
\le\ &c_0^2\int_{X\cap B(z,2r)}\left(\sum_{j=0}^{\infty}\int_{X\cap(B(y,2^{-j+2}r)\setminus B(y,2^{-j+1}r))}\frac{(2^{-j+2}r)^\varepsilon}{\mu(B(y,2^{-j+2}r))}\,d\mu(x)\right)f(y)\,d\mu(y)\\
\le\ &c_0^2\int_{X\cap B(z,2r)}\left(\sum_{j=0}^{\infty}(2^{-j+2}r)^\varepsilon\right)f(y)\,d\mu(y)\\
\le\ &C8^\varepsilon\int_{X\cap B(z,2r)}\left(\sum_{j=0}^{\infty}\int_{2^{-j-1}r}^{2^{-j}r}t^\varepsilon\,\frac{dt}{t}\right)f(y)\,d\mu(y)\\
\le\ &C\int_{X\cap B(z,2r)}\left(\int_0^r t^\varepsilon\,\frac{dt}{t}\right)f(y)\,d\mu(y)\\
=\ &\frac{C}{\varepsilon}r^\varepsilon\int_{X\cap B(z,2r)}f(y)\,d\mu(y).
\end{aligned}$$

Now, by Lemma 3.3, (κ2) and (3.1), we have

$$\begin{aligned}
r^\varepsilon\int_{X\cap B(z,2r)}f(y)\,dy &\le Cr^\varepsilon\mu(B(z,2r))\Phi^{-1}(z,\kappa(z,2r)^{-1})\\
&\le C\mu(B(z,2r))\int_r^{2r}\rho^\varepsilon\Phi^{-1}(z,\kappa(z,\rho)^{-1})\,\frac{d\rho}{\rho}
\end{aligned}$$

if $0<r\le d_X/2$ and, by Lemma 3.3 and (3.3), we have

$$\begin{aligned}
r^\varepsilon\int_{X\cap B(z,2r)}f(y)\,dy &= r^\varepsilon\int_{B(z,d_X)}f(y)\,dy\\
&\le C{d_X}^\varepsilon\mu(B(z,d_X))\Phi^{-1}(z,\kappa(z,d_X)^{-1})\le C\mu(B(z,r))
\end{aligned}$$

if $d_X/2 < r \le d_X$. Therefore

$$\int_{X\cap B(z,r)} I_1(x)\,d\mu(x) \le \frac{C}{\varepsilon}\frac{\mu(B(z,r))}{\lambda_\varepsilon(z,r)}$$

for all $0 < r \le d_X$.

For I_2, first note that $I_2(x) = 0$ if $x \in X$ and $r \ge d_X/2$. Let $0 < r < d_X/2$. Let j_0 be the smallest positive integer such that $2^{j_0} r \ge d_X$. Since

$$I_2(x) \le C\int_{X\setminus B(z,2r)} \frac{d(z,y)^\varepsilon f(y)}{\mu(B(z,d(z,y)))}\,d\mu(y) \quad \text{for} \quad x \in X\cap B(z,r),$$

by Lemma 3.3, we have

$$\begin{aligned}
I_2(x) &\le C\sum_{j=1}^{j_0-1}\int_{B(z,2^{j+1}r)\setminus B(z,2^j r)} \frac{d(z,y)^\varepsilon}{\mu(B(z,d(z,y)))} f(y)\,d\mu(y) \\
&\le C\sum_{j=1}^{j_0-1}(2^{j+1}r)^\varepsilon \frac{1}{\mu(B(z,2^j r))}\int_{X\cap B(z,2^{j+1}r)} f(y)\,d\mu(y) \\
&\le C\sum_{j=1}^{j_0-1}(2^{j+1}r)^\varepsilon \frac{1}{\mu(B(z,2^{j+1} r))}\int_{X\cap B(z,2^{j+1}r)} f(y)\,d\mu(y) \\
&\le C\left(\sum_{j=1}^{j_0-2}(2^{j+1}r)^\varepsilon \Phi^{-1}(x,\kappa(x,2^{j+1}r)^{-1}) + d_X^\varepsilon \Phi^{-1}(x,\kappa(x,d_X)^{-1})\right).
\end{aligned}$$

As in the proof of Lemma 3.5, we obtain

$$\begin{aligned}
I_2(x) &\le C\left(\sum_{j=1}^{j_0-2}\int_{2^j r}^{2^{j+1}r} \rho^\varepsilon \Phi^{-1}(x,\kappa(x,\rho)^{-1})\,\frac{d\rho}{\rho} + \int_{d_X/2}^{d_X} \rho^\varepsilon \Phi^{-1}(x,\kappa(x,\rho)^{-1})\,\frac{d\rho}{\rho}\right) \\
&\le C\int_r^{d_X} \rho^\varepsilon \Phi^{-1}(z,\kappa(z,\rho)^{-1})\,\frac{d\rho}{\rho} \\
&\le \frac{C}{\lambda_\varepsilon(z,r)}
\end{aligned}$$

for all $x \in X\cap B(z,r)$. Hence

$$\int_{X\cap B(z,r)} I_2(x)\,d\mu(x) \le C\frac{\mu(B(z,r))}{\lambda_\varepsilon(z,r)}.$$

Thus this lemma is proved. $\square$

4 Trudinger's inequality for Musielak-Orlicz-Morrey spaces

In this section, we deal with the case $\Gamma(x,t)$ satisfies the uniform log-type condition:

($\Gamma_{\log}$) there exists a constant $c_\Gamma > 0$ such that

$$\Gamma(x,t^2) \le c_\Gamma \Gamma(x,t)$$

for all $x \in X$ and $t \ge 1$.

EXAMPLE 4.1. Let Φ and κ be as in Examples 2.1 and 2.3, respectively. Then

$$\Gamma(x,t) \sim \int_{1/t}^{d_X} \rho^{\alpha-\nu(x)/p(x)} \prod_{j=1}^{k} \left[L_e^{(j)}(1/\rho)\right]^{-(q_j(x)+\beta_j(x))/p(x)} \frac{d\rho}{\rho} \qquad (t \geq 2/d_X),$$

so that it satisfies $(\Gamma_{\log})$ if and only if

$$\alpha p(x) \geq \nu(x) \qquad \text{for all } x \in X.$$

By $(\Gamma_{\log})$, together with Lemma 3.4, we see that $\Gamma(x,t)$ satisfies the uniform doubling condition in t:

LEMMA 4.2 (cf. [20, Lemma 4.2]). *Suppose $\Gamma(x,t)$ satisfies $(\Gamma_{\log})$. For every $a > 1$, there exists $b > 0$ such that $\Gamma(x,at) \leq b\Gamma(x,t)$ for all $x \in X$ and $t > 0$.*

THEOREM 4.3. *Assume that $\Phi(x,t)$ satisfies $(\Phi5)$, $\Gamma(x,t)$ satisfies $(\Gamma_{\log})$. For each $x \in X$, let $\gamma(x) = \sup_{s>0} \Gamma(x,s)$. Suppose $\Psi(x,t) : X \times [0,\infty) \to [0,\infty]$ satisfies the following conditions:*

- $(\Psi1)$ *$\Psi(\cdot,t)$ is measurable on X for each $t \in [0,\infty)$; $\Psi(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in X$;*
- $(\Psi2)$ *there is a constant $A_1' \geq 1$ such that $\Psi(x,t) \leq \Psi(x,A_1's)$ for all $x \in X$ whenever $0 < t < s$;*
- $(\Psi3)$ *$\Psi(x,\Gamma(x,t)/A_2') \leq A_3't$ for all $x \in X$ and $t > 0$ with constants $A_2',\ A_3' \geq 1$ independent of x.*

Then, for $0 < \varepsilon < \alpha$, there exists a constant $C^* > 0$ such that $I_\alpha f(x)/C^* < \gamma(x)$ for a.e. $x \in X$ and

$$\frac{\lambda_\varepsilon(z,r)}{\mu(B(z,r))} \int_{X\cap B(z,r)} \Psi\left(x, \frac{I_\alpha f(x)}{C^*}\right) d\mu(x) \leq 1$$

for all $z \in X$, $0 < r \leq d_X$ and $f \geq 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(X)} \leq 1$.

Proof. Let $f \geq 0$ and $\|f\|_{L^{\Phi,\kappa}(X)} \leq 1$. Fix $x \in X$. For $0 < \delta \leq d_X/2$, Lemma 3.5 implies

$$\begin{aligned}
I_\alpha f(x) &\leq \int_{X\cap B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\, d\mu(y) + C\Gamma\left(x, \frac{1}{\delta}\right) \\
&= \int_{X\cap B(x,\delta)} d(x,y)^{\alpha-\varepsilon} \frac{d(x,y)^\varepsilon f(y)}{\mu(B(x,d(x,y)))}\, d\mu(y) + C\Gamma\left(x, \frac{1}{\delta}\right) \\
&\leq C\left\{\delta^{\alpha-\varepsilon} I_\varepsilon f(x) + \Gamma\left(x, \frac{1}{\delta}\right)\right\}
\end{aligned}$$

with constants $C > 0$ independent of x.

If $I_\varepsilon f(x) \leq 2/d_X$, then we take $\delta = d_X/2$. Then, by Lemma 3.4

$$I_\alpha f(x) \leq C\Gamma\left(x, \frac{2}{d_X}\right).$$

By Lemma 4.2, there exists $C_1^* > 0$ independent of x such that

$$I_\alpha f(x) \le C_1^* \Gamma\left(x, \frac{1}{2A_3'}\right) \qquad \text{if } I_\varepsilon f(x) \le 2/d_X. \tag{4.1}$$

Next, suppose $2/d_X < I_\varepsilon f(x) < \infty$. Let $m = \sup_{s \ge 2/d_X, x\in X} \Gamma(x,s)/s$. By $(\Gamma_{\log})$, $m < \infty$. Define δ by

$$\delta^{\alpha-\varepsilon} = \frac{(d_X/2)^{\alpha-\varepsilon}}{m} \Gamma(x, I_\varepsilon f(x))(I_\varepsilon f(x))^{-1}.$$

Since $\Gamma(x, I_\varepsilon f(x))(I_\varepsilon f(x))^{-1} \le m$, $0 < \delta \le d_X/2$. Then by Lemma 3.4

$$\begin{aligned}\frac{1}{\delta} &\le C\Gamma(x, I_\varepsilon f(x))^{-1/(\alpha-\varepsilon)}(I_\varepsilon f(x))^{1/(\alpha-\varepsilon)} \\ &\le C\Gamma(x, 2/d_X)^{-1/(\alpha-\varepsilon)}(I_\varepsilon f(x))^{1/(\alpha-\varepsilon)} \le C(I_\varepsilon f(x))^{1/(\alpha-\varepsilon)}.\end{aligned}$$

Hence, using $(\Gamma_{\log})$ and Lemma 4.2, we obtain

$$\Gamma\left(x, \frac{1}{\delta}\right) \le \Gamma\big(x, C(I_\varepsilon f(x))^{1/(\alpha-\varepsilon)}\big) \le C\Gamma(x, I_\varepsilon f(x)).$$

By Lemma 4.2 again, we see that there exists a constant $C_2^* > 0$ independent of x such that

$$I_\alpha f(x) \le C_2^* \Gamma\left(x, \frac{1}{2C_{I,\varepsilon}A_3'} I_\varepsilon f(x)\right) \qquad \text{if } 2/d_X < I_\varepsilon f(x) < \infty, \tag{4.2}$$

where $C_{I,\varepsilon}$ is the constant given in Lemma 3.6.

Now, let $C^* = A_1' A_2' \max(C_1^*, C_2^*)$. Then, by (4.1) and (4.2),

$$\frac{I_\alpha f(x)}{C^*} \le \frac{1}{A_1'A_2'} \max\left\{\Gamma\left(x, \frac{1}{2A_3'}\right), \Gamma\left(x, \frac{1}{2C_{I,\varepsilon}A_3'} I_\varepsilon f(x)\right)\right\}$$

whenever $I_\varepsilon f(x) < \infty$. Since $I_\varepsilon f(x) < \infty$ for a.e. $x \in X$ by Lemma 3.6, $I_\alpha f(x)/C^* < \gamma(x)$ a.e. $x \in X$, and by $(\Psi 2)$ and $(\Psi 3)$, we have

$$\begin{aligned}&\Psi\left(x, \frac{I_\alpha f(x)}{C^*}\right) \\ &\le \max\left\{\Psi\left(x, \Gamma\left(x, \frac{1}{2A_3'}\right)/A_2'\right), \Psi\left(x, \Gamma\left(x, \frac{1}{2C_{I,\varepsilon}A_3'} I_\varepsilon f(x)\right)/A_2'\right)\right\} \\ &\le \frac{1}{2} + \frac{1}{2C_{I,\varepsilon}} I_\varepsilon f(x)\end{aligned}$$

for a.e. $x \in X$. Thus, noting that $\lambda_\varepsilon(z,r) \le 1$ and using Lemma 3.6, we have

$$\begin{aligned}&\frac{\lambda_\varepsilon(z,r)}{\mu(B(z,r))} \int_{X\cap B(z,r)} \Psi\left(x, \frac{I_\alpha f(x)}{C^*}\right) d\mu(x) \\ &\qquad \le \frac{1}{2}\lambda_\varepsilon(z,r) + \frac{1}{2C_{I,\varepsilon}} \frac{\lambda_\varepsilon(z,r)}{\mu(B(z,r))} \int_{X\cap B(z,r)} I_\varepsilon f(x)\, d\mu(x) \\ &\qquad \le \frac{1}{2} + \frac{1}{2} = 1\end{aligned}$$

for all $z \in X$ and $0 < r \le d_X$. $\square$

REMARK 4.4. If $\Gamma(x,s)$ is bounded, that is,

$$\sup_{x\in X}\int_0^{d_X}\rho^{\alpha}\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\,d\rho<\infty,$$

then by Lemma 3.5 we see that $I_\alpha|f|$ is bounded for every $f\in L^{\Phi,\kappa}(X)$.

REMARK 4.5. We can not take $\varepsilon=\alpha$ in Theorem 4.3. For details, see [23, Remark 2.8].

As in the proof of [20, Corollary 4.6], we obtain the following corollary applying Theorem 4.3 to special Φ and κ given in Examples 2.1 and 2.3.

COROLLARY 4.6. *Let Φ and κ be as in Examples 2.1 and 2.3.*
Assume that

$$\alpha-\nu(x)/p(x)=0\qquad \text{for all } x\in X.$$

(1) *Suppose there exists an integer $1\le j_0\le k$ such that*

$$\inf_{x\in X}(p(x)-q_{j_0}(x)-\beta_{j_0}(x))>0$$

and

$$\sup_{x\in X}(p(x)-q_j(x)-\beta_j(x))\le 0$$

for all $j\le j_0-1$ in case $j_0\ge 2$. Then for $0<\varepsilon<\alpha$ there exist constants $C^*>0$ and $C^{**}>0$ such that

$$\frac{r^{\nu(z)/p(z)-\varepsilon}}{|B(z,r)|}\int_{X\cap B(z,r)}E_+^{(j_0)}\left(\left(\frac{I_\alpha f(x)}{C^*}\right)^{p(x)/(p(x)-q_{j_0}(x)-\beta_{j_0}(x))}\right.$$
$$\left.\times\prod_{j=1}^{k-j_0}\left(L_e^{(j)}\left(\frac{I_\alpha f(x)}{C^*}\right)\right)^{(q_{j_0+j}(x)+\beta_{j_0+j}(x))/(p(x)-q_{j_0}(x)-\beta_{j_0}(x))}\right)d\mu(x)\le C^{**}$$

for all $z\in X$, $0<r\le d_X$ and $f\ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(X)}\le 1$, where $E^{(1)}(t)=e^t-e$, $E^{(j+1)}(t)=\exp(E^j(t))-e$ and $E_+^{(j)}(t)=\max(E^{(j)}(t),0)$.
(2) *If*

$$\sup_{x\in X}(p(x)-q_j(x)-\beta_j(x))\le 0$$

for all $j=1,\dots,k$, then for $0<\varepsilon<\alpha$ there exist constants $C^*>0$ and $C^{**}>0$ such that

$$\frac{r^{\nu(z)/p(z)-\varepsilon}}{|B(z,r)|}\int_{X\cap B(z,r)}E^{(k+1)}\left(\frac{I_\alpha f(x)}{C^*}\right)d\mu(x)\le C^{**}$$

for all $z\in X$, $0<r\le d_X$ and $f\ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(X)}\le 1$.

5 Continuity for Musielak-Orlicz-Morrey spaces

In this section, we discuss the continuity of Riesz potentials $I_\alpha f$ of functions in Musielak-Orlicz-Morrey spaces under the condition: there are constants $\theta > 0$ and $C_0 > 0$ such that

$$\left| \frac{d(x,y)^\alpha}{\mu(B(x,d(x,y)))} - \frac{d(z,y)^\alpha}{\mu(B(z,d(z,y)))} \right| \le C_0 \left(\frac{d(x,z)}{d(x,y)} \right)^\theta \frac{d(x,y)^\alpha}{\mu(B(x,d(x,y)))} \tag{5.1}$$

whenever $d(x,z) \le d(x,y)/2$.

We consider the functions

$$\omega(x,r) = \int_0^r \rho^\alpha \Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big) \frac{d\rho}{\rho}$$

and

$$\omega_\theta(x,r) = r^\theta \int_r^{d_X} \rho^{\alpha-\theta} \Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big) \frac{d\rho}{\rho}$$

for $\theta > 0$ and $0 < r \le d_X$.

Lemma 5.1 (cf. [20, Lemma 5.1]). *Let $E \subset X$. If $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in E$, then $\omega_\theta(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in E$.*

Lemma 5.2 (cf. [20, Lemma 5.2]). *There exists a constant $C > 0$ such that*

$$\omega(x,2r) \le C\omega(x,r)$$

for all $x \in X$ and $0 < r \le d_X/2$.

Theorem 5.3. *Assume that $\Phi(x,t)$ satisfies (Φ5). Then there exists a constant $C > 0$ such that*

$$|I_\alpha f(x) - I_\alpha f(z)| \le C\{\omega(x,d(x,z)) + \omega(z,d(x,z)) + \omega_\theta(x,d(x,z))\}$$

for all $x, z \in X$ with $d(x,z) \le d_X/4$ and nonnegative $f \in L^{\Phi,\kappa}(X)$ with $\|f\|_{L^{\Phi,\kappa}(X)} \le 1$.

Before giving a proof of Theorem 5.3, we prepare two more lemmas.

Lemma 5.4. *Assume that $\Phi(x,t)$ satisfies (Φ5). Let f be a nonnegative function on X such that $\|f\|_{L^{\Phi,\kappa}(X)} \le 1$. Then there exists a constant $C > 0$ such that*

$$\int_{X \cap B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))} d\mu(y) \le C\omega(x,\delta)$$

for all $x \in X$ and $0 < \delta \le d_X$.

Proof. Let f be a nonnegative μ-measurable function on X with $\|f\|_{L^{\Phi,\kappa}(X)} \le 1$. As usual we start by decomposing $B(x,\delta)$ dyadically:

$$\begin{aligned}
&\int_{X\cap B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\, d\mu(y) \\
=\ & \sum_{j=1}^{\infty} \int_{X\cap(B(x,2^{-j+1}\delta)\setminus B(x,2^{-j}\delta))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\, d\mu(y) \\
\le\ & \sum_{j=1}^{\infty} (2^{-j+1}\delta)^\alpha \frac{1}{\mu(B(x,2^{-j}\delta))} \int_{B(x,2^{-j+1}\delta)} f(y)\, d\mu(y) \\
\le\ & c_0 \sum_{j=1}^{\infty} (2^{-j+1}\delta)^\alpha \frac{1}{\mu(B(x,2^{-j+1}\delta))} \int_{B(x,2^{-j+1}\delta)} f(y)\, d\mu(y).
\end{aligned}$$

By Lemma 3.3, we have

$$\begin{aligned}
\int_{X\cap B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\, d\mu(y) &\le C\sum_{j=1}^{\infty} (2^{-j+1}\delta)^\alpha \Phi^{-1}(x,\kappa(x,2^{-j+1}\delta)^{-1}) \\
&\le C\int_0^\delta \rho^\alpha \Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\, \frac{d\rho}{\rho} \\
&= C\omega(x,\delta).
\end{aligned}$$

□

The following lemma can be proved on the same manner as Lemma 3.5.

LEMMA 5.5. *Assume that $\Phi(x,t)$ satisfies ($\Phi 5$). Let $\theta \in \mathbf{R}$. Let f be a nonnegative function on X such that $\|f\|_{L^{\Phi,\kappa}(X)} \le 1$. Then there exists a constant $C > 0$ such that*

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^{\alpha-\theta} f(y)}{\mu(B(x,d(x,y)))}\, d\mu(y) \le C\delta^{-\theta}\omega_\theta(x,\delta)$$

for all $x \in X$ and $0 < \delta \le d_X/2$.

Proof of Theorem 5.3. Let f be a nonnegative μ-measurable function on X with $\|f\|_{L^{\Phi,\kappa}(X)} \le 1$ and $x, z \in X$ with $d(x,z) \le d_X/4$. Write

$$\begin{aligned}
&I_\alpha f(x) - I_\alpha f(z) \\
=\ & \int_{X\cap B(x,2d(x,z))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\, d\mu(y) - \int_{X\cap B(x,2d(x,z))} \frac{d(z,y)^\alpha f(y)}{\mu(B(z,d(z,y)))}\, d\mu(y) \\
&+ \int_{X\setminus B(x,2d(x,z))} \left(\frac{d(x,y)^\alpha}{\mu(B(x,d(x,y)))} - \frac{d(z,y)^\alpha}{\mu(B(z,d(z,y)))} \right) f(y)\, d\mu(y).
\end{aligned}$$

Using Lemmas 5.2 and 5.4, we have

$$\int_{X\cap B(x,2d(x,z))} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\, d\mu(y) \le C\omega(x,2d(x,z)) \le C\omega(x,d(x,z))$$

and

$$\begin{aligned}\int_{X\cap B(x,2d(x,z))} \frac{d(z,y)^{\alpha} f(y)}{\mu(B(z,d(z,y)))}\,d\mu(y) &\le \int_{X\cap B(z,3d(x,z))} \frac{d(z,y)^{\alpha} f(y)}{\mu(B(z,d(z,y)))}\,d\mu(y)\\ &\le C\omega(z,3d(x,z)) \le C\omega(z,d(x,z)).\end{aligned}$$

On the other hand, by (5.1) and Lemma 5.5, we have

$$\begin{aligned}&\int_{X\setminus B(x,2d(x,z))} \left| \frac{d(x,y)^{\alpha}}{\mu(B(x,d(x,y)))} - \frac{d(z,y)^{\alpha}}{\mu(B(z,d(z,y)))} \right| f(y)\,d\mu(y)\\ \le\ & Cd(x,z)^{\theta} \int_{X\setminus B(x,2d(x,z))} \frac{d(x,y)^{\alpha-\theta} f(y)}{\mu(B(x,d(x,y)))}\,d\mu(y)\\ \le\ & C\omega_{\theta}(x,2d(x,z)) \le C\omega_{\theta}(x,d(x,z)).\end{aligned}$$

Then we have the conclusion. □

In view of Lemma 5.1, we obtain the following corollary.

COROLLARY 5.6. *Assume that* $\Phi(x,t)$ *satisfies* ($\Phi5$).

(a) *Let* $x_0 \in X$ *and suppose* $\omega(x,r) \to 0$ *as* $r \to 0+$ *uniformly in* $x \in X \cap B(x_0,\delta)$ *for some* $\delta > 0$. *Then* $I_{\alpha}f$ *is continuous at* x_0 *for every* $f \in L^{\Phi,\kappa}(X)$.

(b) *Suppose* $\omega(x,r) \to 0$ *as* $r \to 0+$ *uniformly in* $x \in X$. *Then* $I_{\alpha}f$ *is uniformly continuous on* X *for every* $f \in L^{\Phi,\kappa}(X)$.

6 Lemmas for Musielak-Orlicz spaces

For a measurable function $Q(\cdot)$ satisfying

$$0 < Q^- := \inf_{x\in X} Q(x) \le \sup_{x\in X} Q(x) =: Q^+ < \infty, \tag{6.1}$$

we say that a measure μ is lower Ahlfors $Q(x)$-regular if there exists a constant $c_1 > 0$ such that

$$\mu(B(x,r)) \ge c_1 r^{Q(x)}$$

for all $x \in X$ and $0 < r < d_X$. Recall that we say that the measure μ is a doubling measure if there exists a constant $c_0 > 0$ such that $\mu(B(x,2r)) \le c_0\mu(B(x,r))$ for every $x \in X$ and $0 < r < d_X$. Here note that if μ is a doubling measure and $d_X < \infty$, then μ is lower Ahlfors $\log_2 c_0$-regular since

$$\frac{\mu(B(x,r))}{\mu(B(x,d_X))} \ge c_0^{-2} \left(\frac{r}{d_X}\right)^{\log_2 c_0}$$

for all $x \in X$ and $0 < r < d_X$ (see e.g. [4, Lemma 3.3]).

For a locally integrable function f on X, the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap X} |f(y)|\,d\mu(y).$$

As in the proof of [19, Theorem 4.1], we can show the following boundedness of maximal operator on $L^{\Phi}(X)$.

Lemma 6.1 (c.f. [19, Theorem 4.1]). *Suppose that* $\Phi(x,t)$ *satisfies* (Φ5) *and further assume:*

($\Phi 3^*$) $t \mapsto t^{-\varepsilon_0}\phi(x,t)$ *is uniformly almost increasing on* $(0,\infty)$ *for some* $\varepsilon_0 > 0$.

Then the maximal operator M *is bounded from* $L^\Phi(X)$ *into itself, namely, there is a constant* $C > 0$ *such that*

$$\|Mf\|_{L^\Phi(X)} \le C\|f\|_{L^\Phi(X)}$$

for all $f \in L^\Phi(X)$.

We consider the function

$$\gamma(x,t) : X \times (0, d_X) \to (0,\infty)$$

satisfying the following conditions:

(γ1) $\gamma(\,\cdot\,,t)$ is measurable on X for each $0 < t < d_X$ and $\gamma(x,\,\cdot\,)$ is continuous on $(0,d_X)$ for each $x \in X$;

(γ2) there exist constants $\gamma_0 > 0$ and $B_0 \ge 1$ such that

$$B_0^{-1} \le \gamma(x,t) \le B_0 t^{-\gamma_0} \quad \text{for all } x \in X \quad \text{whenever } 0 < t < d_X.$$

(γ3) there exists a constant $B_1 \ge 1$ such that

$$B_1^{-1}\gamma(x,s) \le \gamma(x,t) \le B_1\gamma(x,s) \quad \text{for all } x \in X \text{ and } 1 \le t/s \le 2.$$

Further we consider the function

$$\widetilde{\Gamma}(x,t) : X \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions (Γ1), (Γ2) and (Γ3):

(Γ1) $\widetilde{\Gamma}(\,\cdot\,,t)$ is measurable on X for each $t \ge 0$ and $\widetilde{\Gamma}(x,\,\cdot\,)$ is continuous on $[0,\infty)$ for each $x \in X$;

(Γ2) $\widetilde{\Gamma}(x,\cdot)$ is uniformly almost increasing, namely there exists a constant $B_2 \ge 1$ such that

$$\widetilde{\Gamma}(x,t) \le B_2\widetilde{\Gamma}(x,s) \quad \text{for all } x \in X \quad \text{whenever } 0 \le t < s;$$

(Γ3) For a measurable function $Q(\cdot)$ satisfying (6.1), there exist constants $\alpha_0 > 0, B_3 \ge 1$ and $B_4 \ge 1$ such that

$$t^{\alpha - Q(x)}\phi(x,\gamma(x,t))^{-1} \le B_3\widetilde{\Gamma}(x,1/t)$$

for all $x \in X$ and $\alpha \ge \alpha_0$ whenever $0 < t < d_X$ and

$$\int_t^{d_X} \rho^\alpha \gamma(x,\rho)\,\frac{d\rho}{\rho} \le B_4\widetilde{\Gamma}(x,1/t)$$

for all $x \in X, 0 < t \le d_X/2$ and $\alpha \ge \alpha_0$.

Example 6.2. Let Φ be as in Example 2.1.

(1) Suppose there exists an integer $1 \le j_0 \le k$ such that

$$\inf_{x\in X}(p(x) - q_{j_0}(x) - 1) > 0$$

and

$$\sup_{x\in X}(p(x) - q_j(x) - 1) \le 0$$

for all $j \le j_0 - 1$ in case $j_0 \ge 2$. set

$$\gamma(x,t) = t^{-Q(x)/p(x)} \left(\prod_{j=1}^{j_0-1}[L_e^{(j)}(1/t)]^{-1}\right)[L_e^{(j_0)}(1/t)]^{-(q_{j_0}(x)+1)/p(x)} \left(\prod_{j=j_0+1}^{k}[L_e^{(j)}(1/t)]^{-q_j(x)/p(x)}\right)$$

and

$$\widetilde{\Gamma}(x,t) = [L_e^{(j_0)}(t)]^{(p(x)-q_{j_0}(x)-1)/p(x)} \left(\prod_{j=j_0+1}^{k}[L_e^{(j)}(t)]^{-q_j(x)/p(x)}\right).$$

Then $\gamma(x,t)$ satisfies (γ1), (γ2) and (γ3) and $\widetilde{\Gamma}(x,t)$ satisfies (Γ1), (Γ2) and (Γ3) for all $\alpha \ge Q^+/p^-$.

(2) Suppose that

$$\sup_{x\in X}(p(x) - q_j(x) - 1) \le 0$$

for all $j = 1, \ldots, k$. set

$$\gamma(x,t) = t^{-Q(x)/p(x)} \left(\prod_{j=1}^{k}[L_e^{(j)}(1/t)]^{-1}\right)[L_e^{(k+1)}(1/t)]^{-1/p(x)}$$

and

$$\widetilde{\Gamma}(x,t) = [L_e^{(k+1)}(1/t)]^{1-1/p(x)}.$$

Then $\gamma(x,t)$ satisfies (γ1), (γ2) and (γ3) and $\widetilde{\Gamma}(x,t)$ satisfies (Γ1), (Γ2) and (Γ3) for all $\alpha \ge Q^+/p^-$.

In fact, see the proof of [39, Corollary 4.2].

Lemma 6.3. *Assume that μ is lower Ahlfors $Q(x)$-regular. Suppose that $\Phi(x,t)$ satisfies (Φ5). Let $\alpha \ge \alpha_0$. Then there exists a constant $C > 0$ such that*

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\, d\mu(y) \le C\widetilde{\Gamma}\left(x, \frac{1}{\delta}\right)$$

for all $x \in X$, $0 < \delta \le d_X/2$ and nonnegative $f \in L^\Phi(X)$ with $\|f\|_{L^\Phi(X)} \le 1$.

Proof. Let f be a nonnegative μ-measurable function on X with $\|f\|_{L^\Phi(X)} \le 1$. Let j_0 be the smallest integer j_0 such that $2^{j_0}\delta \ge d_X$. Since

$$B_0^{-1} \le \gamma(x, d(x,y)) \le B_0 d(x,y)^{-\gamma_0}$$

in view of ($\gamma 2$), we have

$$d(x,y) \le B_0^{2/\gamma_0}(B_0\gamma(x,d(x,y)))^{-1/\gamma_0}.$$

Hence, by ($\Phi 3$), ($\Phi 4$) and ($\Phi 5$), we obtain

$$\phi(y,\gamma(x,d(x,y)))^{-1} \le B'\phi(x,\gamma(x,d(x,y)))^{-1}$$

with some constant $B' > 0$. By ($\gamma 3$), ($\Phi 3$), ($\Gamma 2$) and ($\Gamma 3$), we have

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,d\mu(y)\\
\le\ & \int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha \gamma(x,d(x,y))}{\mu(B(x,d(x,y)))}\,d\mu(y)\\
&+A_2\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\frac{\phi(y,f(y))}{\phi(y,\gamma(x,d(x,y)))}\,d\mu(y)\\
\le\ & \sum_{j=1}^{j_0}\int_{B(x,2^j\delta)\setminus B(x,2^{j-1}\delta)} \frac{d(x,y)^\alpha \gamma(x,d(x,y))}{\mu(B(x,d(x,y)))}\,d\mu(y)\\
&+c_0^{-1}A_2B'\int_{X\setminus B(x,\delta)} d(x,y)^{\alpha-Q(x)}\phi(x,\gamma(x,d(x,y)))^{-1}\Phi(y,f(y))\,d\mu(y)\\
\le\ & 2^\alpha B_1\sum_{j=1}^{j_0}(2^{j-1}\delta)^\alpha\gamma(x,2^{j-1}\delta)\int_{B(x,2^j\delta)\setminus B(x,2^{j-1}\delta)}\frac{1}{\mu(B(x,2^{j-1}\delta))}\,d\mu(y)\\
&+c_0^{-1}A_2B_2B_3B'\widetilde{\Gamma}(x,1/\delta)\int_{X\setminus B(x,\delta)}\Phi(y,f(y))\,d\mu(y)\\
\le\ & 2^\alpha c_2B_1\sum_{j=1}^{j_0}(2^{j-1}\delta)^\alpha\gamma(x,2^{j-1}\delta)+c_0^{-1}A_2B_2B_3B'\widetilde{\Gamma}(x,1/\delta).
\end{aligned}$$

Since

$$\int_\delta^{d_X}\rho^\alpha\gamma(x,\rho)\,\frac{d\rho}{\rho} \ge \sum_{j=1}^{j_0-1}\int_{2^{j-1}\delta}^{2^j\delta}\rho^\alpha\gamma(x,\rho)\,\frac{d\rho}{\rho} \ge \frac{\log 2}{B_1}\sum_{j=1}^{j_0-1}(2^{j-1}\delta)^\alpha\gamma(x,2^{j-1}\delta)$$

and

$$\int_\delta^{d_X}\rho^\alpha\gamma(x,\rho)\,\frac{d\rho}{\rho} \ge \int_{d_X/2}^{d_X}\rho^\alpha\gamma(x,\rho)\,\frac{d\rho}{\rho} \ge \frac{\log 2}{2^\alpha B_1}(2^{j_0-1}\delta)^\alpha\gamma(x,2^{j_0-1}\delta),$$

we have

$$\sum_{j=1}^{j_0}(2^{j-1}\delta)^\alpha\gamma(x,2^{j-1}\delta) \le \frac{B_1}{\log 2}(2^\alpha+1)\int_\delta^{d_X}\rho^\alpha\gamma(x,\rho)\,\frac{d\rho}{\rho}.$$

Hence, we obtain

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,d\mu(y)\\
\le\ & (\log 2)^{-1}2^\alpha(2^\alpha+1)c_2B_1^2\int_\delta^{d_X}\rho^\alpha\gamma(x,\rho)\,\frac{d\rho}{\rho}+c_0^{-1}A_2B_2B_3B'\widetilde{\Gamma}(x,1/\delta)\\
\le\ & (\log 2)^{-1}2^\alpha(2^\alpha+1)c_2B_1^2B_4\widetilde{\Gamma}(x,1/\delta)+c_0^{-1}A_2B_2B_3B'\widetilde{\Gamma}(x,1/\delta)\\
=\ & ((\log 2)^{-1}2^\alpha(2^\alpha+1)c_2B_1^2B_4+c_0^{-1}A_2B_2B_3B')\widetilde{\Gamma}(x,1/\delta),
\end{aligned}$$

as required. $\square$

Lemma 6.4 (cf. [39, Lemma 3.3]). *Let $\alpha \ge \alpha_0$. Then there exists a constant $C' > 0$ such that $\widetilde{\Gamma}(x, 2/d_X) \ge C'$ for all $x \in X$.*

Lemma 6.5 (cf. [39, Lemma 3.4]). *Suppose $\widetilde{\Gamma}(x,t)$ satisfies the uniform log-type condition:*

($\widetilde{\Gamma}_{\log}$) *there exists a constant $c_\Gamma > 0$ such that*

$$c_\Gamma^{-1}\widetilde{\Gamma}(x,t) \le \widetilde{\Gamma}(x,t^2) \le c_\Gamma \widetilde{\Gamma}(x,t)$$

for all $x \in X$ and $t > 0$.

Then, for every $a > 1$, there exists $b > 0$ such that $\widetilde{\Gamma}(x,at) \le b\widetilde{\Gamma}(x,t)$ for all $x \in X$ and $t > 0$.

7 Trudinger's inequality for Musielak-Orlicz spaces

Theorem 7.1. *Suppose that μ is lower Ahlfors $Q(x)$-regular. Assume that $\Phi(x,t)$ satisfies ($\Phi5$) and ($\Phi3^*$). Further, assume that $\widetilde{\Gamma}(x,t)$ satisfies ($\widetilde{\Gamma}_{\log}$). For each $x \in X$, let $\widetilde{\gamma}(x) = \sup_{s>0} \widetilde{\Gamma}(x,s)$. Suppose $\widetilde{\Psi}(x,t) : X \times [0,\infty) \to [0,\infty]$ satisfies the following conditions:*

($\widetilde{\Psi}1$) *$\widetilde{\Psi}(\cdot,t)$ is measurable on X for each $t \in [0,\infty)$ and $\widetilde{\Psi}(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in X$;*

($\widetilde{\Psi}2$) *there is a constant $B_5 \ge 1$ such that $\widetilde{\Psi}(x,t) \le \widetilde{\Psi}(x,B_5 s)$ for all $x \in X$ whenever $0 < t < s$;*

($\widetilde{\Psi}3$) *there are constants B_6, $B_7 \ge 1$ and $t_0 > 0$ such that $\widetilde{\Psi}(x, \widetilde{\Gamma}(x,t)/B_6) \le B_7 t$ for all $x \in X$ and $t \ge t_0$.*

Then there exist constants $c_1, c_2 > 0$ such that $I_\alpha f(x)/c_1 \le \widetilde{\gamma}(x)$ for μ-a.e. $x \in X$ and

$$\int_X \widetilde{\Psi}\left(x, \frac{I_\alpha f(x)}{c_1}\right) d\mu(x) \le c_2$$

for all $\alpha \ge \alpha_0$ and nonnegative functions $f \in L^\Phi(X)$ satisfying $\|f\|_{L^\Phi(X)} \le 1$.

Proof. Let f be a nonnegative μ-measurable function on X with $\|f\|_{L^\Phi(X)} \le 1$. Note from Lemma 6.1 that

$$\int_X Mf(x)\, d\mu(x) \le \mu(X) + A_1 A_2 \int_X \Phi(x, Mf(x))\, d\mu(x) \le C_M. \tag{7.1}$$

Fix $x \in X$. For $0 < \delta \le d_X/2$, Lemma 6.3 implies

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\, d\mu(y) + \int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\, d\mu(y) \\ &\le C\left\{\delta^\alpha Mf(x) + \widetilde{\Gamma}\left(x, \frac{1}{\delta}\right)\right\} \end{aligned}$$

with a constant $C > 0$ independent of x.

If $Mf(x) \le 2/d_X$, then we take $\delta = d_X/2$. Then, by Lemma 6.4

$$I_\alpha f(x) \le C\widetilde{\Gamma}\left(x, \frac{2}{d_X}\right).$$

By Lemma 6.5 and (Γ2), there exists $C_1^* > 0$ independent of x such that

$$I_\alpha f(x) \le C_1^* \widetilde{\Gamma}\left(x, t_0\right) \qquad \text{if } Mf(x) \le 2/d_X. \tag{7.2}$$

Next, suppose $2/d_X < Mf(x) < \infty$. Let $m = \sup_{s \ge 2/d_X, x \in X} \widetilde{\Gamma}(x,s)/s$. By ($\widetilde{\Gamma}_{\log}$), $m < \infty$. Define δ by

$$\delta^\alpha = \frac{(d_X/2)^\alpha}{m} \widetilde{\Gamma}(x, Mf(x))(Mf(x))^{-1}.$$

Since $\widetilde{\Gamma}(x, Mf(x))(Mf(x))^{-1} \le m$, $0 < \delta \le d_X/2$. Then by Lemma 6.4 and (Γ2)

$$\begin{aligned}\frac{1}{\delta} &= \frac{m^{1/\alpha}}{d_X/2} \widetilde{\Gamma}(x, Mf(x))^{-1/\alpha}(Mf(x))^{1/\alpha} \\ &\le \frac{m^{1/\alpha}}{d_X/2} B_2^{1/\alpha} \widetilde{\Gamma}(x, 2/d_X)^{-1/\alpha}(Mf(x))^{1/\alpha} \le C(Mf(x))^{1/\alpha}.\end{aligned}$$

Hence, using (Γ2), ($\widetilde{\Gamma}_{\log}$) and Lemma 6.5, we obtain

$$\widetilde{\Gamma}\left(x, \frac{1}{\delta}\right) \le B_2 \widetilde{\Gamma}\big(x, C(Mf(x))^{1/\alpha}\big) \le C\widetilde{\Gamma}(x, Mf(x)).$$

By Lemma 6.5 again, we see from (Γ2) that there exists a constant $C_2^* > 0$ independent of x such that

$$I_\alpha f(x) \le C_2^* \widetilde{\Gamma}\left(x, \frac{t_0 d_X}{2} Mf(x)\right) \qquad \text{if } 2/d_X < Mf(x) < \infty. \tag{7.3}$$

Now, let $c_1 = B_5 B_6 \max(C_1^*, C_2^*)$. Then, by (7.2) and (7.3),

$$\frac{I_\alpha f(x)}{c_1} \le \frac{1}{B_5 B_6} \max\left\{\widetilde{\Gamma}\left(x, t_0\right), \widetilde{\Gamma}\left(x, \frac{t_0 d_X}{2} Mf(x)\right)\right\}$$

whenever $Mf(x) < \infty$. Since $Mf(x) < \infty$ for μ-a.e. $x \in X$ by Lemma 6.1, $I_\alpha f(x)/c_1 \le \widetilde{\gamma}(x)$ μ-a.e. $x \in X$, and by ($\widetilde{\Psi}$2) and ($\widetilde{\Psi}$3), we have

$$\begin{aligned}&\widetilde{\Psi}\left(x, \frac{I_\alpha f(x)}{c_1}\right) \\ &\le \max\left\{\widetilde{\Psi}\left(x, \widetilde{\Gamma}\left(x, t_0\right)/B_6\right), \widetilde{\Psi}\left(x, \widetilde{\Gamma}\left(x, \frac{t_0 d_X}{2} Mf(x)\right)/B_6\right)\right\} \\ &\le B_7 t_0 + \frac{B_7 t_0 d_X}{2} Mf(x)\end{aligned}$$

for μ-a.e. $x \in X$. Thus, we have by (7.1)

$$\int_X \widetilde{\Psi}\left(x, \frac{I_\alpha f(x)}{c_1}\right) d\mu(x) \le B_7 t_0 \mu(X) + \frac{B_7 t_0 d_X}{2} \int_X Mf(x)\, d\mu(x)$$
$$\le B_7 t_0 \mu(X) + \frac{B_7 t_0 d_X C_M}{2} = c_2.$$

□

We obtain the following corollary applying Theorem 7.1 to special Φ given in Example 2.1,

Corollary 7.2. *Let Φ be as in Example 2.1. Asuume that μ is lower Ahlfors $Q(x)$-regular.*

(1) *Suppose there exists an integer $1 \le j_0 \le k$ such that*

$$\inf_{x \in X} (p(x) - q_{j_0}(x) - 1) > 0 \tag{7.4}$$

and

$$\sup_{x \in X} (p(x) - q_j(x) - 1) \le 0 \tag{7.5}$$

for all $j \le j_0 - 1$ in case $j_0 \ge 2$. Then there exist constants $c_1, c_2 > 0$ such that

$$\int_X E_+^{(j_0)} \left(\left(\frac{I_\alpha f(x)}{c_1}\right)^{p(x)/(p(x)-q_{j_0}(x)-1)} \times \prod_{j=1}^{k-j_0} \left(L_e^{(j)}\left(\frac{I_\alpha f(x)}{c_1}\right)\right)^{q_{j_0+j}(x)/(p(x)-q_{j_0}(x)-1)} \right) d\mu(x) \le c_2$$

for all $\alpha \ge Q^+/p^-$ and nonnegative functions $f \in L^\Phi(X)$ satisfying $\|f\|_{L^\Phi(X)} \le 1$.

(2) *If*

$$\sup_{x \in X} (p(x) - q_j(x) - 1) \le 0$$

for all $j = 1, \ldots, k$, then there exist constants $c_1, c_2 > 0$ such that

$$\int_X E^{(k+1)} \left(\left(\frac{I_\alpha f(x)}{c_1}\right)^{p(x)/(p(x)-1)} \right) d\mu(x) \le c_2$$

for all $\alpha \ge Q^+/p^-$ and nonnegative functions $f \in L^\Phi(X)$ satisfying $\|f\|_{L^\Phi(X)} \le 1$.

8 Continuity for Musielak-Orlicz spaces

For a measurable function $Q(\cdot)$ satsfying (6.1), we consider the functions

$$\widetilde{\omega}(x, r) = \int_0^r \rho^\alpha \Phi^{-1}\big(x, \rho^{-Q(x)}\big) \frac{d\rho}{\rho}$$

and

$$\widetilde{\omega}_\theta(x,r) = r^\theta \int_r^{d_X} \rho^{\alpha-\theta}\Phi^{-1}\big(x,\rho^{-Q(x)}\big)\,\frac{d\rho}{\rho}$$

for $\theta > 0$ and $0 < r \le d_X$.

As in the proof of Theorem 5.3, we can obtain the continuity of Riesz potentials $I_\alpha f$ of functions in Musielak-Orlicz spaces under the condition (5.1).

THEOREM 8.1. *Asuume that μ is lower Ahlfors $Q(x)$-regular. Suppose that $\Phi(x,t)$ satisfies ($\Phi5$). Suppose that (5.1) holds. Then there exists a constant $C > 0$ such that*

$$|I_\alpha f(x) - I_\alpha f(z)| \le C\{\widetilde{\omega}(x,d(x,z)) + \widetilde{\omega}(z,d(x,z)) + \widetilde{\omega}_\theta(x,d(x,z))\}$$

for all $x,z \in X$ with $0 < d(x,z) \le d_X/2$ whenever $f \in L^\Phi(X)$ is a nonnegative function on X satisfying $\|f\|_{L^\Phi(X)} \le 1$.

COROLLARY 8.2. *Asuume that μ is lower Ahlfors $Q(x)$-regular. Suppose that $\Phi(x,t)$ satisfies ($\Phi5$). Suppose that (5.1) holds.*

(a) *Let $x_0 \in X$ and suppose $\widetilde{\omega}(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in B(x_0,\delta)\cap X$ for some $\delta > 0$. Then $I_\alpha f$ is continuous at x_0 for every $f \in L^\Phi(X)$.*

(b) *Suppose $\widetilde{\omega}(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in X$. Then $I_\alpha f$ is uniformly continuous on X for every $f \in L^\Phi(X)$.*

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