Trudinger's inequality and continuity of potentials on Musielak-Orlicz-Morrey spaces

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Abstract

In this paper we are concerned with Trudinger's inequality and continuity for general potentials of functions in Musielak-Orlicz-Morrey spaces.

1 Introduction

A famous Trudinger inequality ([34]) insists that Sobolev functions in $W^{1,N}(G)$ satisfy finite exponential integrability, where G is an open bounded set in $\mathbf{R}^N$ (see also [2], [4], [28], [35]). Great progress on Trudinger type inequalities has been made for Riesz potentials of order α $(0 < \alpha < N)$ in the limiting case $\alpha p = N$ (see e.g. [5], [6], [7], [8], [33]). In [3], [20] and [24], Trudinger type exponential integrability was studied on Orlicz spaces, as extensions of [5], [6] and [8], and also on generalized Morrey spaces $L^{1,\varphi}$ in [16] and [17]. For Morrey spaces, which were introduced to estimate solutions of partial differential equations, we refer to [27] and [31]. Further, Trudinger type exponential integrability was also studied on Orlicz-Morrey spaces (see [25] and [30]).

In the mean time, variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth condition. These spaces have attracted more and more attention, in connection with the study of elasticity, fluid mechanics; see [32]. Trudinger type exponential integrability on variable exponent Lebesgue spaces $L^{p(\cdot)}$ was investigated in [9], [10] and [11]. For the two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$, see [19]. These spaces are special cases of so-called Musielak-Orlicz spaces ([29]).

Trudinger type exponential integrability for variable exponent Morrey spaces was also studied in [23], and then the result was extended to the two variable exponents Morrey spaces in [18]. In [18], Riesz kernel of variable order is considered. All the above spaces are special cases of what we call "Musielak-Orlicz-Morrey spaces".

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On the other hand, beginning with Sobolev's embedding theorem (see e.g. [1], [2]), continuity properties of Riesz potentials or Sobolev functions have been studied by many authors. Continuity of Riesz potentials of functions in Orlicz spaces was studied in [8], [14], [15], [21] and [24] (cf. also [22]). Then such continuity was investigated on generalized Morrey spaces $L^{1,\varphi}$ in [16] and [17], on Orlicz-Morrey spaces in [26], on variable exponent Lebesgue spaces in [9], [10] and [12], on two variable exponents Lebesgue spaces in [19], on variable exponent Morrey spaces in [26] and on two variable exponents Morrey spaces in [18].

Our aim in this paper is to give a general version of Trudinger type exponential integrability and continuity for potentials of functions in Musielak-Orlicz-Morrey spaces. We consider a general potential kernel of "variable order". By treating such general setting, we can obtain new results (e.g., Corollary 4.6 and Corollary 5.6 + Example 5.8) which have not been found in the literature.

2 Preliminaries

We denote by $B(x,r)$ the ball $\{y \in \mathbf{R}^N : |y-x| < r\}$ with center x and of radius $r > 0$ and by $|B(x,r)|$ its Lebesgue measure, i.e. $|B(x,r)| = \sigma_N r^N$, where σ_N is the volume of the unit ball in $\mathbf{R}^N$.

Throughout this paper, we fix a bounded open set G. Let $d_G = \operatorname{diam} G$.

We consider a function

$$\Phi(x,t) = t\phi(x,t) : G \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions $(\Phi 1)$ – $(\Phi 4)$:

$(\Phi 1)$ $\phi(\,\cdot\,,t)$ is measurable on G for each $t \ge 0$ and $\phi(x,\,\cdot\,)$ is continuous on $[0,\infty)$ for each $x \in G$;

$(\Phi 2)$ there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \phi(x,1) \le A_1 \quad \text{for all } x \in G;$$

$(\Phi 3)$ $\phi(x,\cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \ge 1$ such that

$$\phi(x,t) \le A_2 \phi(x,s) \quad \text{for all } x \in G \quad \text{whenever } 0 \le t < s;$$

$(\Phi 4)$ there exists a constant $A_3 \ge 1$ such that

$$\phi(x,2t) \le A_3 \phi(x,t) \quad \text{for all } x \in G \text{ and } t > 0.$$

Note that $(\Phi 2)$, $(\Phi 3)$ and $(\Phi 4)$ imply

$$0 < \inf_{x \in G} \phi(x,t) \le \sup_{x \in G} \phi(x,t) < \infty$$

for each $t > 0$. Let $\bar{\phi}(x,t) = \sup_{0 \le s \le t} \phi(x,s)$ and

$$\overline{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)\,dr$$

for $x \in G$ and $t \geq 0$. Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \leq \overline{\Phi}(x,t) \leq A_2\Phi(x,t) \tag{2.1}$$

for all $x \in G$ and $t \geq 0$.

By $(\Phi3)$, we see that

$$\Phi(x,at) \geq A_2^{-1}a\Phi(x,t) \quad \text{if } a \geq 1.$$

We shall also consider the following condition:

$(\Phi5)$ for every $\gamma > 0$, there exists a constant $B_\gamma \geq 1$ such that

$$\phi(x,t) \leq B_\gamma \phi(y,t)$$

whenever $|x-y| \leq \gamma t^{-1/N}$ and $t \geq 1$.

Example 2.1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1,\dots,k$, be measurable functions on G such that

(P1) $1 \leq p^- := \inf_{x\in G} p(x) \leq \sup_{x\in G} p(x) =: p^+ < \infty$

and

(Q1) $-\infty < q_j^- := \inf_{x\in G} q_j(x) \leq \sup_{x\in G} q_j(x) =: q_j^+ < \infty$

for all $j = 1,\dots,k$.

Set $L_a(t) = \log(a+t)$ for $a \geq e$ and $t \geq 0$, $L_a^{(1)}(t) = L_a(t)$, $L_a^{(j+1)}(t) = L_a(L_a^{(j)}(t))$ and

$$\Phi(x,t) = t^{p(x)} \prod_{j=1}^{k} (L_a^{(j)}(t))^{q_j(x)}.$$

Then, $\Phi(x,t)$ satisfies $(\Phi1)$, $(\Phi2)$ and $(\Phi4)$. It satisfies $(\Phi3)$ if there is a constant $K \geq 0$ such that $K(p(x)-1) + q_j(x) \geq 0$ for all $x \in G$ and $j = 1,\dots,k$; in particular if $p^- > 1$ or $q_j^- \geq 0$ for all $j = 1,\dots,k$.

$\Phi(x,t)$ satisfies $(\Phi5)$ if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x)-p(y)| \leq \frac{C_p}{L_e(1/|x-y|)}$$

with a constant $C_p \geq 0$ and

(Q2) $q_j(\cdot)$ is j-log-Hölder continuous, namely

$$|q_j(x)-q_j(y)| \leq \frac{C_{q_j}}{L_e^{(j)}(1/|x-y|)}$$

with constants $C_{q_j} \geq 0$, $j = 1,\dots k$.

We also consider a function $\kappa(x,r) : G \times (0, d_G) \to (0,\infty)$ satisfying the following conditions:

(κ1) $\kappa(x,\cdot)$ is measurable for each $x \in G$;

(κ2) $\kappa(x,\cdot)$ is uniformly almost increasing on $(0, d_G)$, namely there exists a constant $Q_1 \ge 1$ such that

$$\kappa(x,r) \le Q_1 \kappa(x,s)$$

for all $x \in G$ whenever $0 < r < s < d_G$;

(κ3) there is a constant $Q_2 \ge 1$ such that

$$Q_2^{-1} \min(1,\, r^N) \le \kappa(x,r) \le Q_2$$

for all $x \in G$ and $0 < r < d_G$.

EXAMPLE 2.2. Let $\nu(\cdot)$ and $\beta_j(\cdot)$, $j = 1, \dots k$ be measurable functions on G such that $\inf_{x\in G} \nu(x) > 0$, $\sup_{x\in G} \nu(x) \le N$ and $-c_1(N - \nu(x)) \le \beta_j(x) \le c_2$ for all $x \in G,\ j = 1, \dots, k$ and some constants $c_1,\ c_2 > 0$. Then

$$\kappa(x,r) = r^{\nu(x)} \prod_{j=1}^{k} (L_e^{(j)}(1/r))^{\beta_j(x)}$$

satisfies (κ1), (κ2) and (κ3).

For a locally integrable function f on G, define the $L^{\Phi,\kappa}$ norm

$$\|f\|_{L^{\Phi,\kappa}(G)} = \inf\left\{\lambda > 0 : \sup_{x\in G, 0<r<d_G} \frac{\kappa(x,r)}{|B(x,r)|} \int_{G\cap B(x,r)} \overline{\Phi}(y, |f(y)|/\lambda)\, dy \le 1\right\}.$$

Let $L^{\Phi,\kappa}(G)$ denote the set of all functions f such that $\|f\|_{L^{\Phi,\kappa}(G)} < \infty$ (cf. [30]), which we call a Musielak-Orlicz-Morrey space. Note that $L^{\Phi,\kappa}(G)$ is the Musielak-Orlicz space $L^{\Phi}(G)$ if $\kappa(x,r) = r^N$ (cf. [29]).

3 Lemmas

Throughout this paper, let C denote various constants independent of the variables in question.

Set

$$\Phi^{-1}(x,s) = \sup\{t > 0\,;\, \Phi(x,t) < s\}$$

for $x \in G$ and $s > 0$.

LEMMA 3.1 ([13, Lemma 5.1]). *$\Phi^{-1}(x,\cdot)$ is non-decreasing;*

$$\Phi^{-1}(x,\lambda s) \le A_2 \lambda \Phi^{-1}(x,s) \tag{3.1}$$

for all $x \in G,\ s > 0$ and $\lambda \ge 1$;

$$A_2^{-1} t \le \Phi^{-1}(x, \Phi(x,t)) \tag{3.2}$$

for all $x \in G$ and $t > 0$; and

$$\min\left\{1,\ \frac{s}{A_1A_2}\right\} \le \Phi^{-1}(x,s) \le \max\{1,\ A_1A_2 s\} \tag{3.3}$$

for all $x \in G$ and $s > 0$.

LEMMA 3.2. *There exists a constant $C > 0$ such that*

$$C^{-1} \le \Phi^{-1}(x, \kappa(x,r)^{-1}) \le C r^{-N} \tag{3.4}$$

for all $x \in G$ and $0 < r < d_G$.

Proof. By (κ3),

$$Q_2^{-1} \le \kappa(x,r)^{-1} \le Q_2 \max(1, r^{-N})$$

for $x \in G$ and $0 < r < d_G$. Hence, by (3.3), we obtain (3.4). □

LEMMA 3.3 (cf. [13, Lemma 5.3]). *Assume that Φ satisfies (Φ5). Then there exists a constant $C > 0$ such that*

$$\int_{G\cap B(x,r)} f(y)\, dy \le C|B(x,r)|\Phi^{-1}(x, \kappa(x,r)^{-1})$$

for all $x \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(G)} \le 1$.

Proof. Let f be a nonnegative measurable function satisfying $\|f\|_{L^{\Phi,\kappa}(G)} \le 1$. Let $f_1 = f\chi_{\{x: f(x)\ge 1\}}$ and $f_2 = f - f_1$. Since

$$\Phi\left(x, \frac{1}{|B(x,r)|}\int_{G\cap B(x,r)} f_1(y)\, dy\right) \le C\kappa(x,r)^{-1}$$

by [13, Lemma 3.1] and (2.1), we see from (3.1) and (3.2) that

$$\int_{G\cap B(x,r)} f_1(y)\, dy \le C|B(x,r)|\Phi^{-1}(x, \kappa(x,r)^{-1})$$

for all $x \in G$ and $0 < r < d_G$.

On the other hand, by the previous lemma, we see that

$$\int_{G\cap B(x,r)} f_2(y)\, dy \le C|B(x,r)|\Phi^{-1}(x, \kappa(x,r)^{-1})$$

for all $x \in G$ and $0 < r < d_G$, so that we obtain the required result. □

As a potential kernel, we consider a function

$$J(x,r) : G \times (0, d_G] \to (0, \infty)$$

satisfying the following conditions:

(J1) $J(\cdot, r)$ is measurable on G for each $r \in (0, d_G]$;

(J2) $J(x,\cdot)$ is non-increasing on $(0,d_G)$ and $J(x,r) < \lim_{\rho\to 0+} J(x,\rho)$ for all $r>0$, for each $x \in G$;

(J3) $J(x,r) \le C_J r^{-\sigma}$ for $x \in G$ and $0 < r \le d_G$ with constants $0 \le \sigma < N$ and $C_J > 0$.

By (J3), $\int_0^{d_G} J(x,\rho)\rho^{N-1}\,d\rho \le J_0 < \infty$. Set

$$\overline{J}(x,r) = \frac{N}{r^N}\int_0^r J(x,\rho)\rho^{N-1}d\rho$$

for $x \in G$ and $0 < r \le d_G$. Then $\overline{J}(x,\cdot)$ is strictly decreasing and continuous. Further, $J(x,r) \le \overline{J}(x,r) \le C'_J r^{-\sigma}$ for all $x \in G$ and $0 < r \le d_G$. Note that

$$d(-\overline{J}(x,\cdot))(\rho) \le N\rho^{-1}\overline{J}(x,\rho)d\rho \tag{3.5}$$

as measures.

We also assume:

(J4) there is $r_0 \in (0,d_G)$ such that

$$\inf_{x\in G} J(x,r_0) > 0 \quad \text{and} \quad \inf_{x\in G} \frac{\overline{J}(x,r_0)}{\overline{J}(x,d_G)} > 1.$$

Example 3.4. Let $\alpha(\cdot)$ be a measurable function on G such that

$$0 < \alpha^- := \inf_{x\in G}\alpha(x) \le \sup_{x\in G}\alpha(x) =: \alpha^+ < N.$$

Then, $J(x,r) = r^{\alpha(x)-N}$ satisfies (J1) – (J4) (with $\sigma = N - \alpha^-$). In particular, it satisfies (J4) with any $r_0 \in (0,d_G)$.

We consider the function

$$\Gamma(x,s) = \begin{cases} \displaystyle\int_{1/s}^{d_G} \rho^N \Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\,d(-\overline{J}(x,\cdot))(\rho) & \text{if } s \ge 1/r_0, \\ \Gamma(x,1/r_0)r_0 s & \text{if } 0 \le s \le 1/r_0 \end{cases}$$

for every $x \in G$, where r_0 is the number given in (J4). $\Gamma(x,\cdot)$ is strictly increasing and continuous for each $x \in G$.

Lemma 3.5. *There exist positive constants C' and C'' such that*

(a) $\Gamma(x,s) \le C' s^\sigma$ *for all $x \in G$ and $s \ge 1/r_0$ with σ in condition* (J3)*;*

(b) $\Gamma(x,1/r_0) \ge C'' > 0$ *for all $x \in G$.*

Proof. By (3.4) and (J3),

$$\Gamma(x,s) \le C\int_{1/s}^{d_G} d(-\overline{J}(x,\cdot))(\rho) \le C\overline{J}(x,1/s) \le C's^{\sigma}$$

for all $x \in G$ and $s \ge 1/r_0$; and

$$\begin{aligned}\Gamma(x,1/r_0) &\ge C^{-1}\int_{r_0}^{d_G} \rho^N d(-\overline{J}(x,\cdot))(\rho) \ge C^{-1}r_0^N \int_{r_0}^{d_G} d(-\overline{J}(x,\cdot))(\rho)\\ &= C^{-1}r_0^N\big(\overline{J}(x,r_0) - \overline{J}(x,d_G)\big) \ge C'' > 0,\end{aligned}$$

where we used (J4) to obtain the inequalities in the last line. □

LEMMA 3.6. *There exists a constant $C > 0$ such that*

$$\int_{G\setminus B(x,\delta)} J(x,|x-y|)f(y)\,dy \le C\Gamma\left(x,\frac{1}{\delta}\right)$$

for all $x \in G$, $0 < \delta \le r_0$ and nonnegative $f \in L^{\Phi,\kappa}(G)$ with $\|f\|_{L^{\Phi,\kappa}(G)} \le 1$.

Proof. By integration by parts, Lemma 3.3, (3.4), (J4) and Lemma 3.5(b), we have

$$\begin{aligned}\int_{G\setminus B(x,\delta)} J(x,|x-y|)f(y)\,dy &\le \int_{G\setminus B(x,\delta)} \overline{J}(x,|x-y|)f(y)\,dy\\ &\le C\Big\{d_G^N\overline{J}(x,d_G)\Phi^{-1}\big(x,\kappa(x,d_G)^{-1}\big)\\ &\qquad + \int_{\delta}^{d_G} \rho^N\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)d(-\overline{J}(x,\cdot))(\rho)\Big\}\\ &\le C\big\{\Gamma(x,1/r_0)+\Gamma(x,1/\delta)\big\} \le C\Gamma(x,1/\delta).\end{aligned}$$

□

LEMMA 3.7. *Let $0 < \varepsilon < N$ and define*

$$I_\varepsilon f(x) = \int_G |x-y|^{\varepsilon-N}f(y)\,dy$$

for a nonnegative measurable function f on G and

$$\lambda_\varepsilon(z,r) = \frac{1}{1+\displaystyle\int_r^{d_G}\rho^{\varepsilon}\Phi^{-1}(z,\kappa(z,\rho)^{-1})\frac{d\rho}{\rho}}$$

for $z \in G$. Then there exists a constant $C_{I,\varepsilon} > 0$ such that

$$\frac{\lambda_\varepsilon(z,r)}{|B(z,r)|}\int_{G\cap B(z,r)} I_\varepsilon f(x)dx \le C_{I,\varepsilon}$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(G)} \le 1$.

Proof. Let $z \in G$. Let $f(x) = 0$ for $x \in \mathbf{R}^N \setminus G$ and write

$$\begin{aligned} I_\varepsilon f(x) &= \int_{B(z,2r)} |x-y|^{\varepsilon - N} f(y)\,dy + \int_{G\setminus B(z,2r)} |x-y|^{\varepsilon-N} f(y)\,dy \\ &= I_1(x) + I_2(x) \end{aligned}$$

for $x \in G$. By Fubini's theorem,

$$\begin{aligned} \int_{G\cap B(z,r)} I_1(x)\,dx &= \int_{B(z,2r)} \left(\int_{G\cap B(z,r)} |x-y|^{\varepsilon-N}\,dx \right) f(y)\,dy \\ &\le \int_{B(z,2r)} \left(\int_{B(y,3r)} |x-y|^{\varepsilon-N}\,dx \right) f(y)\,dy \\ &\le C \int_{B(z,2r)} \left(\int_0^{3r} t^\varepsilon \frac{dt}{t} \right) f(y)\,dy \\ &\le \frac{C}{\varepsilon} r^\varepsilon \int_{B(z,2r)} f(y)dy. \end{aligned}$$

Now, by Lemma 3.3, ($\kappa2$) and (3.1) we have

$$\begin{aligned} r^\varepsilon \int_{B(z,2r)} f(y)\,dy &\le C r^\varepsilon |B(z,2r)| \Phi^{-1}(z, \kappa(z,2r)^{-1}) \\ &\le C|B(z,r)| \int_r^{2r} \rho^\varepsilon \Phi^{-1}(z,\kappa(z,\rho)^{-1}) \frac{d\rho}{\rho} \end{aligned}$$

if $0 < r < d_G/2$ and, by Lemma 3.3 and (3.4), we have

$$\begin{aligned} r^\varepsilon \int_{B(z,2r)} f(y)\,dy &= r^\varepsilon \int_{B(z,d_G)} f(y)\,dy \\ &\le C{d_G}^\varepsilon |B(z,d_G)| \Phi^{-1}(z,\kappa(z,d_G)^{-1}) \le C|B(z,r)| \end{aligned}$$

if $d_G/2 \le r < d_G$. Therefore

$$\int_{G\cap B(z,r)} I_1(x)\,dx \le \frac{C}{\varepsilon} \frac{|B(z,r)|}{\lambda_\varepsilon(z,r)}$$

for all $0 < r < d_G$.

For I_2, first note that $I_2(x) = 0$ if $x \in G$ and $r \ge d_G/2$. Let $0 < r < d_G/2$. Since

$$I_2(x) \le C \int_{G\setminus B(z,2r)} |z-y|^{\varepsilon-N} f(y)\,dy \quad \text{for} \quad x \in G \cap B(z,r),$$

by integration by parts and Lemma 3.3, we have

$$\begin{aligned} I_2(x) &\le C \left\{ {d_G}^\varepsilon \Phi^{-1}(z,\kappa(z,d_G)^{-1}) + \int_{2r}^{d_G} \rho^\varepsilon \Phi^{-1}(z,\kappa(z,\rho)^{-1}) \frac{d\rho}{\rho} \right\} \\ &\le \frac{C}{\lambda_\varepsilon(z,r)} \end{aligned}$$

for all $x \in G \cap B(z,r)$. Hence

$$\int_{G\cap B(z,r)} I_2(x)\,dx \le C \frac{|B(z,r)|}{\lambda_\varepsilon(z,r)}.$$

Thus this lemma is proved. □

4 Trudinger's inequality

In this section, we deal with the case $\Gamma(x,r)$ satisfies the uniform log-type condition:

($\Gamma_{\log}$) there exists a constant $c_\Gamma > 0$ such that

$$\Gamma(x, s^2) \le c_\Gamma \Gamma(x, s) \tag{4.1}$$

for all $x \in G$ and $s \ge 1$.

EXAMPLE 4.1. Let Φ, κ and J be as in Examples 2.1, 2.2 and 3.4, respectively. Then

$$\Gamma(x,s) \sim \int_{1/s}^{d_G} \rho^{\alpha(x)-\nu(x)/p(x)} \prod_{j=1}^{k} \left[L_e^{(j)}(1/\rho)\right]^{-\{q_j(x)+\beta_j(x)\}/p(x)} \frac{d\rho}{\rho} \qquad (s \ge 1/r_0),$$

so that it satisfies ($\Gamma_{\log}$) if and only if

$$\alpha(x)p(x) \ge \nu(x) \qquad \text{for all } x \in G.$$

(Here $h_1(x,s) \sim h_2(x,s)$ means that $C^{-1}h_2(x,s) \le h_1(x,s) \le Ch_2(x,s)$ for a constant $C > 0$.)

By ($\Gamma_{\log}$), together with Lemma 3.5, we see that $\Gamma(x,s)$ satisfies the uniform doubling condition in s:

LEMMA 4.2. *For every $a > 1$, there exists $b > 0$ such that $\Gamma(x, as) \le b\Gamma(x,s)$ for all $x \in G$ and $s > 0$.*

Proof. If $0 < s < a^{-1}r_0^{-1}$, then

$$\Gamma(x, as) = \Gamma(x, 1/r_0)r_0 as = a\Gamma(x,s).$$

If $a^{-1}r_0^{-1} \le s \le a$, then by Lemma 3.5 we see that $C_1 \le \Gamma(x,s) \le C_2$ with positive constants C_1, C_2 independent of x. Finally, if $s > a$, then we see from ($\Gamma_{\log}$) that

$$\Gamma(x, as) \le \Gamma(x, s^2) \le c_\Gamma \Gamma(x,s).$$

□

For a nonnegative measurable function f on G, its J-potential Jf is defined by

$$Jf(x) = \int_G J(x, |x-y|)f(y)\,dy \qquad (x \in G).$$

Now we consider the following condition (Jε):

(Jε) there exists $0 < \varepsilon < N - \sigma$ such that $r \mapsto r^{N-\varepsilon}J(x,r)$ is uniformly almost increasing on $(0, d_G)$ for σ in condition (J3).

EXAMPLE 4.3. Let J be as in Example 3.4. It satisfies (Jε) with $0 < \varepsilon < \alpha^-$.

THEOREM 4.4. *Assume that* Φ *satisfies* (Φ5), Γ *satisfies* ($\Gamma_{\log}$) *and* J *satisfies* (Jε). *For each* $x \in G$, *let* $\gamma(x) = \sup_{s>0} \Gamma(x,s)$. *Suppose* $\Psi(x,t) : G \times [0,\infty) \to [0,\infty]$ *satisfies the following conditions:*

(Ψ1) $\Psi(\cdot,t)$ *is measurable on* G *for each* $t \in [0,\infty)$; $\Psi(x,\cdot)$ *is continuous on* $[0,\infty)$ *for each* $x \in G$;

(Ψ2) *there is a constant* $A'_1 \geq 1$ *such that* $\Psi(x,t) \leq \Psi(x,A'_1 s)$ *for all* $x \in G$ *whenever* $0 < t < s$;

(Ψ3) $\Psi(x,\Gamma(x,s)/A'_2) \leq A'_3 s$ *for all* $x \in G$ *and* $s > 0$ *with constants* $A'_2,\ A'_3 \geq 1$ *independent of* x.

Then, for ε *given in* (Jε), *there exists a constant* $C^* > 0$ *such that* $Jf(x)/C^* < \gamma(x)$ *for a.e.* $x \in G$ *and*

$$\frac{\lambda_\varepsilon(z,r)}{|B(z,r)|}\int_{G\cap B(z,r)} \Psi\left(x, \frac{Jf(x)}{C^*}\right)\,dx \leq 1$$

for all $z \in G$, $0 < r < d_G$ *and* $f \geq 0$ *satisfying* $\|f\|_{L^{\Phi,\kappa}(G)} \leq 1$.

Proof. Let $f \geq 0$ and $\|f\|_{L^{\Phi,\kappa}(G)} \leq 1$. Set $f = 0$ outside G. Fix $x \in G$. For $0 < \delta \leq r_0$, Lemma 3.6, (Jε) and (J3) imply

$$\begin{aligned} Jf(x) &\leq \int_{B(x,\delta)} J(x,|x-y|)f(y)\,dy + C\Gamma\left(x,\frac{1}{\delta}\right) \\ &= \int_{B(x,\delta)} |x-y|^{N-\varepsilon}J(x,|x-y|)|x-y|^{\varepsilon-N}f(y)\,dy + C\Gamma\left(x,\frac{1}{\delta}\right) \\ &\leq C\left\{\delta^{N-\varepsilon}J(x,\delta)I_\varepsilon f(x) + \Gamma\left(x,\frac{1}{\delta}\right)\right\} \\ &\leq C\left\{\delta^{N-\sigma-\varepsilon}I_\varepsilon f(x) + \Gamma\left(x,\frac{1}{\delta}\right)\right\} \end{aligned}$$

with constants $C > 0$ independent of x.

If $I_\varepsilon f(x) \leq 1/r_0$, then we take $\delta = r_0$. Then, by Lemma 3.5(b)

$$Jf(x) \leq C\Gamma\left(x,\frac{1}{r_0}\right).$$

By Lemma 4.2, there exists $C_1^* > 0$ independent of x such that

$$Jf(x) \leq C_1^*\Gamma\left(x,\frac{1}{2A'_3}\right) \qquad \text{if } I_\varepsilon f(x) \leq 1/r_0. \tag{4.2}$$

Next, suppose $1/r_0 < I_\varepsilon f(x) < \infty$. Let $m = \sup_{s\geq 1/r_0, x\in G}\Gamma(x,s)/s$. By ($\Gamma_{\log}$), $m < \infty$. Define δ by

$$\delta^{N-\sigma-\varepsilon} = \frac{r_0^{N-\sigma-\varepsilon}}{m}\Gamma(x,I_\varepsilon f(x))(I_\varepsilon f(x))^{-1}.$$

Since $\Gamma(x, I_\varepsilon f(x))(I_\varepsilon f(x))^{-1} \le m$, $0 < \delta \le r_0$. Then by Lemma 3.5(b)

$$\begin{aligned}\frac{1}{\delta} &\le C\Gamma(x, I_\varepsilon f(x))^{-1/(N-\sigma-\varepsilon)}(I_\varepsilon f(x))^{1/(N-\sigma-\varepsilon)} \\ &\le C\Gamma(x, 1/r_0)^{-1/(N-\sigma-\varepsilon)}(I_\varepsilon f(x))^{1/(N-\sigma-\varepsilon)} \le C(I_\varepsilon f(x))^{1/(N-\sigma-\varepsilon)}.\end{aligned}$$

Hence, using $(\Gamma_{\log})$ and Lemma 4.2, we obtain

$$\Gamma\left(x, \frac{1}{\delta}\right) \le \Gamma\big(x, C(I_\varepsilon f(x))^{1/(N-\sigma-\varepsilon)}\big) \le C\Gamma(x, I_\varepsilon f(x)).$$

By Lemma 4.2 again, we see that there exists a constant $C_2^* > 0$ independent of x such that

$$Jf(x) \le C_2^*\Gamma\left(x, \frac{1}{2C_{I,\varepsilon}A_3'} I_\varepsilon f(x)\right) \qquad \text{if } 1/r_0 < I_\varepsilon f(x) < \infty, \tag{4.3}$$

where $C_{I,\varepsilon}$ is the constant given in Lemma 3.7.

Now, let $C^* = A_1'A_2'\max(C_1^*, C_2^*)$. Then, by (4.2) and (4.3),

$$\frac{Jf(x)}{C^*} \le \frac{1}{A_1'A_2'}\max\left\{\Gamma\left(x, \frac{1}{2A_3'}\right), \Gamma\left(x, \frac{1}{2C_{I,\varepsilon}A_3'} I_\varepsilon f(x)\right)\right\} \tag{4.4}$$

whenever $I_\varepsilon f(x) < \infty$. Since $I_\varepsilon f(x) < \infty$ for a.e. $x \in G$ by Lemma 3.7, $Jf(x)/C^* < \gamma(x)$ a.e. $x \in G$, and by $(\Psi2)$ and $(\Psi3)$, we have

$$\begin{aligned}&\Psi\left(x, \frac{Jf(x)}{C^*}\right) \\ &\le \max\left\{\Psi\left(x, \Gamma\left(x, \frac{1}{2A_3'}\right)/A_2'\right), \Psi\left(x, \Gamma\left(x, \frac{1}{2C_{I,\varepsilon}A_3'} I_\varepsilon f(x)\right)/A_2'\right)\right\} \\ &\le \frac{1}{2} + \frac{1}{2C_{I,\varepsilon}} I_\varepsilon f(x)\end{aligned}$$

for a.e. $x \in G$. Thus, noting that $\lambda_\varepsilon(z, r) \le 1$ and using Lemma 3.7, we have

$$\begin{aligned}&\frac{\lambda_\varepsilon(z,r)}{|B(z,r)|}\int_{G\cap B(z,r)} \Psi\left(x, \frac{Jf(x)}{C^*}\right) dx \\ &\qquad \le \frac{1}{2}\lambda_\varepsilon(z,r) + \frac{1}{2C_{I,\varepsilon}}\frac{\lambda_\varepsilon(z,r)}{|B(z,r)|}\int_{G\cap B(z,r)} I_\varepsilon f(x)\, dx \\ &\qquad \le \frac{1}{2} + \frac{1}{2} = 1\end{aligned}$$

for all $z \in G$ and $0 < r < d_G$. □

REMARK 4.5. If $\Gamma(x, s)$ is bounded, that is,

$$\sup_{x\in G}\int_0^{d_G} \rho^N \Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big)\, d(-\overline{J}(x,\cdot))(\rho) < \infty,$$

then by Lemma 3.6 we see that $J|f|$ is bounded for every $f \in L^{\Phi,\kappa}(G)$. In particular, if $\lambda_{N-\sigma}(x,r)^{-1}$ is bounded, that is,

$$\sup_{x\in G}\int_0^{d_G} \rho^{N-\sigma}\Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big)\,\frac{d\rho}{\rho} < \infty,$$

then $\Gamma(x, s)$ is bounded by (J3), and hence $J|f|$ is bounded for every $f \in L^{\Phi,\kappa}(G)$.

Applying Theorem 4.4 to special Φ, κ and J given in Examples 2.1, 2.2 and 3.4, we obtain the following corollary, which is an extension of [18, Corollary 5.3]. In fact, [18, Corollary 5.3] is a case $k = 1$ of Corollary 4.6.

COROLLARY 4.6. *Let Φ and κ be as in Examples 2.1 and 2.2 and let α be as in Example 3.4.*

Set

$$I_{\alpha(\cdot)}f(x) = \int_G |x-y|^{\alpha(x)-N} f(y)\,dy$$

for a nonnegative locally integrable function f on G.

Assume that

$$\alpha(x) - \nu(x)/p(x) = 0 \qquad \text{for all } x \in G.$$

(1) *Suppose there exists an integer $1 \le j_0 \le k$ such that*

$$\inf_{x\in G}(p(x) - q_{j_0}(x) - \beta_{j_0}(x)) > 0 \tag{4.5}$$

and

$$\sup_{x\in G}(p(x) - q_j(x) - \beta_j(x)) \le 0 \tag{4.6}$$

for all $j \le j_0 - 1$ in case $j_0 \ge 2$. Then for $0 < \varepsilon < \alpha^-$ there exist constants $C^* > 0$ and $C^{**} > 0$ such that

$$\frac{r^{\nu(z)/p(z)-\varepsilon}}{|B(z,r)|} \int_{G\cap B(z,r)} E_+^{(j_0)}\left(\left(\frac{I_{\alpha(\cdot)}f(x)}{C^*}\right)^{p(x)/(p(x)-q_{j_0}(x)-\beta_{j_0}(x))} \right.$$
$$\left. \times \prod_{j=1}^{k-j_0} \left(L_e^{(j)}\left(\frac{I_{\alpha(\cdot)}f(x)}{C^*}\right)\right)^{(q_{j_0+j}(x)+\beta_{j_0+j}(x))/(p(x)-q_{j_0}(x)-\beta_{j_0}(x))} \right) dx \le C^{**}$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(G)} \le 1$, where $E^{(1)}(t) = e^t - e$, $E^{(j+1)}(t) = \exp(E^j(t)) - e$ and $E_+^{(j)}(t) = \max(E^{(j)}(t), 0)$.

(2) *If*

$$\sup_{x\in G}(p(x) - q_j(x) - \beta_j(x)) \le 0$$

for all $j = 1, \ldots, k$, then for $0 < \varepsilon < \alpha^-$ there exist constants $C^* > 0$ and $C^{**} > 0$ such that

$$\frac{r^{\nu(z)/p(z)-\varepsilon}}{|B(z,r)|} \int_{G\cap B(z,r)} E^{(k+1)}\left(\frac{I_{\alpha(\cdot)}f(x)}{C^*}\right) dx \le C^{**}$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(G)} \le 1$.

REMARK 4.7. [16, Remark 2.8] shows that we cannot take $\varepsilon = \alpha^-$ in the above corollary.

Proof of Corollary 5.6. By Example 4.1,

$$\Gamma(x,s) \sim \int_{1/s}^{d_G} \prod_{j=1}^{k} \left[L_e^{(j)}(1/\rho)\right]^{-\{q_j(x)+\beta_j(x)\}/p(x)} \frac{d\rho}{\rho}$$

for $s \ge 1/r_0$. We shall show

$$\Gamma(x,s) \le C_1 \Gamma_1(x,s) \tag{4.7}$$

for $s \ge 1/r_0$, where

$$\Gamma_1(x,s) = \left[L_e^{(j_0)}(s)\right]^{1-\{q_{j_0}(x)+\beta_{j_0}(x)\}/p(x)} \prod_{j=j_0+1}^{k} \left[L_e^{(j)}(s)\right]^{-\{q_j(x)+\beta_j(x)\}/p(x)}.$$

To prove the assertion of (1), assume (4.5) and (4.6). Let $\rho > 1/s$. By (4.6), $\left[L_e^{(j)}(1/\rho)\right]^{-\{q_j(x)+\beta_j(x)\}/p(x)} \le \left[L_e^{(j)}(1/\rho)\right]^{-1}$ for $1 \le j \le j_0 - 1$. By (4.5), we find $\varepsilon_0 > 0$ such that $\inf_{x\in G}\{1 - \{q_{j_0}(x) + \beta_{j_0}(x)\}/p(x)\} > \varepsilon_0$. Since

$$t \mapsto \left[L_e^{(j_0)}(t)\right]^{1-\{q_{j_0}(x)+\beta_{j_0}(x)\}/p(x)-\varepsilon_0} \prod_{j=j_0+1}^{k} \left[L_e^{(j)}(t)\right]^{-\{q_j(x)+\beta_j(x)\}/p(x)}$$

is uniformly almost increasing,

$$\begin{aligned}
\Gamma(x,s) &\le C \int_{1/s}^{d_G} \prod_{j=1}^{k} \left[L_e^{(j)}(1/\rho)\right]^{-\{q_j(x)+\beta_j(x)\}/p(x)} \frac{d\rho}{\rho} \\
&\le C \left[L_e^{(j_0)}(s)\right]^{1-\{q_{j_0}(x)+\beta_{j_0}(x)\}/p(x)-\varepsilon_0} \prod_{j=j_0+1}^{k} \left[L_e^{(j)}(s)\right]^{-\{q_j(x)+\beta_j(x)\}/p(x)} \\
&\quad \times \int_{1/s}^{d_G} \left(\prod_{j=1}^{j_0-1} \left[L_e^{(j)}(1/\rho)\right]^{-1} \right) \left[L_e^{(j_0)}(1/\rho)\right]^{-1+\varepsilon_0} \frac{d\rho}{\rho} \\
&\le C_1 \Gamma_1(x,s),
\end{aligned}$$

which shows (4.7).

Now, set

$$\psi(x,t) = t^{p(x)/\{p(x)-q_{j_0}(x)-\beta_{j_0}(x)\}} \prod_{i=1}^{k-j_0} \left[L_e^{(i)}(t)\right]^{\{q_{j_0+i}(x)+\beta_{j_0+i}(x)\}/\{p(x)-q_{j_0}(x)-\beta_{j_0}(x)\}}$$

for $x \in G$ and $t > 0$. Then

$$\psi(x, \Gamma_1(x,s)) \le C_2 L_e^{(j_0)}(s)$$

for $s \ge 1/r_0$.

Since $\inf_{x\in G} p(x)/\{p(x) - q_{j_0}(x) - \beta_{j_0}(x)\} > 0$, there are constants $0 < \theta \le 1$ and $C_3 \ge 1$ such that

$$\psi(x, at) \le C_3 a^\theta \psi(x,t) \tag{4.8}$$

for all $x \in G$, $t > 0$ and $0 < a \le 1$. Hence, choosing $A' \ge 1$ such that $C_3^2 C_2 (C_1/A')^\theta \le 1$, we have

$$\begin{aligned}
\psi(x, \Gamma(x,s)/A') &\le C_3 \psi(x, (C_1/A')\Gamma_1(x,s)) \\
&\le C_3^2 (C_1/A')^\theta \psi(x, \Gamma_1(x,s)) \le C_3^2 (C_1/A')^\theta C_2 L_e^{(j_0)}(s) \le L_e^{(j_0)}(s)
\end{aligned}$$

for $s \ge 1/r_0$. Thus,

$$E^{(j_0)}(\psi(x, \Gamma(x,s)/A')) \le s \quad \text{for } s \ge 1/r_0. \tag{4.9}$$

Let $u_0 > 0$ be the unique solution of the equation $e^u - e = u$. Then $E(u) \ge u_0$ if and only if $u \ge u_0$. Choose $t_0 > 0$ such that $\psi(x,t) \ge u_0$ for $t \ge t_0$ and define

$$\Psi(x,t) = \begin{cases} E^{(j_0)}(\psi(x,t)) & \text{for } t \ge t_0, \\ \Psi(x,t_0)\dfrac{t}{t_0} & \text{for } 0 < t < t_0. \end{cases}$$

Then, $\Psi(x,t)$ satisfies (Ψ1), (Ψ2) (with $A_1' = C_3^{1/\theta}$, say) and (Ψ3), in view of (4.8) and (4.9).

In the present situation, we see that

$$\lambda_{\varepsilon'}(z,r) \sim r^{\nu(z)/p(z)-\varepsilon'} \prod_{j=1}^{k} \left[L_e^{(j)}(1/r)\right]^{\{q_j(z)+\beta_j(z)\}/p(z)}$$

for $0 < \varepsilon' < \alpha^-$, so that

$$r^{\nu(z)/p(z)-\varepsilon} \le C_4 \lambda_{\varepsilon'}(z,r)$$

if $0 < \varepsilon < \varepsilon' < \alpha^-$. Thus, given $0 < \varepsilon < \alpha^-$, Theorem 4.4 implies the existence of a constant $C^* > 0$ such that

$$\frac{r^{\nu(z)/p(z)-\varepsilon}}{|B(z,r)|} \int_{G \cap B(z,r)} \Psi\left(x, \frac{I_{\alpha(\cdot)}f(x)}{C^*}\right) dx \le C_4$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(G)} \le 1$. Let $S_f = \{x \in G : I_{\alpha(\cdot)}f(x) \ge C^* t_0\}$. Then

$$\begin{aligned} &\frac{r^{\nu(z)/p(z)-\varepsilon}}{|B(z,r)|} \int_{G \cap B(z,r)} E_+^{(j_0)}\left(\psi\left(x, \frac{I_{\alpha(\cdot)}f(x)}{C^*}\right)\right) dx \\ &\le \frac{C_5}{|B(z,r)|} \int_{B(z,r)\setminus S_f} dx + \frac{r^{\nu(z)/p(z)-\varepsilon}}{|B(z,r)|} \int_{S_f \cap B(z,r)} \Psi\left(x, \frac{I_{\alpha(\cdot)}f(x)}{C^*}\right) dx \\ &\le C_5 + C_4 = C^{**} \end{aligned}$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(G)} \le 1$, which shows the assertion of (1).

The case (2) can be considered as the case (1) with $j_0 = k+1$ and $q_{k+1}(x) = \beta_{k+1}(x) \equiv 0$. □

5 Continuity

In this section, we discuss the continuity of potentials Jf under the condition

(J5) there are $0 < \theta \le 1$ and $C > 0$ such that

$$|J(x,r) - J(z,s)| \le C\left(\frac{|x-z|}{r}\right)^{\theta} \overline{J}(x,r) \quad \text{whenever} \quad |r-s| \le |x-z| \le r/2$$

for $x, z \in G$, $0 < r,\ s < d_G$.

We consider the functions

$$\omega(x,r) = \int_0^r \rho^{N-1}\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\,\overline{J}(x,\rho)d\rho$$

and

$$\omega_\theta(x,r) = r^\theta \int_r^{d_G} \rho^{N-1-\theta}\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\,\overline{J}(x,\rho)d\rho$$

for $\theta > 0$ and $0 < r \le d_G$.

LEMMA 5.1. *Let $E \subset G$. If $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in E$, then $\omega_\theta(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in E$.*

Proof. Suppose $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in E$. Given $\varepsilon > 0$ there is $\delta > 0$ ($\delta \le d_G$) such that $\omega(x,\delta) < \varepsilon/2$ for all $x \in E$. Set $g(x,\rho) = \rho^{N-1}\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\,\overline{J}(x,\rho)$. By Lemma 3.2 and (J3),

$$C_\delta := \sup_{x\in G,\,\delta\le\rho\le d_G} g(x,\rho) < \infty.$$

If $0 < r \le \delta$ and $x \in E$, then

$$\begin{aligned}\omega_\theta(x,r) = r^\theta \int_r^{d_G} \rho^{-\theta} g(x,\rho)d\rho &\le \int_r^\delta g(x,\rho)d\rho + \left(\frac{r}{\delta}\right)^\theta \int_\delta^{d_G} g(x,\rho)d\rho \\ &\le \omega(x,\delta) + \left(\frac{r}{\delta}\right)^\theta C_\delta d_G < \frac{\varepsilon}{2} + \left(\frac{r}{\delta}\right)^\theta C_\delta d_G.\end{aligned}$$

Choosing $\delta' > 0$ ($\delta' \le \delta$) such that $(\delta'/\delta)^\theta C_\delta d_G < \varepsilon/2$, we see that $\omega_\theta(x,r) < \varepsilon$ for all $x \in E$ and $0 < r \le \delta'$, which means that $\omega_\theta(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in E$. □

LEMMA 5.2. *There exists a constant $C > 0$ such that*

$$\omega(x,2r) \le C\omega(x,r)$$

for all $x \in G$ and $0 < r \le d_G/2$.

Proof. By (κ2), (3.1) and the fact that $\overline{J}(x,\cdot)$ is strictly decreasing, we have

$$\begin{aligned}\omega(x,2r) &= \int_0^{2r} \rho^{N-1}\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\overline{J}(x,\rho)d\rho \\ &= C\int_0^r \rho^{N-1}\Phi^{-1}\big(x,\kappa(x,2\rho)^{-1}\big)\overline{J}(x,2\rho)d\rho \\ &\le C\int_0^r \rho^{N-1}\Phi^{-1}\big(x,Q_1\kappa(x,\rho)^{-1}\big)\overline{J}(x,\rho)d\rho \\ &\le C\int_0^r \rho^{N-1}\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\overline{J}(x,\rho)d\rho = C\omega(x,r),\end{aligned}$$

as required. □

THEOREM 5.3. *Suppose that J satisfies* (J5). *Then there exists a constant $C > 0$ such that*

$$|Jf(x) - Jf(z)| \le C\{\omega(x, |x-z|) + \omega(z, |x-z|) + \omega_\theta(x, |x-z|)\}$$

for all $x, z \in G$ with $|x-z| < d_G/4$ and nonnegative $f \in L^{\Phi,\kappa}(G)$ with $\|f\|_{L^{\Phi,\kappa}(G)} \le 1$.

Before giving a proof of Theorem 5.3, we prepare two more lemmas.

LEMMA 5.4. *There exists a constant $C > 0$ such that*

$$\int_{B(x,r)} J(x, |x-y|) f(y)\, dy \le C\omega(x, r)$$

for all $x \in G$, $0 < r \le d_G$ and nonnegative $f \in L^{\Phi,\kappa}(G)$ with $\|f\|_{L^{\Phi,\kappa}(G)} \le 1$.

Proof. By integration by parts, Lemma 3.3 and (3.5), we have

$$\begin{aligned}
&\int_{B(x,r)} J(x, |x-y|) f(y)\, dy \le \int_{B(x,r)} \overline{J}(x, |x-y|) f(y)\, dy \\
\le\ & C\Big\{ r^N \overline{J}(x, r) \Phi^{-1}\big(x, \kappa(x,r)^{-1}\big) \\
&\quad + \int_0^r \rho^N \Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big) d(-\overline{J}(x, \cdot))(\rho) \Big\} \\
\le\ & C\Big\{ r^N \overline{J}(x, r) \Phi^{-1}\big(x, \kappa(x,r)^{-1}\big) + \omega(x, r) \Big\}.
\end{aligned}$$

In view of (κ2) and (3.1), we have

$$\begin{aligned}
\omega(x, r) &\ge \Phi^{-1}\big(x, Q_1^{-1}\kappa(x,r)^{-1}\big) \overline{J}(x, r) \int_0^r \rho^{N-1} d\rho \\
&\ge C r^N \overline{J}(x, r) \Phi^{-1}\big(x, \kappa(x,r)^{-1}\big).
\end{aligned}$$

Hence we have the required inequality. □

LEMMA 5.5. *Let $0 < \theta \le 1$. Then there exists a constant $C > 0$ such that*

$$\int_{G \setminus B(x,r)} |x-y|^{-\theta} \overline{J}(x, |x-y|) f(y)\, dy \le C r^{-\theta} \omega_\theta(x, r)$$

for all $x \in G$, $0 < r \le d_G/2$ and nonnegative $f \in L^{\Phi,\kappa}(G)$ with $\|f\|_{L^{\Phi,\kappa}(G)} \le 1$.

Proof. Let $\widetilde{J}(x, r) = r^{-\theta} \overline{J}(x, r)$. Then, $\widetilde{J}(x, \cdot)$ is continuous, strictly decreasing and by (3.5)

$$d(-\widetilde{J}(x, \cdot))(\rho) = \theta \rho^{-\theta-1} \overline{J}(x, \rho)\, d\rho + \rho^{-\theta} d(-\overline{J}(x, \cdot)(\rho) \le (N+1) \rho^{-\theta-1} \overline{J}(x, \rho)\, d\rho$$

as measures. Hence, by integration by parts and Lemma 3.3, we have

$$\begin{aligned}
&\int_{G \setminus B(x,r)} |x-y|^{-\theta} \overline{J}(x, |x-y|) f(y)\, dy \\
\le\ & C\Big\{ d_G^{N-\theta} \overline{J}(x, d_G) \Phi^{-1}\big(x, \kappa(x, d_G)^{-1}\big) \\
&\quad + \int_r^{d_G} \rho^{N-\theta-1} \Phi^{-1}\big(x, \kappa(x,\rho)^{-1}\big) \overline{J}(x, \rho) d\rho \Big\}.
\end{aligned}$$

In view of (κ2) and (3.1), we have

$$\begin{aligned}
&\int_r^{d_G} \rho^{N-\theta-1}\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\overline{J}(x,\rho)d\rho \\
\geq\ & d_G^{-\theta}\Phi^{-1}\big(x,Q_1^{-1}\kappa(x,d_G)^{-1}\big)\overline{J}(x,d_G)\int_{d_G/2}^{d_G}\rho^{N-1}d\rho \\
\geq\ & C d_G^{N-\theta}\overline{J}(x,d_G)\Phi^{-1}\big(x,\kappa(x,d_G)^{-1}\big)
\end{aligned}$$

if $r \leq d_G/2$. Hence

$$\begin{aligned}
&\int_{G\setminus B(x,r)} |x-y|^{-\theta}\overline{J}(x,|x-y|)f(y)\,dy \\
\leq\ & C\int_r^{d_G} \rho^{N-1-\theta}\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\overline{J}(x,\rho)d\rho = Cr^{-\theta}\omega_\theta(x,r),
\end{aligned}$$

as required. □

Proof of Theorem 5.3. Let $f \in L^{\Phi,\kappa}(G)$ be nonnegative and $\|f\|_{L^{\Phi,\kappa}(G)} \leq 1$. Write

$$\begin{aligned}
& Jf(x) - Jf(z) \\
=\ & \int_{B(x,2|x-z|)} J(x,|x-y|)f(y)\,dy - \int_{B(x,2|x-z|)} J(z,|z-y|)f(y)\,dy \\
& + \int_{G\setminus B(x,2|x-z|)} (J(x,|x-y|) - J(z,|z-y|))f(y)\,dy
\end{aligned}$$

for $x, z \in G$. By Lemma 5.4 and Lemma 5.2, we have

$$\int_{B(x,2|x-z|)} J(x,|x-y|)f(y)\,dy \leq C\omega(x,|x-z|),$$

and

$$\begin{aligned}
\int_{B(x,2|x-z|)} J(z,|z-y|)f(y)\,dy &\leq \int_{B(z,3|x-z|)} J(z,|z-y|)f(y)\,dy \\
&\leq C\omega(z,|x-z|).
\end{aligned}$$

On the other hand, we have by (J5), Lemma 5.5 and Lemma 5.2,

$$\begin{aligned}
&\int_{G\setminus B(x,2|x-z|)} |J(x,|x-y|) - J(z,|z-y|)|f(y)\,dy \\
\leq\ & C|x-z|^{\theta}\int_{G\setminus B(x,2|x-z|)} |x-y|^{-\theta}\overline{J}(x,|x-y|)f(y)\,dy \\
\leq\ & C\omega_\theta(x,2|x-z|) \leq C\omega_\theta(x,|x-z|)
\end{aligned}$$

if $|x-z| < d_G/4$.

Thus we have the conclusion of the theorem. □

In view of Lemma 5.1, we obtain

Corollary 5.6. *Assume that J satisfies* (J5).

(a) *Let $x_0 \in G$ and suppose $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in B(x_0,\delta)\cap G$ for some $\delta > 0$. Then Jf is continuous at x_0 for every $f \in L^{\Phi,\kappa}(G)$.*

(b) *Suppose $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in G$. Then Jf is uniformly continuous on G for every $f \in L^{\Phi,\kappa}(G)$.*

Remark 5.7. Let $E \subset G$. If there exist $\delta \in (0, d_G)$ and a measurable function $h(r)$ on $(0,\delta)$ such that

$$\Phi^{-1}\big(x, \kappa(x,r)^{-1}\big)\, \overline{J}(x,r) \le h(r)$$

for all $x \in E$ and $0 < r < \delta$ and

$$\int_0^\delta \rho^{N-1} h(\rho)\, d\rho < \infty,$$

then $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in E$.

In this case, $\Gamma(x,s)$ is bounded on $E \times (0,\infty)$.

Applying Theorem 5.3 to special Φ, κ and J given in Examples 2.1, 2.2 and 3.4, we obtain the following Example, which is an extension of [18, section 6]. In [18, section 6], a case $k = 1$ is dealt with .

Example 5.8 (cf. [18, section 6]). Let Φ, κ and J be as in Examples 2.1, 2.2 and 3.4. J satisfies (J5) if α is θ-Hölder continuous. Since

$$\omega(x,r) \sim \int_0^r \rho^{\alpha(x)-\nu(x)/p(x)} \prod_{j=1}^{k} \big[L_e^{(j)}(1/\rho)\big]^{-\{q_j(x)+\beta_j(x)\}/p(x)} \frac{d\rho}{\rho},$$

$\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in E$ $(E \subset G)$ if either

$$\inf_{x\in E}\left(\alpha(x) - \frac{\nu(x)}{p(x)}\right) > 0,$$

or

$$\inf_{x\in E}\left(\alpha(x) - \frac{\nu(x)}{p(x)}\right) = 0, \quad \sup_{x\in E} \frac{q_j(x)+\beta_j(x)}{p(x)} \le 1,\ j = 1,\dots,j_0-1,$$

and

$$\inf_{x\in E} \frac{q_{j_0}(x)+\beta_{j_0}(x)}{p(x)} > 1$$

for some $1 \le j_0 \le k$.

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