Boundedness of maximal operators and Sobolev's inequality on Musielak-Orlicz-Morrey spaces

Fumi-Yuki Maeda, Yoshihiro Mizuta,
Takao Ohno and Tetsu Shimomura

Abstract

Our aim in this paper is to deal with the boundedness of the Hardy-Littlewood maximal operator on Musielak-Orlicz-Morrey spaces. As an application of the boundedness of the maximal operator, we establish a generalization of Sobolev's inequality for general potentials of functions in Musielak-Orlicz-Morrey spaces.

1 Introduction

For a locally integrable function f on $\mathbf{R}^N$, the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is the ball in $\mathbf{R}^N$ with center x and of radius $r > 0$ and $|B(x,r)|$ denotes its Lebesgue measure. The mapping $f \mapsto Mf$ is called the maximal operator.

The maximal operator is a classical tool in harmonic analysis and studying Sobolev functions and partial differential equations and plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on (see [4, 9, 10, 25], etc.).

It is well known that the maximal operator is bounded on the Lebesgue space $L^p(\mathbf{R}^N)$ if $p > 1$ (see [25]). In [5] and [19], the boundedness of the maximal operator was generalized by replacing Lebesgue space by Morrey space, where Morrey space was introduced to estimate solutions of partial differential equations. For Morrey spaces, we refer to [17] and [23]; also cf. [16]. Further, the boundedness of the maximal operator was also studied on Orlicz-Morrey spaces (see [20, 21, 22]).

In the mean time, variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard

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growth condition. These spaces have attracted more and more attention, in connection with the study of elasticity, fluid mechanics; see [24]. Boundedness of the maximal operator on variable exponent Lebesgue spaces $L^{p(\cdot)}$ was investigated in [6] and [7], and then their results were extended to the two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$ in [11] and [14]. These spaces are special cases of so-called Musielak-Orlicz spaces ([18]). For general Musielak-Orlicz spaces, Diening [8] gave a sufficient condition for the maximal operator to be bounded. However that condition is not easy to verify for the above special cases.

The boundedness of the maximal operator was also studied for variable exponent Morrey spaces (see [3, 12, 15]). All the above spaces are special cases of what we call "the Musielak-Orlicz-Morrey spaces". Our first aim in this paper is to show that the maximal operator M is bounded on Musielak-Orlicz-Morrey spaces.

One of important applications of the boundedness of the maximal operator is Sobolev's inequality; in the classical case,

$$\|I_\alpha * f\|_{p^*} \le C\|f\|_p$$

for $f \in L^p(\mathbf{R}^N)$, $0 < \alpha < N$ and $1 < p < N/\alpha$, where I_α is the Riesz kernel of order α and $1/p^* = 1/p - \alpha/N$ (see, e.g. [2, Theorem 3.1.4]).

Sobolev's inequality for Morrey spaces was given by D. R. Adams [1] (also [5] and [19]): For $0 < \alpha < N$, $1 < p < N/\alpha$ and $0 < \lambda < N - \alpha p$,

$$\|I_\alpha * f\|_{q,\lambda} \le C\|f\|_{p,\lambda} \quad \text{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N-\lambda}.$$

This result was extended to Orlicz-Morrey spaces and generalized Riesz kernel by E. Nakai [20]. On the other hand, variable exponent versions were discussed on bounded open sets in [3], [12], [15], etc.. In [3] and [12], Riesz kernel of variable order is also considered. Variable exponent version on $\mathbf{R}^N$ has been given in [13].

As an application of the boundedness of M, we shall give a general version of Sobolev's inequality for potentials of functions in Musielak-Orlicz-Morrey spaces. We consider a general potential kernel of "variable order".

2 Preliminaries

We consider a function

$$\Phi(x,t) = t\phi(x,t) : \mathbf{R}^N \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions $(\Phi1)$ – $(\Phi4)$:

$(\Phi1)$ $\phi(\,\cdot\,,t)$ is measurable on $\mathbf{R}^N$ for each $t \ge 0$ and $\phi(x,\,\cdot\,)$ is continuous on $[0,\infty)$ for each $x \in \mathbf{R}^N$;

$(\Phi2)$ there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \phi(x,1) \le A_1 \quad \text{for all } x \in \mathbf{R}^N;$$

($\Phi 3$) $\phi(x,\cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \ge 1$ such that

$$\phi(x,t) \le A_2 \phi(x,s) \quad \text{for all } x \in \mathbf{R}^N \quad \text{whenever } 0 \le t < s;$$

($\Phi 4$) there exists a constant $A_3 \ge 1$ such that

$$\phi(x,2t) \le A_3 \phi(x,t) \quad \text{for all } x \in \mathbf{R}^N \text{ and } t > 0.$$

Note that ($\Phi 2$), ($\Phi 3$) and ($\Phi 4$) imply

$$0 < \inf_{x \in \mathbf{R}^N} \phi(x,t) \le \sup_{x \in \mathbf{R}^N} \phi(x,t) < \infty$$

for each $t > 0$.

If $\Phi(x,\cdot)$ is convex for each $x \in \mathbf{R}^N$, then ($\Phi 3$) holds with $A_2 = 1$; namely $\phi(x,\cdot)$ is non-decreasing for each $x \in \mathbf{R}^N$.

Let $\bar{\phi}(x,t) = \sup_{0 \le s \le t} \phi(x,s)$ and

$$\overline{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)\,dr$$

for $x \in \mathbf{R}^N$ and $t \ge 0$. Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \le \overline{\Phi}(x,t) \le A_2 \Phi(x,t) \tag{2.1}$$

for all $x \in \mathbf{R}^N$ and $t \ge 0$.

By ($\Phi 3$), we see that

$$\Phi(x,at) \begin{cases} \le A_2 a \Phi(x,t) & \text{if } 0 \le a \le 1 \\ \ge A_2^{-1} a \Phi(x,t) & \text{if } a \ge 1. \end{cases} \tag{2.2}$$

We shall also consider the following conditions:

($\Phi 5$) for every $\gamma > 0$, there exists a constant $B_\gamma \ge 1$ such that

$$\phi(x,t) \le B_\gamma \phi(y,t)$$

whenever $|x-y| \le \gamma t^{-1/N}$ and $t \ge 1$;

($\Phi 6$) there exist a function $g \in L^1(\mathbf{R}^N)$ and a constant $B_\infty \ge 1$ such that $0 \le g(x) < 1$ for all $x \in \mathbf{R}^N$ and

$$B_\infty^{-1}\Phi(x,t) \le \Phi(x',t) \le B_\infty \Phi(x,t)$$

whenever $|x'| \ge |x|$ and $g(x) \le t \le 1$.

Example 2.1. Let $p(\cdot)$ and $q(\cdot)$ be measurable functions on $\mathbf{R}^N$ such that

(P1) $1 \le p^- := \operatorname{ess\,inf}_{x\in\mathbf{R}^N} p(x) \le \operatorname{ess\,sup}_{x\in\mathbf{R}^N} p(x) =: p^+ < \infty$

and

(Q1) $-\infty < q^- := \operatorname{ess\,inf}_{x\in\mathbf{R}^N} q(x) \le \operatorname{ess\,sup}_{x\in\mathbf{R}^N} q(x) =: q^+ < \infty.$

Then, $\Phi_{p(\cdot),q(\cdot),a}(x,t) = t^{p(x)}(\log(a+t))^{q(x)}$ $(a \ge e)$ satisfies (Φ1), (Φ2) and (Φ4). It satisfies (Φ3) if $p^- > 1$ or $q^- \ge 0$. As a matter of fact, it satisfies (Φ3) if and only if $q(x) \ge 0$ at points x where $p(x) = 1$ and

$$\sup_{x:p(x)>1,q(x)<0} q(x)\log(p(x)-1) < \infty.$$

$\Phi_{p(\cdot),q(\cdot),a}(x,t)$ satisfies (Φ5) if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x)-p(y)| \le \frac{C_p}{\log(1/|x-y|)} \quad \text{for } |x-y| \le \frac{1}{2}$$

with a constant $C_p \ge 0$,

and

(Q2) $q(\cdot)$ is log-log-Hölder continuous, namely

$$|q(x)-q(y)| \le \frac{C_q}{\log(\log(1/|x-y|))} \quad \text{for } |x-y| \le e^{-2}$$

with a constant $C_q \ge 0$.

$\Phi_{p(\cdot),q(\cdot),a}(x,t)$ satisfies (Φ6) with $g(x) = 1/(1+|x|)^{N+1}$ if $p(\cdot)$ is log-Hölder continuous at ∞, namely if it satisfies

(P3) $|p(x)-p(x')| \le \dfrac{C_\infty}{\log(e+|x|)}$ whenever $|x'| \ge |x|$ with a constant $C_\infty \ge 0$.

In fact, if $1/(1+|x|)^{N+1} < t \le 1$, then $t^{-|p(x)-p(x')|} \le e^{(N+1)C_\infty}$ for $|x'| \ge |x|$ and $(\log(a+t))^{|q(x)-q(x')|} \le (\log(a+1))^{q^+-q^-}$.

Given $\Phi(x,t)$ as above, the associated Musielak-Orlicz space

$$L^\Phi(\mathbf{R}^N) = \left\{ f \in L^1_{loc}(\mathbf{R}^N)\,;\, \int_{\mathbf{R}^N} \Phi\big(y,|f(y)|\big)\,dy < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_\Phi = \inf\left\{ \lambda > 0\,;\, \int_{\mathbf{R}^N} \overline{\Phi}\big(y,|f(y)|/\lambda\big)\,dy \le 1 \right\}$$

(cf. [18]).

We also consider a function $\kappa(x,r) : \mathbf{R}^N \times (0,\infty) \to (0,\infty)$ satisfying the following conditions:

(κ1) there is a constant $Q_1 \ge 1$ such that

$$\kappa(x, 2r) \le Q_1 \kappa(x, r)$$

for all $x \in \mathbf{R}^N$ and $r > 0$;

(κ2) $r \mapsto r^{-\varepsilon}\kappa(x, r)$ is uniformly almost increasing on $(0, \infty)$ for some $\varepsilon > 0$, namely there exists a constant $Q_2 \ge 1$ such that

$$r^{-\varepsilon}\kappa(x, r) \le Q_2 s^{-\varepsilon}\kappa(x, s)$$

for all $x \in \mathbf{R}^N$ whenever $0 < r < s$;

(κ3) there is a constant $Q_3 \ge 1$ such that

$$Q_3^{-1} \min(1, r^N) \le \kappa(x, r) \le Q_3 \max(1, r^N)$$

for all $x \in \mathbf{R}^N$ and $r > 0$.

EXAMPLE 2.2. Let $\nu(\cdot)$ and $\beta(\cdot)$ be functions on $\mathbf{R}^N$ such that $\inf_{x\in\mathbf{R}^N} \nu(x) > 0$, $\sup_{x\in\mathbf{R}^N} \nu(x) \le N$ and $-c(N - \nu(x)) \le \beta(x) \le c(N - \nu(x))$ for all $x \in \mathbf{R}^N$ and some constant $c > 0$. Then $\kappa(x, r) = r^{\nu(x)}(\log(e + r + 1/r))^{\beta(x)}$ satisfies (κ1), (κ2) and (κ3).

Condition (κ2) implies that $\kappa(x, \cdot)$ is uniformly almost increasing on $(0, \infty)$ and $\kappa(x, r) \to \infty$ uniformly as $r \to \infty$. Further, if $\kappa(x, \cdot)$ is measurable for every $x \in \mathbf{R}^N$, then (κ2) implies

$$\int_r^\infty \frac{1}{\kappa(x,\rho)} \frac{d\rho}{\rho} \le \frac{Q_2}{\varepsilon} \frac{1}{\kappa(x, r)} \tag{2.3}$$

for all $x \in \mathbf{R}^N$ and $r > 0$.

REMARK 2.3. Conversely, if $\kappa(x, r)$ satisfies (κ1) and

$$\int_r^\infty \frac{1}{\kappa(x,\rho)} \frac{d\rho}{\rho} \le Q \frac{1}{\kappa(x, r)}$$

for all $x \in \mathbf{R}^N$ and $r > 0$, then we can show that $\kappa(x, r)$ satisfies (κ2) with $\varepsilon = 1/Q$.

Given $\Phi(x, t)$ and $\kappa(x, r)$, we define the Musielak-Orlicz-Morrey space $L^{\Phi,\kappa}(\mathbf{R}^N)$ by

$$L^{\Phi,\kappa}(\mathbf{R}^N) = \left\{ f \in L^1_{loc}(\mathbf{R}^N)\,;\ \sup_{x\in\mathbf{R}^N,\ r>0} \frac{\kappa(x,r)}{|B(x,r)|} \int_{B(x,r)} \Phi\big(y, |f(y)|\big)\, dy < \infty \right\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{\Phi,\kappa} = \inf \left\{ \lambda > 0\,;\ \sup_{x\in\mathbf{R}^N,\ r>0} \frac{\kappa(x,r)}{|B(x,r)|} \int_{B(x,r)} \overline{\Phi}\big(y, |f(y)|/\lambda\big)\, dy \le 1 \right\}$$

(cf. [20]).

Note that $L^{\Phi,\kappa}(\mathbf{R}^N) = L^\Phi(\mathbf{R}^N)$ if $\kappa(x, r) = r^N$.

Proposition 2.4.

$$L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N) \subset L^{\Phi,\kappa}(\mathbf{R}^N).$$

Proof. Let $f \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. We may assume that $\|f\|_\infty \le 1$.

If $0 < r \le 1$, then by (κ3), (Φ2) and (Φ3),

$$\frac{\kappa(x,r)}{|B(x,r)|} \int_{B(x,r)} \Phi\big(y, |f(y)|\big)\, dy \le Q_3 A_1 A_2 < \infty.$$

If $r > 1$, then by (κ3), (Φ2) and (2.2)

$$\frac{\kappa(x,r)}{|B(x,r)|} \int_{B(x,r)} \Phi\big(y, |f(y)|\big)\, dy \le \frac{Q_3 r^N}{|B(x,r)|} A_1 A_2 \int_{\mathbf{R}^N} |f(y)|\, dy \le C\|f\|_1 < \infty.$$

Hence $f \in L^{\Phi,\kappa}(\mathbf{R}^N)$. □

3 Lemmas

For a nonnegative $f \in L^1_{loc}(\mathbf{R}^N)$, let

$$I(f;x,r) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$$

and

$$J(f;x,r) = \frac{1}{|B(x,r)|} \int_{B(x,r)} \Phi\big(y, f(y)\big)\, dy$$

in this section.

Lemma 3.1. *Suppose $\Phi(x,t)$ satisfies (Φ5). Then there exists a constant $C > 0$ such that*

$$\Phi\big(x, I(f;x,r)\big) \le C J(f;x,r)$$

for all $x \in \mathbf{R}^N$, $r > 0$ and for all nonnegative $f \in L^1_{loc}(\mathbf{R}^N)$ such that $f(y) \ge 1$ or $f(y) = 0$ for each $y \in \mathbf{R}^N$ and $\|f\|_{\Phi,\kappa} \le 1$.

Proof. Given f as in the statement of the lemma, $x \in \mathbf{R}^N$ and $r > 0$, set $I = I(f;x,r)$ and $J = J(f;x,r)$. Note that $\|f\|_{\Phi,\kappa} \le 1$ implies $J \le 2A_3\kappa(x,r)^{-1}$ by (2.1).

By (Φ2) and (2.2), $\Phi\big(y, f(y)\big) \ge (A_1A_2)^{-1} f(y)$, since $f(y) \ge 1$ or $f(y) = 0$. Hence $I \le A_1A_2J$. Thus, if $J \le 1$, then

$$\Phi(x, I) \le (A_1A_2J)A_2\phi(x, A_1A_2) \le CJ.$$

Next, suppose $J > 1$. Since $\Phi(x,t) \to \infty$ as $t \to \infty$, there exists $K \ge 1$ such that

$$\Phi(x, K) = \Phi(x, 1)J.$$

Then $K \le A_2 J$ by (2.2). With this K, we have

$$\int_{B(x,r)} f(y)\, dy \le K|B(x,r)| + A_2 \int_{B(x,r)} f(y) \frac{\phi\big(y, f(y)\big)}{\phi(y, K)}\, dy.$$

Since $\kappa(x,r)J \le 2A_3$, $\kappa(x,r) < 2A_3$. Since $\kappa(x,r) \to \infty$ uniformly as $r \to \infty$, there is $R > 0$ such that $\kappa(y,\rho) > 2A_3$ for all $y \in \mathbf{R}^N$ and $\rho > R$. Then $0 < r \le R$, so that

$$1 \le K \le A_2 J \le 2A_2A_3\kappa(x,r)^{-1} \le Cr^{-N}$$

with a constant $C > 0$ by (κ3). Hence, by (Φ5) there is $\beta > 0$, independent of f, x, r, such that

$$\phi(x,K) \le \beta\phi(y,K) \quad \text{for all } y \in B(x,r).$$

Thus, we have

$$\begin{aligned}\int_{B(x,r)} f(y)\,dy &\le K|B(x,r)| + \frac{A_2\beta}{\phi(x,K)}\int_{B(x,r)} f(y)\phi\big(y,f(y)\big)\,dy \\ &= K|B(x,r)| + A_2\beta|B(x,r)|\frac{J}{\phi(x,K)} \\ &= K|B(x,r)|\left(1+\frac{A_2\beta}{\phi(x,1)}\right) \le K|B(x,r)|\left(1+A_1A_2\beta\right).\end{aligned}$$

Therefore

$$I \le (1+A_1A_2\beta)K,$$

so that by (Φ2), (Φ3) and (Φ4)

$$\Phi(x,I) \le C\Phi(x,K) \le CJ$$

with constants $C > 0$ independent of f, x, r, as required. □

LEMMA 3.2. *Suppose* $\Phi(x,t)$ *satisfies* (Φ6). *Then there exists a constant* $C > 0$ *such that*

$$\Phi\big(x, I(f;x,r)\big) \le C\left\{J(f;x,r) + \Phi(x,g(x))\right\}$$

for all $x \in \mathbf{R}^N$, $r > 0$ *and for all nonnegative* $f \in L^1_{loc}(\mathbf{R}^N)$ *such that* $g(y) \le f(y) \le 1$ *or* $f(y) = 0$ *for each* $y \in \mathbf{R}^N$, *where* g *is the function appearing in* (Φ6).

Proof. Given f as in the statement of the lemma, $x \in \mathbf{R}^N$ and $r > 0$, let $I = I(f;x,r)$ and $J = J(f;x,r)$.

By Jensen's inequality, we have

$$\overline{\Phi}(x,I) \le \frac{1}{|B(x,r)|}\int_{B(x,r)} \overline{\Phi}\big(x,f(y)\big)\,dy.$$

In view of (2.1),

$$\Phi(x,I) \le 2A_2A_3\frac{1}{|B(x,r)|}\int_{B(x,r)} \Phi\big(x,f(y)\big)\,dy.$$

If $|x| \ge |y|$, then $\Phi\big(x,f(y)\big) \le B_\infty\Phi\big(y,f(y)\big)$ by (Φ6).

Let $|x| < |y|$. If $g(x) < f(y)$, then $\Phi\big(x,f(y)\big) \le B_\infty\Phi\big(y,f(y)\big)$ by (Φ6) again. If $g(x) \ge f(y)$, then $\Phi\big(x,f(y)\big) \le A_2\Phi\big(x,g(x)\big)$ by (Φ3). Hence,

$$\Phi\big(x,f(y)\big) \le C\left\{\Phi\big(y,f(y)\big) + \Phi\big(x,g(x)\big)\right\}$$

in any case. Therefore, we obtain the required inequality. □

4 Boundedness of the maximal operator

THEOREM 4.1. *Suppose that* $\Phi(x,t)$ *satisfies* ($\Phi5$), ($\Phi6$) *and further assume:*

($\Phi3^*$) $t \mapsto t^{-\varepsilon_0}\phi(x,t)$ *is uniformly almost increasing on* $(0,\infty)$ *for some* $\varepsilon_0 > 0$.

Then the maximal operator M *is bounded from* $L^{\Phi,\kappa}(\mathbf{R}^N)$ *into itself, namely, there is a constant* $C > 0$ *such that*

$$\|Mf\|_{\Phi,\kappa} \le C\|f\|_{\Phi,\kappa}$$

for all $f \in L^{\Phi,\kappa}(\mathbf{R}^N)$.

We use the following result which is a special case of the theorem when $\Phi(x,t) = t^{p_0}$ $(p_0 > 1)$ (see [19, Theorem 1]):

LEMMA 4.2. *Let* $p_0 > 1$. *Then there exists a constant* $C > 0$ *for which the following holds: If* f *is a measurable function such that*

$$\int_{B(x,r)} |f(y)|^{p_0}\, dy \le |B(x,r)|\kappa(x,r)^{-1}$$

for all $x \in \mathbf{R}^N$ *and* $r > 0$, *then*

$$\int_{B(x,r)} [Mf(y)]^{p_0}\, dy \le C|B(x,r)|\kappa(x,r)^{-1}$$

for all $x \in \mathbf{R}^N$ *and* $r > 0$.

REMARK 4.3. In the proof of [19, Theorem 1], a condition like (2.3) is used. Modifying its proof, we can prove this result without the measurability of $\kappa(\cdot,r)$.

Proof of Theorem 4.1. Set $p_0 = 1 + \varepsilon_0$ for $\varepsilon_0 > 0$ in condition ($\Phi3^*$) and consider the function

$$\Phi_0(x,t) = \Phi(x,t)^{1/p_0}.$$

Then $\Phi_0(x,t)$ also satisfies all the conditions (Φj), $j = 1,2,\dots,6$. In fact, it trivially satisfies (Φj) for $j = 1,2,4,5,6$ with the same g for ($\Phi6$). Since

$$\Phi_0(x,t) = t\phi_0(x,t) \quad \text{with} \quad \phi_0(x,t) = \left[t^{-\varepsilon_0}\phi(x,t)\right]^{1/p_0},$$

condition ($\Phi3^*$) implies that $\Phi_0(x,t)$ satisfies ($\Phi3$).

Let $f \ge 0$ and $\|f\|_{\Phi,\kappa} \le 1$. Let $f_1 = f\chi_{\{x:f(x)\ge 1\}}$, $f_2 = f\chi_{\{x:g(x)\le f(x)<1\}}$ with g in ($\Phi6$) and $f_3 = f - f_1 - f_2$, where χ_E is the characteristic function of E.

Since $\Phi(x,t) \ge 1/(A_1A_2)$ for $t \ge 1$,

$$\Phi_0(x,t) \le (A_1A_2)^{1-1/p_0}\Phi(x,t)$$

if $t \ge 1$. Hence there is a constant $\lambda > 0$ such that $\|f_1\|_{\Phi_0,\kappa} \le \lambda$ whenever $\|f\|_{\Phi,\kappa} \le 1$. Applying Lemma 3.1 to Φ_0 and f_1/λ, we have

$$\Phi_0\big(x, Mf_1(x)\big) \le CM\Phi_0\big(\cdot, f_1(\cdot)\big)(x),$$

so that

$$\Phi\big(x, Mf_1(x)\big) \le C\Big[M\Phi_0\big(\cdot, f(\cdot)\big)(x)\Big]^{p_0} \tag{4.1}$$

for all $x \in \mathbf{R}^N$ with a constant $C > 0$ independent of f.

Next, applying Lemma 3.2 to Φ_0 and f_2, we have

$$\Phi_0(x, Mf_2(x)) \le C\left[M\Phi_0\big(\cdot, f_2(\cdot)\big)(x) + \Phi_0\big(x, g(x)\big)\right].$$

Noting that $\Phi_0(x, g(x)) \le Cg(x)$ by (2.2) and (Φ2), we have

$$\Phi\big(x, Mf_2(x)\big) \le C\left\{\Big[M\Phi_0\big(\cdot, f(\cdot)\big)(x)\Big]^{p_0} + g(x)^{p_0}\right\} \tag{4.2}$$

for all $x \in \mathbf{R}^N$ with a constant $C > 0$ independent of f.

Since $0 \le f_3 \le g \le 1$, $0 \le Mf_3 \le Mg \le 1$. Hence we have

$$\Phi\big(x, Mf_3(x)\big) \le A_2\Phi_0(x, Mg(x))^{p_0} \le C[Mg(x)]^{p_0} \tag{4.3}$$

for all $x \in \mathbf{R}^N$ with a constant $C > 0$ independent of f.

Combining (4.1), (4.2) and (4.3), and noting that $g(x) \le Mg(x)$ for a.e. $x \in \mathbf{R}^N$, we obtain

$$\Phi\big(x, Mf(x)\big) \le C\left\{\Big[M\Phi_0\big(\cdot, f(\cdot)\big)(x)\Big]^{p_0} + [Mg(x)]^{p_0}\right\} \tag{4.4}$$

for a.e. $x \in \mathbf{R}^N$ with a constant $C > 0$ independent of f.

In view of (2.1),

$$\int_{B(x,r)} \Phi_0(y, f(y))^{p_0}\, dy = \int_{B(x,r)} \Phi(y, f(y))\, dy \le 2A_3|B(x,r)|\kappa(x,r)^{-1}$$

for all $x \in \mathbf{R}^N$ and $r > 0$. Hence, applying Lemma 4.2 to $(2A_3)^{-1/p_0}\Phi_0(y, f(y))$, we have

$$\int_{B(x,r)} \Big[M\Phi_0\big(\cdot, f(\cdot)\big)(y)\Big]^{p_0} dy \le C|B(x,r)|\kappa(x,r)^{-1}$$

with a constant $C > 0$ independent of x, r and f.

Applying Proposition 2.4 with $\Phi(x,t) = t^{p_0}$ and Lemma 4.2 to g, we obtain

$$\int_{B(x,r)} [Mg(y)]^{p_0}\, dy \le C|B(x,r)|\kappa(x,r)^{-1}$$

for all $x \in \mathbf{R}^N$ and $r > 0$.

Thus, by (4.4), we finally obtain

$$\int_{B(x,r)} \Phi\big(y, Mf(y)\big)\, dy \le C|B(x,r)|\kappa(x,r)^{-1}$$

for all $x \in \mathbf{R}^N$ and $r > 0$. This completes the proof of the theorem. $\square$

Taking $\kappa(x,r) = r^N$ in the above theorem, we have

COROLLARY 4.4. *If $\Phi(x,t)$ satisfies the same conditions as in Theorem 4.1, then the maximal operator M is bounded from $L^{\Phi}(\mathbf{R}^N)$ into itself, namely, there is a constant $C>0$ such that*

$$\|Mf\|_{\Phi} \le C\|f\|_{\Phi}$$

for $f \in L^{\Phi}(\mathbf{R}^N)$.

EXAMPLE 4.5. Let $p_j(\cdot)$, $j=1,\dots,m$, satisfy (P1), (P2) and (P3), and $q_j(\cdot)$, $j=1,\dots,m$, satisfy (Q1) and (Q2). Further assume that $p_j^- > 1$ for all j. For positive numbers b_j, $j=1,\dots,m$, set

$$\Phi_{\{p_j(\cdot)\},\{q_j(\cdot)\},\{b_j\}}(x,t) = \sum_{j=1}^{m} b_j t^{p_j(x)} (\log(e+t))^{q_j(x)}.$$

This function satisfies all the conditions $(\Phi1)$ – $(\Phi5)$ and $(\Phi6)$ with $g(x) = 1/(1+|x|)^{N+1}$. It satisfies $(\Phi3^*)$ for $0 < \varepsilon_0 < \min_j p_j^- - 1$.

5 Lemmas for Sobolev's inequality

We begin with the following lemma:

LEMMA 5.1. *Let $F(x,t)$ be a positive function on $\mathbf{R}^N \times (0,\infty)$ satisfying the following conditions:*

(F1) *$F(x,\cdot)$ is continuous on $(0,\infty)$ for each $x \in \mathbf{R}^N$;*

(F2) *$t \mapsto t^{-\varepsilon}F(x,t)$ is uniformly almost increasing for $\varepsilon > 0$; namely there exists a constant $K_1 \ge 1$ such that*

$$t^{-\varepsilon}F(x,t) \le K_1 s^{-\varepsilon} F(x,s) \quad \text{for all } x \in \mathbf{R}^N \quad \text{whenever } 0<t<s;$$

(F3) *there exists a constant $K_2 \ge 1$ such that*

$$K_2^{-1} \le F(x,1) \le K_2 \quad \text{for all } x \in \mathbf{R}^N.$$

Set

$$F^{-1}(x,s) = \sup\{t>0\,;\, F(x,t) < s\}$$

for $x \in \mathbf{R}^N$ and $s>0$. Then:

(1) *$F^{-1}(x,\cdot)$ is non-decreasing.*

(2)

$$F^{-1}(x,\lambda s) \le (K_1\lambda)^{1/\varepsilon} F^{-1}(x,s) \tag{5.1}$$

for all $x \in \mathbf{R}^N$, $s>0$ and $\lambda \ge 1$.

(3)

$$F(x, F^{-1}(x,t)) = t \tag{5.2}$$

for all $x \in \mathbf{R}^N$ and $t>0$.

(4)

$$K_1^{-1/\varepsilon} t \le F^{-1}(x, F(x,t)) \le K_1^{2/\varepsilon} t \tag{5.3}$$

for all $x \in \mathbf{R}^N$ *and* $t > 0$.

(5)

$$\min\left\{1, \left(\frac{s}{K_1 K_2}\right)^{1/\varepsilon}\right\} \le F^{-1}(x,s) \le \max\{1, (K_1 K_2 s)^{1/\varepsilon}\} \tag{5.4}$$

for all $x \in \mathbf{R}^N$ *and* $s > 0$.

Proof. (1) is obvious from the definition of $F^{-1}(x,s)$ and (3) is an easy consequence of the definition of $F^{-1}(x,s)$ and the continuity of $F(x,\cdot)$.

(2) Let $\lambda \ge 1$ and $0 < t < F^{-1}(x, \lambda s)$. Then there is t' with $t < t' \le F^{-1}(x, \lambda s)$ such that $F(x,t') < \lambda s$. Then by (F2)

$$s > \frac{1}{\lambda} F(x,t') \ge F(x, t'/(K_1\lambda)^{1/\varepsilon}),$$

so that $t'/(K_1\lambda)^{1/\varepsilon} \le F^{-1}(x,s)$. Letting $t \to F^{-1}(x,\lambda s)$, we obtain (5.1).

(4) If $F(x,t') < K_1^{-1} F(x,t)$, then $t' < t$ by (F2). Hence

$$F^{-1}(x, K_1^{-1} F(x,t)) \le t.$$

Then, using (5.1), we have

$$F^{-1}(x, F(x,t)) \le K_1^{2/\varepsilon} F^{-1}(x, K_1^{-1} F(x,t)) \le K_1^{2/\varepsilon} t.$$

On the other hand, if $s < K_1^{-1/\varepsilon} t$, then $s < t$, so that by (F2)

$$F(x,s) < (K_1^{-1/\varepsilon})^{\varepsilon} K_1 F(x,t) = F(x,t).$$

Hence $F^{-1}(x, F(x,t)) \ge s$. Letting $s \to K_1^{-1/\varepsilon} t$, we have

$$F^{-1}(x, F(x,t)) \ge K_1^{-1/\varepsilon} t.$$

(5) First consider the case $F^{-1}(x,s) < 1$. Then, for any t with $F^{-1}(x,s) < t < 1$, we find by (F2) and (F3)

$$s \le F(x,t) \le K_1 K_2\, t^{\varepsilon},$$

so that

$$\left(\frac{s}{K_1 K_2}\right)^{1/\varepsilon} \le F^{-1}(x,s) \le 1.$$

In the case $F^{-1}(x,s) > 1$, for every t with $1 < t < F^{-1}(x,s)$ there exists $\underline{t}$ with $t < \underline{t} \le F^{-1}(x,s)$ such that $F(x,\underline{t}) < s$. In view of (F2) and (F3), we have

$$\frac{1}{K_1 K_2} \underline{t}^{\varepsilon} \le F(x,\underline{t}) < s,$$

so that

$$1 < t^{\varepsilon} < \underline{t}^{\varepsilon} \le K_1 K_2 s.$$

Letting $t \to F^{-1}(x,s)$, we have the second inequality in (5.4). □

REMARK 5.2. $F(x,t) = \Phi(x,t)$ satisfies (F1), (F2) and (F3) with $\varepsilon = 1$. $F(x,t) = \kappa(x,t)$ satisfies (F2) and (F3).

Hereafter, we assume (Φ5), (Φ6) and

(κ4) $\kappa(x,\cdot)$ is continuous for each $x \in \mathbf{R}^N$,

i.e., condition (F1) for $F = \kappa$.

Set $g^*(x) = \max\big(g(x),\, Mg(x)\big)$ for the function g appearing in condition (Φ6). We consider the function

$$w(x) := \kappa^{-1}(x, \Phi(x, ag^*(x))^{-1}), \quad x \in \mathbf{R}^N,$$

where $0 < a \le 1$.

LEMMA 5.3. *There exists a constant $C > 0$ (which may depend on a) such that*

$$\int_{B(x,r)} f(y)dy \le C|B(x,r)|\Phi^{-1}(x, \kappa(x,r)^{-1})$$

for all $x \in \mathbf{R}^N$, $0 < r \le w(x)$ and $f \ge 0$ satisfying $\|f\|_{\Phi,\kappa} \le 1$.

Proof. Let f be a nonnegative measurable function satisfying $\|f\|_{\Phi,\kappa} \le 1$. Set $f_1 = f\chi_{\{x:f(x)\ge 1\}}$, $f_2 = f\chi_{\{x:g(x)\le f(x)<1\}}$ and $f_3 = f - f_1 - f_2$. Let

$$I_i = \frac{1}{|B(x,r)|}\int_{B(x,r)} f_i(y)\,dy, \quad i = 1,2,3,$$

$I = I_1 + I_2 + I_3$ and

$$J = \frac{1}{|B(x,r)|}\int_{B(x,r)} \Phi(y, f(y))\,dy.$$

By Lemma 3.1,

$$\Phi(x, I_1) \le CJ \le C\kappa(x,r)^{-1}$$

and by Lemma 3.2,

$$\Phi(x, I_2) \le C(J + \Phi(x, g(x))) \le C(\kappa(x,r)^{-1} + \Phi(x, g(x)))$$

with constants $C > 0$ independent of x, r, f.

As to I_3, since $I_3 \le Mf_3(x) \le Mg(x)$, we have

$$\Phi(x, I_3) \le A_2\Phi(x, Mg(x)).$$

Hence

$$\Phi(x, I) \le C\big(\kappa(x,r)^{-1} + \Phi(x, g^*(x))\big) \quad \text{for all } x \in \mathbf{R}^N. \tag{5.5}$$

If $0 < r \le w(x)$, then by (κ2) and (5.2)

$$\kappa(x,r) \le C\kappa(x, w(x)) = C\Phi(x, ag^*(x))^{-1},$$

so that $\Phi(x, ag^*(x)) \le C\kappa(x,r)^{-1}$. By ($\Phi$4), $\Phi(x, g^*(x)) \le C\Phi(x, ag^*(x))$ (with $C > 0$ which may depend on a), and hence $\Phi(x, I) \le C\kappa(x,r)^{-1}$ by (5.5), which implies

$$I \le C\Phi^{-1}(x, \kappa(x,r)^{-1})$$

by Lemma 5.1 with $F = \Phi$. Thus we obtain the required inequality.

□

We consider a continuous function $\Phi_\infty(t) = t\phi_\infty(t) : [0,\infty) \to [0,\infty)$ such that $\phi_\infty(t) > 0$ for $t > 0$, $\phi_\infty(t)$ is almost increasing on $[0,\infty)$ and satisfies the doubling condition. We further assume:

($\Phi_\infty 1$) There exists a constant $\tilde{B}_\infty \geq 1$ such that

$$\tilde{B}_\infty^{-1}\Phi(x,t) \leq \Phi_\infty(t) \leq \tilde{B}_\infty\Phi(x,t) \quad \text{whenever } g(x) \leq t \leq 1$$

for $g(x)$ in condition ($\Phi 6$).

Note that if $\Phi_\infty(t)$ is continuous on $[0,\infty)$ and if there exists a sequence $\{x_n\}$ such that $|x_n| \to \infty$ and $\lim_{n\to\infty} \Phi(x_n,t) = \Phi_\infty(t)$ for all $t > 0$, then it satisfies the above conditions.

Lemma 5.4. *Assume:*

($\Phi_\infty 2$) *There exists a constant $c_\infty \geq 1$ such that*

$$\Phi_\infty(g^*(x)) \leq c_\infty(1+|x|)^{-N}$$

for all $x \in \mathbf{R}^N$.

Then there are constants $C_1 > 0$ and $C_2 > 0$, which are independent of a, such that

$$w(x) \geq C_1(1+|x|) \quad \text{and} \quad g^*(y) \leq C_2\Phi_\infty^{-1}(\kappa(x,1+|y|)^{-1}) \tag{5.6}$$

for all $x,\ y \in \mathbf{R}^N$.

Proof. By ($\Phi 3$), ($\Phi_\infty 1$) and ($\Phi_\infty 2$),

$$\Phi(x,ag^*(x)) \leq A_2\Phi(x,g^*(x)) \leq A_2\tilde{B}_\infty\Phi_\infty(g^*(x)) \leq A_2\tilde{B}_\infty c_\infty(1+|x|)^{-N}.$$

Hence, using ($\kappa 3$) and Lemma 5.1 with $F = \kappa$, we have

$$\begin{aligned} w(x) &= \kappa^{-1}(x,\Phi(x,ag^*(x))^{-1}) \\ &\geq \kappa^{-1}(x,C(1+|x|)^N) \geq \kappa^{-1}(x,C\kappa(x,1+|x|)) \geq C_1(1+|x|) \end{aligned}$$

with a constant $C_1 > 0$ independent of x and a.

Next, by ($\kappa 3$) and ($\Phi_\infty 2$)

$$\Phi_\infty(g^*(y)) \leq c_\infty Q_3\kappa(x,1+|y|)^{-1}.$$

Hence by Lemma 5.1 with $F(x,t) = \Phi_\infty(t)$, we have

$$g^*(y) \leq C_2\Phi_\infty^{-1}(\kappa(x,1+|y|)^{-1})$$

with $C_2 > 0$ independent of $x,\ y$.

□

Remark 5.5. Condition ($\Phi_\infty 2$) is satisfied if $g(x) = (1+|x|)^{-\gamma}$ with $\gamma > N$. In fact, $g^*(x) = Mg(x) \leq \min\{1, C(1+|x|)^{-\gamma}\}$ in this case, so that

$$\Phi_\infty(g^*(x)) = g^*(x)\phi_\infty(g^*(x)) \leq C(1+|x|)^{-\gamma}\phi_\infty(1) \leq C(1+|x|)^{-N}.$$

LEMMA 5.6. *Assume* $(\Phi_\infty 2)$ *and*

$(\Phi_\infty\kappa)$ $r \mapsto r^\gamma \Phi_\infty^{-1}(\kappa(x,r)^{-1})$ *is uniformly almost increasing on* $[1,\infty)$ *for some* $0<\gamma<N$.

Then there exists a constant $C>0$ *(independent of* a*) such that*

$$\int_{B(x,r)} f(y)dy \le Cr^N \Phi_\infty^{-1}(\kappa(x,r)^{-1})$$

for all $x \in \mathbf{R}^N$, $r \ge w(x)$ *and* $f \ge 0$ *satisfying* $\|f\|_{\Phi,\kappa} \le 1$.

Proof. Let f be a nonnegative measurable function satisfying $\|f\|_{\Phi,\kappa} \le 1$.

Given $x \in \mathbf{R}^N$, set

$$k(y) = \min\left\{1,\, C_2\Phi_\infty^{-1}(\kappa(x,1+|y|)^{-1})\right\}$$

with $C_2>0$ given in Lemma 5.4. Then by $(\Phi 3)$

$$\int_{B(x,r)} f(y)\,dy \le \int_{B(x,r)} k(y)\,dy + A_2\int_{B(x,r)} f(y)\frac{\phi(y,f(y))}{\phi(y,k(y))}\,dy.$$

If $r \ge w(x)$, then $r \ge C_1(1+|x|)$ by (5.6), so that $|y| < |x|+r \le (1+1/C_1)r-1$ for $y \in B(x,r)$. Hence

$$\begin{aligned}\int_{B(x,r)} k(y)\,dy &\le C_2\int_{B(0,(1+1/C_1)r)} \Phi_\infty^{-1}(\kappa(x,1+|y|)^{-1})\,dy\\ &= C\int_0^{(1+1/C_1)r} \rho^N\Phi_\infty^{-1}(\kappa(x,1+\rho)^{-1})\,\frac{d\rho}{\rho}.\end{aligned}$$

Noting that $1+(1+1/C_1)r \le (1+2/C_1)r$ and using $(\kappa 2)$, $(\Phi_\infty\kappa)$ and (5.1) with $F(x,t)=\Phi_\infty(t)$, we have

$$\begin{aligned}\int_{B(x,r)} k(y)\,dy &\le Cr^\gamma\Phi_\infty^{-1}(\kappa(x,r)^{-1})\int_0^{(1+1/C_1)r} \rho^{N-\gamma}\,\frac{d\rho}{\rho}\\ &= Cr^N\Phi_\infty^{-1}(\kappa(x,r)^{-1}).\end{aligned}$$

Since $g(y) \le g^*(y) \le k(y) \le 1$ by (5.6),

$$\phi(y,k(y)) \ge \tilde{B}_\infty^{-1}\phi_\infty(k(y)) \tag{5.7}$$

for all $y \in \mathbf{R}^N$ by $(\Phi_\infty 1)$.

Since $1+|y| < (1+1/C_1)r$ for $y \in B(x,r)$, $(\kappa 2)$ and $(\kappa 1)$ imply $\kappa(x,1+|y|) \le C\kappa(x,r)$, and hence by Lemma 5.1 with $F(x,t)=\Phi_\infty(t)$

$$\Phi_\infty^{-1}(\kappa(x,1+|y|)^{-1}) \ge C\Phi_\infty^{-1}(\kappa(x,r)^{-1})$$

for all $y \in B(x,r)$ with a constant $C>0$ (independent of x, y and r). Hence,

$$k(y) \ge \min\left\{1,\, C\Phi_\infty^{-1}(\kappa(x,r)^{-1})\right\},$$

so that by the doubling condition for ϕ_∞

$$\begin{aligned}\phi_\infty(k(y)) &\geq C\min\{1, \phi_\infty(\Phi_\infty^{-1}(\kappa(x,r)^{-1}))\} \\ &= C\min\left\{1, \frac{1}{\kappa(x,r)\Phi_\infty^{-1}(\kappa(x,r)^{-1})}\right\}\end{aligned}$$

with a constant $C > 0$. Thus, in view of (5.7),

$$\frac{1}{\phi(y,k(y))} \leq C\max\{1, \kappa(x,r)\Phi_\infty^{-1}(\kappa(x,r)^{-1})\},$$

and hence

$$\begin{aligned}&\int_{B(x,r)} f(y)\frac{\phi(y,f(y))}{\phi(y,k(y))}\,dy \\ &\leq C\max\{1, \kappa(x,r)\Phi_\infty^{-1}(\kappa(x,r)^{-1})\}\int_{B(x,r)}\Phi(y,f(y))\,dy \\ &\leq C|B(x,r)|\max\{\kappa(x,r)^{-1}, \Phi_\infty^{-1}(\kappa(x,r)^{-1})\}.\end{aligned}$$

Since $r \geq C_1$ as seen above, $\kappa(x,r)^{-1}$ is bounded by (κ3), so that $\Phi_\infty^{-1}(\kappa(x,r)^{-1}) \geq C\kappa(x,r)^{-1}$ by (5.4) with $F(x,t) = \Phi_\infty(t)$. Therefore

$$\int_{B(x,r)} f(y)\frac{\phi(y,f(y))}{\phi(y,k(y))}\,dy \leq C|B(x,r)|\Phi_\infty^{-1}(\kappa(x,r)^{-1}).$$

This completes the proof.

□

6 Sobolev's inequality

As a potential kernel, we consider a function

$$J(x,r) : \mathbf{R}^N \times (0,\infty) \to [0,\infty)$$

satisfying the following conditions:

(J1) $J(\cdot,r)$ is measurable on $\mathbf{R}^N$ for each $r \in (0,\infty)$;

(J2) $J(x,\cdot)$ is non-increasing on $(0,\infty)$ for each $x \in \mathbf{R}^N$;

(J3) $\int_0^1 J(x,r)r^{N-1}dr < \infty$ for every $x \in \mathbf{R}^N$.

Example 6.1. Let $\alpha(\cdot)$ be a measurable function on $\mathbf{R}^N$ such that

$$0 < \alpha^- := \inf_{x\in\mathbf{R}^N}\alpha(x) \leq \sup_{x\in\mathbf{R}^N}\alpha(x) =: \alpha^+ < N.$$

Then, $J(x,r) = r^{\alpha(x)-N}$ satisfies (J1), (J2) and (J3).

For a nonnegative measurable function f on $\mathbf{R}^N$, its J-potential Jf is defined by

$$Jf(x) = \int_{\mathbf{R}^N} J(x, |x-y|) f(y)\, dy.$$

Set

$$\overline{J}(x,r) = \frac{N}{r^N} \int_0^r J(x,\rho)\rho^{N-1} d\rho$$

for $x \in \mathbf{R}^N$ and $r > 0$. Then $J(x,r) \le \overline{J}(x,r)$ for all $x \in \mathbf{R}^N$ and $r > 0$. Further, $\overline{J}(x,\cdot)$ is non-increasing and continuous on $(0,\infty)$ for each $x \in \mathbf{R}^N$. Also, set

$$Y_J(x,r) = r^N \overline{J}(x,r)$$

for $x \in \mathbf{R}^N$ and $r > 0$.

We consider a function $\Psi(x,t) : \mathbf{R}^N \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Ψ1) $\Psi(\cdot,t)$ is measurable on $\mathbf{R}^N$ for each $t \ge 0$ and $\Psi(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in \mathbf{R}^N$;

(Ψ2) $\Psi(x,\cdot)$ is uniformly almost increasing on $[0,\infty)$, namely there is a constant $A_4 \ge 1$ such that $\Psi(x,t) \le A_4 \Psi(x,t')$ for all $x \in \mathbf{R}^N$, whenever $0 \le t < t'$;

(Ψ3) there exists a constant $A_5 \ge 1$ such that

$$\Psi\left(x, tY_J\big(x, \kappa^{-1}(x, \Phi(x,t)^{-1})\big)\right) \le A_5 \Phi(x,t)$$

for all $x \in \mathbf{R}^N$ and $t > 0$.

Now we consider the following conditions ($\Phi\kappa J$) and ($\Phi_\infty \kappa J$):

($\Phi\kappa J$) $r \mapsto r^\varepsilon Y_J(x,r)\Phi^{-1}(x, \kappa(x,r)^{-1})$ is uniformly almost decreasing on $(0,\infty)$ for some $\varepsilon > 0$;

($\Phi_\infty\kappa J$) $r \mapsto r^\varepsilon Y_J(x,r)\Phi_\infty^{-1}(\kappa(x,r)^{-1})$ is uniformly almost decreasing on $[1,\infty)$ for some $\varepsilon > 0$.

Lemma 6.2. (1) *Assume* ($\Phi\kappa J$). *Then there exists a constant* $C > 0$ *such that*

$$\int_r^\infty \rho^N \Phi^{-1}(x, \kappa(x,\rho)^{-1})\, d(-\overline{J}(x,\cdot))(\rho) \le C Y_J(x,r) \Phi^{-1}(x, \kappa(x,r)^{-1}) \tag{6.1}$$

for all $r > 0$ *and* $x \in \mathbf{R}^N$.

(2) *Assume* ($\Phi_\infty\kappa J$). *Then, given* $r_0 > 0$, *there exists a constant* $C > 0$ *such that*

$$\int_r^\infty \rho^N \Phi_\infty^{-1}(\kappa(x,\rho)^{-1})\, d(-\overline{J}(x,\cdot))(\rho) \le C Y_J(x,r) \Phi_\infty^{-1}(\kappa(x,r)^{-1}) \tag{6.2}$$

for all $r \ge r_0$ *and* $x \in \mathbf{R}^N$.

Proof. From the definition of $\overline{J}(x,r)$, we see that

$$d(-\overline{J}(x,\cdot))(\rho) \le N\overline{J}(x,\rho)\frac{d\rho}{\rho}$$

as measures. Hence by $(\Phi\kappa J)$,

$$\begin{aligned}
&\int_r^\infty \rho^N \Phi^{-1}(x,\kappa(x,\rho)^{-1})\, d(-\overline{J}(x,\cdot))(\rho) \\
&\le N\int_r^\infty \rho^{N-1}\Phi^{-1}(x,\kappa(x,\rho)^{-1})\overline{J}(x,\rho)\, d\rho \\
&\le Cr^\varepsilon Y_J(x,r)\Phi^{-1}(x,\kappa(x,r)^{-1})\int_r^\infty \rho^{-\varepsilon-1}\, d\rho \\
&= \frac{C}{\varepsilon} Y_J(x,r)\Phi^{-1}(x,\kappa(x,r)^{-1}),
\end{aligned}$$

which shows (6.1).

Note that $r \mapsto r^\varepsilon Y_J(x,r)\Phi_\infty^{-1}(\kappa(x,r)^{-1})$ is uniformly almost decreasing on $[r_0,\infty)$. Then we can show (6.2) just as (6.1).

□

Recall that $w(x) = \kappa^{-1}(x, \Phi(x, ag^*(x))^{-1})$ with $0 < a \le 1$.

Lemma 6.3. *Assume $(\Phi\kappa J)$. Then there exists a constant $C > 0$ (which may depend on a) such that*

$$\int_{B(x,w(x))\setminus B(x,\delta)} J(x,|x-y|)f(y)dy \le CY_J(x,\delta)\Phi^{-1}(x,\kappa(x,\delta)^{-1})$$

for all $x \in \mathbf{R}^N$, $0 < \delta \le w(x)$ and $f \ge 0$ satisfying $\|f\|_{\Phi,\kappa} \le 1$.

Proof. By the integration by parts, Lemmas 5.3 and 6.2, we have

$$\begin{aligned}
\int_{B(x,w(x))\setminus B(x,\delta)} J(x,|x-y|)f(y)dy &\le \int_{B(x,w(x))\setminus B(x,\delta)} \overline{J}(x,|x-y|)f(y)dy \\
&\le C\Big\{ w(x)^N\overline{J}(x,w(x))\Phi^{-1}(x,\kappa(x,w(x))^{-1}) \\
&\qquad + \int_\delta^{w(x)} \rho^N\Phi^{-1}(x,\kappa(x,\rho)^{-1})\, d(-\overline{J}(x,\cdot))(\rho)\Big\} \\
&\le CY_J(x,\delta)\Phi^{-1}(x,\kappa(x,\delta)^{-1}),
\end{aligned}$$

where we used the fact that $r \mapsto r^N\overline{J}(x,r)\Phi^{-1}(x,\kappa(x,r)^{-1})$ is also uniformly almost decreasing. □

Theorem 6.4. *Suppose $\Phi(x,t)$ satisfies $(\Phi5)$, $(\Phi3^*)$ and $(\Phi6)$. For the function $\Phi_\infty(t)$ as in the previous section, assume $(\Phi_\infty 1)$, $(\Phi_\infty 2)$ and $(\Phi_\infty\kappa)$. Further assume $(\Phi\kappa J)$ and $(\Phi_\infty\kappa J)$. Then there exists a constant $C > 0$ such that*

$$\sup_{x\in\mathbf{R}^N,\ r>0} \frac{\kappa(x,r)}{|B(x,r)|}\int_{B(x,r)} \Psi(y, Jf(y)/C)dy \le 1$$

for all $f \ge 0$ satisfying $\|f\|_{\Phi,\kappa} \le 1$.

Proof. Let f be a nonnegative measurable function such that $\|f\|_{\Phi,\kappa} \le 1$. By Theorem 4.1, there is a constant $\lambda_0 \ge 1$ such that $\|Mf\|_{\Phi,\kappa} \le \lambda_0$.

Note that $Mg \in L^{\Phi,\kappa}(\mathbf{R}^N)$ by Proposition 2.4 and Theorem 4.1. Set $\lambda = \|g^*\|_{\Phi,\kappa} = \|Mg\|_{\Phi,\kappa}$ and

$$a = \min\left\{1, \frac{1}{4A_2A_3A_4^2A_5\lambda}\right\}. \tag{6.3}$$

Let

$$J_1(x) = \int_{B(x,w(x))} J(x,|x-y|)f(y)dy$$

and

$$J_2(x) = \int_{\mathbf{R}^N\setminus B(x,w(x))} J(x,|x-y|)f(y)dy.$$

Also, set

$$v(x) = \kappa^{-1}(x, \Phi(x, bMf(x))^{-1})$$

with

$$b = \frac{1}{4A_2A_3A_4^2A_5\lambda_0}. \tag{6.4}$$

First, note that

$$\int_{B(x,\delta)} J(x,|x-y|)f(y)dy \le C(N)Y_J(x,\delta)Mf(x)$$

for any $\delta > 0$. Thus, if $v(x) \ge w(x)$, then

$$J_1(x) \le C(N)Y_J(x,v(x))Mf(x).$$

If $v(x) < w(x)$, then by Lemma 6.3

$$\begin{aligned} J_1(x) &\le C(N)Y_J(x,v(x))Mf(x) + \int_{B(x,w(x))\setminus B(x,v(x))} J(x,|x-y|)f(y)\,dy \\ &\le C\big\{Y_J(x,v(x))Mf(x) + Y_J(x,v(x))\Phi^{-1}(x,\kappa(x,v(x))^{-1})\big\}. \end{aligned}$$

Since $\kappa(x,v(x)) = \Phi(x,bMf(x))^{-1}$,

$$\Phi^{-1}(x,\kappa(x,v(x))^{-1}) = \Phi^{-1}\big(x,\Phi(x,bMf(x))\big) \le A_2^2bMf(x)$$

by (5.3). Therefore

$$J_1(x) \le C_0Y_J(x,v(x))[bMf(x)]$$

in any case with a constant $C_0 > 0$ independent of x and f. Hence

$$\Psi(x, J_1(x)/C_0) \le A_4A_5\Phi(x,bMf(x))$$

by $(\Psi2)$ and $(\Psi3)$. By (2.2), (2.1) and (6.4),

$$\Phi(x,bMf(x)) \le A_2b\lambda_0\Phi(x,Mf(x)/\lambda_0) \le 2A_2A_3b\lambda_0\overline{\Phi}(x,Mf(x)/\lambda_0).$$

Hence by (6.4)

$$\Psi(x, J_1(x)/C_0) \le \frac{1}{2A_4}\overline{\Phi}(x, Mf(x)/\lambda_0). \tag{6.5}$$

Next, we treat $J_2(x)$. By the integration by parts, $(\Phi_\infty\kappa J)$, Lemma 5.6 and Lemma 6.2,

$$\begin{aligned}
J_2(x) &\le \int_{\mathbf{R}^N\setminus B(x,w(x))} \overline{J}(x, |x-y|)f(y)dy \\
&\le C\Big\{w(x)^N\Phi_\infty^{-1}(\kappa(x,w(x))^{-1})\overline{J}(x,w(x)) \\
&\qquad + \int_{w(x)}^\infty \rho^N\Phi_\infty^{-1}(\kappa(x,\rho)^{-1})d(-\overline{J}(x,\cdot))(\rho)\Big\} \\
&\le CY_J(x,w(x))\Phi_\infty^{-1}(\kappa(x,w(x))^{-1}).
\end{aligned}$$

Since $\kappa(x,w(x)) = \Phi(x, ag^*(x))^{-1}$,

$$\kappa(x,w(x))^{-1} = \Phi(x,ag^*(x)) \le A_2\Phi(x,g^*(x)) \le A_2B_\infty\Phi_\infty(g^*(x))$$

by $(\Phi_\infty 1)$, so that

$$\Phi_\infty^{-1}(\kappa(x,w(x))^{-1}) \le Cg^*(x)$$

by Lemma 5.1 with $F(x,t) = \Phi_\infty(t)$. Thus there is a constant $C_0' > 0$ such that

$$J_2(x) \le C_0'Y_J(x, \kappa^{-1}(x, \Phi(x,ag^*(x))^{-1}))[ag^*(x)],$$

which implies

$$\Psi(x, J_2(x)/C_0') \le A_4A_5\Phi(x, ag^*(x))$$

by $(\Psi 2)$ and $(\Psi 3)$. Now, by (2.2), (2.1) and (6.3),

$$\Phi(x, ag^*(x)) \le aA_2\lambda\Phi(x, g^*(x)/\lambda) \le 2aA_2A_3\lambda\overline{\Phi}(x, g^*(x)/\lambda).$$

Hence, by (6.3)

$$\Psi(x, J_2(x)/C_0') \le \frac{1}{2A_4}\overline{\Phi}(x, g^*(x)/\lambda). \tag{6.6}$$

Thus, by (6.5), (6.6) and $(\Psi 2)$, we have

$$\Psi\big(x, Jf(x)/(C_0+C_0')\big) \le \frac{1}{2}\big\{\overline{\Phi}(x, Mf(x)/\lambda_0) + \overline{\Phi}(x, g^*(x)/\lambda)\big\}.$$

Hence

$$\sup_{x\in\mathbf{R}^N, r>0}\frac{\kappa(x,r)}{|B(x,r)|}\int_{B(x,r)}\Psi\big(y, Jf(y)/(C_0+C_0')\big)dy \le \frac{1}{2}+\frac{1}{2} = 1,$$

as required. □

Taking $\kappa(x,r) = r^N$ in Theorem 6.4, we have

Corollary 6.5. *Suppose $\Phi(x,t)$ satisfies* $(\Phi 5)$, $(\Phi 3^*)$ *and* $(\Phi 6)$. *For the function $\Phi_\infty(t)$ as in the previous section, assume* $(\Phi_\infty 1)$, $(\Phi_\infty 2)$,

($\Phi_\infty 3$) $r \mapsto r^\gamma \Phi_\infty^{-1}(r^{-N})$ *is almost increasing for some* $0 < \gamma < N$,

(ΦJ) $r \mapsto r^\varepsilon Y_J(x,r)\Phi^{-1}(x,r^{-N})$ *is uniformly almost decreasing on* $(0,\infty)$ *for some* $\varepsilon > 0$,

($\Phi_\infty J$) $r \mapsto r^\varepsilon Y_J(x,r)\Phi_\infty^{-1}(r^{-N})$ *is uniformly almost decreasing on* $[1,\infty)$ *for some* $\varepsilon > 0$.

Suppose that $\Psi(x,t)$ *satisfies* ($\Psi 1$), ($\Psi 2$) *and that there exists a constant* $A^* \ge 1$ *such that*

$$\Psi\left(x,\, tY_J(x, \Phi(x,t)^{-1/N})\right) \le A^*\Phi(x,t)$$

for all $t > 0$.

Then there exists a constant $C > 0$ *such that*

$$\int_{\mathbf{R}^N} \Psi(x, Jf(x)/C)dx \le 1$$

for all $f \ge 0$ *satisfying* $\int_{\mathbf{R}^N} \Phi(x,f(x))\,dx \le 1$.

Example 6.6 (cf. [13]). Let

$$\Phi(x,t) = t^{p(x)}(\log(e+t))^{q(x)}$$

with functions $p(\cdot)$ and $q(\cdot)$ on $\mathbf{R}^N$ satisfying (P1), (P2), (P3), (Q1) and (Q2) in Example 2.1. Assume further that $p^- > 1$. Then $\Phi(x,t)$ satisfies ($\Phi 3^*$).

Let

$$\kappa(x,r) = r^{\nu(x)}(\log(e+r+1/r))^{\beta(x)}$$

with functions $\nu(\cdot)$ and $\beta(\cdot)$ on $\mathbf{R}^N$ satisfying conditions in Example 2.2.

For these Φ and κ,

$$\kappa^{-1}(x, \Phi(x,t)^{-1}) \approx \left[t^{p(x)}(\log(e+t))^{q(x)}(\log(e+t+1/t))^{\beta(x)}\right]^{-1/\nu(x)}.$$

(Here $h_1(x,t) \approx h_2(x,t)$ means that $C^{-1}h_2(x,t) \le h_1(x,t) \le Ch_2(x,t)$ for a constant $C > 0$.)

If $J(x,r) = r^{\alpha(x)-N}$ ($0 < \alpha^- \le \alpha^+ < N$), then $Y_J(x,r) = (N/\alpha(x))r^{\alpha(x)} \approx r^{\alpha(x)}$, so that

$$\begin{aligned} tY_J\big(x,&\kappa^{-1}(x,\Phi(x,t)^{-1})\big) \\ &\approx t^{1-p(x)\alpha(x)/\nu(x)}(\log(e+t))^{-\alpha(x)q(x)/\nu(x)}(\log(e+t+1/t))^{-\alpha(x)\beta(x)/\nu(x)}. \end{aligned}$$

Thus, if

$$\inf_{x\in\mathbf{R}^N}\left(\frac{\nu(x)}{p(x)} - \alpha(x)\right) > 0, \tag{6.7}$$

we may take

$$\Psi(x,t) = \left[t(\log(e+t))^{q(x)/p(x)}(\log(e+t+1/t))^{\alpha(x)\beta(x)/\nu(x)}\right]^{p^*(x)},$$

where

$$\frac{1}{p^*(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{\nu(x)}.$$

Also, we may take $\Phi_\infty(t) = t^{p(\infty)}$, where $p(\infty) = \lim_{|x|\to\infty} p(x)$, which exists by (P3). Then, ($\Phi_\infty 1$) and ($\Phi_\infty 2$) are satisfied. (Note that $g(x) = 1/(1+|x|)^{N+1}$; cf. Remark 5.5.) Also, ($\Phi_\infty\kappa$) is satisfied since $\nu^+ \le N$ and $p(\infty) \ge p^- > 1$. Condition ($\Phi\kappa J$) is satisfied for these special Φ, κ and J under condition (6.7). Finally condition ($\Phi_\infty\kappa J$) is satisfied if

$$\inf_{x\in\mathbf{R}^N}\left(\frac{\nu(x)}{p(\infty)} - \alpha(x)\right) > 0.$$

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4-24 Furue-higashi-machi, Nishi-ku
Hiroshima 733-0872, Japan
E-mail : fymaeda@h6.dion.ne.jp
and
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8521, Japan
E-mail : yomizuta@hiroshima-u.ac.jp
and
Faculty of Education and Welfare Science
Oita University
Dannoharu Oita-city 870-1192, Japan
E-mail : t-ohno@oita-u.ac.jp
and
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524, Japan
E-mail : tshimo@hiroshima-u.ac.jp