Approximate identities and Young type inequalities in Musielak-Orlicz spaces

Fumi-Yuki Maeda, Yoshihiro Mizuta,
Takao Ohno and Tetsu Shimomura

October 2, 2012

Abstract

Our aim in this paper is to deal with approximate identities and Young type inequalities in Musielak-Orlicz spaces.

1 Introduction

Let κ be an integrable function on $\mathbf{R}^N$. For each $t > 0$, define the function κ_t by $\kappa_t(x) = t^{-N}\kappa(x/t)$. Note that by a change of variables, $\|\kappa_t\|_{L^1(\mathbf{R}^N)} = \|\kappa\|_{L^1(\mathbf{R}^N)}$. We say that the family $\{\kappa_t\}_{t>0}$ is an *approximate identity* if $\int_{\mathbf{R}^N} \kappa(x)\, dx = 1$. Define the radial majorant of κ to be the function

$$\hat{\kappa}(x) = \sup_{|y|\geq|x|} |\kappa(y)|.$$

If $\hat{\kappa}$ is integrable, we say that the family $\{\kappa_t\}_{t>0}$ is *of potential-type*.

It is well known (see, e.g., [9]) that if $\{\kappa_t\}_{t>0}$ is a potential-type approximate identity, then $\kappa_t * f \to f$ in $L^p(\mathbf{R}^N)$ as $t \to 0$ for every $f \in L^p(\mathbf{R}^N)$ $(p \geq 1)$.

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions (see [3]). Cruz-Uribe and Fiorenza [1] gave sufficient conditions for the convergence of approximate identities in variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbf{R}^N)$ when $p(\cdot)$ is a variable exponent satisfying the log-Hölder conditions on $\mathbf{R}^N$, locally and at ∞, as an extension of [2], [9], etc. In fact they proved the following:

Theorem A. *Let $\{\kappa_t\}_{t>0}$ be an approximate identity. Suppose that either*

2000 Mathematics Subject Classification : Primary 46E30

Key words and phrases : approximate identities, Musielak-Orlicz space, Young type inequality

(1) $\{\kappa_t\}_{t>0}$ *is of potential-type, or*

(2) $\kappa \in L^{(p^-)'}(\mathbf{R}^N)$ *and has compact support, where* $p^- := \inf_{x\in\mathbf{R}^N} p(x)$ (≥ 1) *and* $1/p^- + 1/(p^-)' = 1$.

Then

$$\sup_{0<t\leq 1} \|\kappa_t * f\|_{L^{p(\cdot)}(\mathbf{R}^N)} \leq C\|f\|_{L^{p(\cdot)}(\mathbf{R}^N)}$$

and

$$\lim_{t\to 0} \|\kappa_t * f - f\|_{L^{p(\cdot)}(\mathbf{R}^N)} = 0$$

for all $f \in L^{p(\cdot)}(\mathbf{R}^N)$.

Recently, Theorem A was extended to the two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^N)$ in [4]. These spaces are special cases of so-called Musielak-Orlicz spaces ([8]).

Our aim in this paper is to extend these results to Musielak-Orlicz spaces $L^\Phi(\mathbf{R}^N)$ (see Section 2 for the definition of Φ). As a related topic, we also give a Young type inequality for convolution with respect to the norm in $L^\Phi(\mathbf{R}^N)$.

2 Preliminaries

We consider a function

$$\Phi(x,t) = t\phi(x,t) : \mathbf{R}^N \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions ($\Phi1$) – ($\Phi4$):

($\Phi1$) $\phi(\,\cdot\,,t)$ is measurable on $\mathbf{R}^N$ for each $t \geq 0$ and $\phi(x,\,\cdot\,)$ is continuous on $[0,\infty)$ for each $x \in \mathbf{R}^N$;

($\Phi2$) there exists a constant $A_1 \geq 1$ such that

$$A_1^{-1} \leq \phi(x,1) \leq A_1 \quad \text{for all } x \in \mathbf{R}^N;$$

($\Phi3$) $\phi(x,\cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \geq 1$ such that

$$\phi(x,t) \leq A_2\phi(x,s) \quad \text{for all } x \in \mathbf{R}^N \quad \text{whenever } 0 \leq t < s;$$

($\Phi4$) there exists a constant $A_3 \geq 1$ such that

$$\phi(x,2t) \leq A_3\phi(x,t) \quad \text{for all } x \in \mathbf{R}^N \text{ and } t > 0.$$

Note that ($\Phi2$), ($\Phi3$) and ($\Phi4$) imply

$$0 < \inf_{x\in\mathbf{R}^N} \phi(x,t) \leq \sup_{x\in\mathbf{R}^N} \phi(x,t) < \infty$$

for each $t > 0$.

If $\Phi(x,\cdot)$ is convex for each $x \in \mathbf{R}^N$, then ($\Phi3$) holds with $A_2 = 1$; namely $\phi(x,\cdot)$ is non-decreasing for each $x \in \mathbf{R}^N$.

EXAMPLE 2.1. Let $p_1(\cdot)$, $p_2(\cdot)$, $q_1(\cdot)$ and $q_2(\cdot)$ be measurable functions on $\mathbf{R}^N$ such that

(P1) $1 \le p_j^- := \inf_{x\in\mathbf{R}^N} p_j(x) \le \sup_{x\in\mathbf{R}^N} p_j(x) =: p_j^+ < \infty$, $j = 1, 2$

and

(Q1) $-\infty < q_j^- := \inf_{x\in\mathbf{R}^N} q_j(x) \le \sup_{x\in\mathbf{R}^N} q_j(x) =: q_j^+ < \infty$, $j = 1, 2$.

Then,

$$\Phi(x,t) = (1+t)^{p_1(x)}(1+1/t)^{-p_2(x)}(\log(e+t))^{q_1(x)}(\log(e+1/t))^{-q_2(x)}$$

satisfies $(\Phi 1)$, $(\Phi 2)$ and $(\Phi 4)$. It satisfies $(\Phi 3)$ if $p_j^- > 1$, $j = 1, 2$ or $q_j^- \ge 0$, $j = 1, 2$. As a matter of fact, it satisfies $(\Phi 3)$ if and only if $p_j(\cdot)$, $q_j(\cdot)$ satisfies the following conditions:

(1) $q_j(x) \ge 0$ at points x where $p_j(x) = 1$, $j = 1, 2$;

(2) $\sup_{x: p_j(x)>1}\{\min(q_j(x), 0)\log(p_j(x) - 1)\} < \infty$, $j = 1, 2$.

Let $\bar\phi(x,t) = \sup_{0\le s\le t}\phi(x,s)$ and

$$\overline{\Phi}(x,t) = \int_0^t \bar\phi(x,r)\,dr$$

for $x \in \mathbf{R}^N$ and $t \ge 0$. Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \le \overline{\Phi}(x,t) \le A_2\Phi(x,t) \tag{2.1}$$

for all $x \in \mathbf{R}^N$ and $t \ge 0$. In fact, the first inequality is seen as follows:

$$\overline{\Phi}(x,t) \ge \int_{t/2}^t \bar\phi(x,r)\,dr \ge \frac{t}{2}\phi(x,t/2) \ge \frac{1}{2A_3}\Phi(x,t).$$

Corresponding to $(\Phi 2)$ and $(\Phi 4)$, we have by (2.1)

$$(2A_1A_3)^{-1} \le \overline{\Phi}(x,1) \le A_1A_2 \quad\text{and}\quad \overline{\Phi}(x,2t) \le 2A_3\overline{\Phi}(x,t) \tag{2.2}$$

for all $x \in \mathbf{R}^N$ and $t > 0$.

Given $\Phi(x,t)$ as above, the associated Musielak-Orlicz space

$$L^\Phi(\mathbf{R}^N) = \left\{f \in L^1_{loc}(\mathbf{R}^N)\,;\, \int_{\mathbf{R}^N}\Phi\big(y,|f(y)|\big)\,dy < \infty\right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(\mathbf{R}^N)} = \inf\left\{\lambda > 0\,;\, \int_{\mathbf{R}^N}\overline{\Phi}\big(y,|f(y)|/\lambda\big)\,dy \le 1\right\}$$

(cf. [8]).

By (2.2), we have the following lemma (see [7]).

Lemma 2.2.
$$\|f\|_{L^\Phi(\mathbf{R}^N)} \le 2\left(\int_{\mathbf{R}^N} \overline{\Phi}(x,|f(x)|)\,dx\right)^\sigma \tag{2.3}$$
with $\sigma = \log 2/\log(2A_3)$, *if* $\|f\|_{L^\Phi(\mathbf{R}^N)} \le 1$.

We shall also consider the following conditions:

($\Phi5$) for every $\gamma > 0$, there exists a constant $B_\gamma \ge 1$ such that
$$\phi(x,t) \le B_\gamma \phi(y,t)$$
whenever $|x-y| \le \gamma t^{-1/N}$ and $t \ge 1$;

($\Phi6$) there exist a function $g \in L^1(\mathbf{R}^N)$ and a constant $B_\infty \ge 1$ such that $0 \le g(x) < 1$ for all $x \in \mathbf{R}^N$ and
$$B_\infty^{-1}\Phi(x,t) \le \Phi(x',t) \le B_\infty \Phi(x,t)$$
whenever $|x'| \ge |x|$ and $g(x) \le t \le 1$.

If $\Phi(x,t)$ satisfies ($\Phi5$) (resp. ($\Phi6$)), then so does $\overline{\Phi}(x,t)$ with $\overline{B}_\gamma = 2A_2A_3B_\gamma$ in place of B_γ (resp. $\overline{B}_\infty = 2A_2A_3B_\infty$ in place of B_∞).

Example 2.3. Let $\Phi(x,t)$ be as in Example 2.1. It satisfies ($\Phi5$) if

(P2) $p_1(\cdot)$ is log-Hölder continuous, namely
$$|p_1(x) - p_1(y)| \le \frac{C_p}{\log(1/|x-y|)} \quad \text{for } |x-y| \le \frac{1}{2}$$
with a constant $C_p \ge 0$,

and

(Q2) $q_1(\cdot)$ is log-log-Hölder continuous, namely
$$|q_1(x) - q_1(y)| \le \frac{C_q}{\log(\log(1/|x-y|))} \quad \text{for } |x-y| \le e^{-2}$$
with a constant $C_q \ge 0$.

$\Phi(x,t)$ satisfies ($\Phi6$) with $g(x) = 1/(1+|x|)^{N+1}$ if

(P3) $p_2(\cdot)$ is log-Hölder continuous at ∞, namely
$$|p_2(x) - p_2(x')| \le \frac{C_{p,\infty}}{\log(e+|x|)} \quad \text{whenever } |x'| \ge |x|$$
with a constant $C_{p,\infty} \ge 0$,

and

(Q3) $q_2(\cdot)$ is log-log-Hölder continuous at ∞, namely

$$|q_2(x) - q_2(x')| \le \frac{C_{q,\infty}}{\log(e + \log(e + |x|))} \quad \text{whenever } |x'| \ge |x|$$

with a constant $C_{q,\infty} \ge 0$.

In fact, if $1/(1+|x|)^{N+1} < t \le 1$, then $(1+t)^{|p_1(x)-p_1(x')|} \le 2^{p_1^+-1}$, $(1+1/t)^{|p_2(x)-p_2(x')|} \le e^{(N+1)C_{p,\infty}}$, $(\log(e+t))^{|q_1(x)-q_1(x')|} \le (\log(e+1))^{q_1^+-q_1^-}$ and $(\log(e+1/t))^{|q_2(x)-q_2(x')|} \le C(N, C_{q,\infty})$ for $|x'| \ge |x|$.

3 The case of potential-type

Throughout this paper, let C denote various positive constants independent of the variables in question.

First, we recall the following classical result (see, e.g., Stein [9]).

LEMMA 3.1. *Let $1 \le p < \infty$ and $\{\kappa_t\}_{t>0}$ be a potential-type approximate identity. Then, $\kappa_t * f$ converges to f in $L^p(\mathbf{R}^N)$ for every $f \in L^p(\mathbf{R}^N)$.*

We denote by $B(x,r)$ the open ball centered at $x \in \mathbf{R}^N$ and with radius $r > 0$. For a measurable set E, we denote by $|E|$ the Lebesgue measure of E.

For a nonnegative $f \in L^1_{loc}(\mathbf{R}^N)$, $x \in \mathbf{R}^N$ and $r > 0$, let

$$I(f;x,r) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy$$

and

$$J(f;x,r) = \frac{1}{|B(x,r)|} \int_{B(x,r)} \overline{\Phi}\big(y, f(y)\big)\,dy$$

in this section.

The following lemmas are due to [5, 6].

LEMMA 3.2 ([5, Lemma 2.1], [6, Lemma 3.1]). *Suppose $\Phi(x,t)$ satisfies (Φ5). Then there exists a constant $C > 0$ such that*

$$\overline{\Phi}\big(x, I(f;x,r)\big) \le C J(f;x,r)$$

for all $x \in \mathbf{R}^N$, $r > 0$ and for all nonnegative $f \in L^1_{loc}(\mathbf{R}^N)$ such that $f(y) \ge 1$ or $f(y) = 0$ for each $y \in \mathbf{R}^N$ and $\|f\|_{L^\Phi(\mathbf{R}^N)} \le 1$.

LEMMA 3.3 ([5, Lemma 2.2], [6, Lemma 3.2]). *Suppose $\Phi(x,t)$ satisfies (Φ6). Then there exists a constant $C > 0$ such that*

$$\overline{\Phi}\big(x, I(f;x,r)\big) \le C\left\{J(f;x,r) + g(x)\right\}$$

for all $x \in \mathbf{R}^N$, $r > 0$ and for all nonnegative $f \in L^1_{loc}(\mathbf{R}^N)$ such that $g(y) \le f(y) \le 1$ or $f(y) = 0$ for each $y \in \mathbf{R}^N$, where g is the function appearing in (Φ6).

By using Lemmas 3.2 and 3.3, we show the following theorem.

THEOREM 3.4. *Suppose* $\Phi(x,t)$ *satisfies* $(\Phi5)$ *and* $(\Phi6)$. *If* $\{\kappa_t\}_{t>0}$ *is of potential-type, then*

$$\|\kappa_t * f\|_{L^\Phi(\mathbf{R}^N)} \le C\|\hat{\kappa}\|_{L^1(\mathbf{R}^N)}\|f\|_{L^\Phi(\mathbf{R}^N)}$$

for all $t>0$ *and* $f \in L^\Phi(\mathbf{R}^N)$.

Proof. Suppose $\|\hat{\kappa}\|_{L^1(\mathbf{R}^N)} = 1$ and let f be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{L^\Phi(\mathbf{R}^N)} \le 1$. Write

$$\begin{aligned} f &= f\chi_{\{y\in\mathbf{R}^N: f(y)\ge 1\}} + f\chi_{\{y\in\mathbf{R}^N: g(y)<f(y)<1\}} + f\chi_{\{y\in\mathbf{R}^N: f(y)\le g(y)\}} \\ &= f_1 + f_2 + f_3, \end{aligned}$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbf{R}^N$ and g is the function appearing in $(\Phi6)$.

Since $\hat{\kappa}_t$ is a radial function, we write $\hat{\kappa}_t(r)$ for $\hat{\kappa}_t(x)$ when $|x| = r$. First note that

$$\begin{aligned} |\kappa_t * f_j(x)| &\le \int_{\mathbf{R}^N} \hat{\kappa}_t(|x-y|) f_j(y)\,dy \\ &= \int_0^\infty I(f_j;x,r)|B(x,r)|\,d(-\hat{\kappa}_t(r)), \end{aligned}$$

$j = 1,2$ and

$$\int_{\mathbf{R}^N} |B(x,r)|\,d(-\hat{\kappa}_t(r)) = \|\hat{\kappa}_t\|_{L^1(\mathbf{R}^N)} = 1,$$

so that Jensen's inequality yields

$$\overline{\Phi}(x, |\kappa_t * f_j(x)|) \le \int_0^\infty \overline{\Phi}\,(x, I(f_j;x,r))\,|B(x,r)|\,d(-\hat{\kappa}_t(r)),$$

$j = 1,2$.

Hence, by Lemma 3.2

$$\overline{\Phi}(x, |\kappa_t * f_1(x)|) \le C\int_0^\infty J(f_1;x,r)|B(x,r)|\,d(-\hat{\kappa}_t(r)) \le C(\hat{\kappa}_t * h)(x),$$

where $h(y) = \overline{\Phi}(y, f(y))$. The usual Young inequality for convolution gives

$$\begin{aligned} \int_{\mathbf{R}^N} \overline{\Phi}(x, |\kappa_t * f_1(x)|)\,dx &\le C\int_{\mathbf{R}^N} (\hat{\kappa}_t * h)(x)\,dx \\ &\le C\|\hat{\kappa}_t\|_{L^1(\mathbf{R}^N)}\|h\|_{L^1(\mathbf{R}^N)} \le C. \end{aligned}$$

Similarly, noting that $g \in L^1(\mathbf{R}^N)$ and applying Lemma 3.3, we derive the same result for f_2.

Finally, noting that $|\kappa_t * f_3(x)| \le \|\kappa_t\|_{L^1(\mathbf{R}^N)} \le 1$, we obtain

$$\begin{aligned}\int_{\mathbf{R}^N} \overline{\Phi}(x, |\kappa_t * f_3(x)|)\, dx &\le C \int_{\mathbf{R}^N} |\kappa_t * f_3(x)|\, dx \\ &\le C\|\kappa_t\|_{L^1(\mathbf{R}^N)} \|g\|_{L^1(\mathbf{R}^N)} \le C.\end{aligned}$$

Thus

$$\int_{\mathbf{R}^N} \overline{\Phi}(x, |\kappa_t * f(x)|)\, dx \le C,$$

which implies the required assertion. □

THEOREM 3.5. *Suppose* $\Phi(x,t)$ *satisfies* $(\Phi5)$ *and* $(\Phi6)$. *Let* $\{\kappa_t\}_{t>0}$ *be a potential-type approximate identity. Then* $\kappa_t * f$ *converges to* f *in* $L^{\Phi}(\mathbf{R}^N)$:

$$\lim_{t\to 0} \|\kappa_t * f - f\|_{L^{\Phi}(\mathbf{R}^N)} = 0$$

for every $f \in L^{\Phi}(\mathbf{R}^N)$.

Proof. Given $\varepsilon > 0$, we find a bounded function h in $L^{\Phi}(\mathbf{R}^N)$ with compact support such that $\|f - h\|_{L^{\Phi}(\mathbf{R}^N)} < \varepsilon$. By Theorem 3.4 we have

$$\begin{aligned}\|\kappa_t * f - f\|_{L^{\Phi}(\mathbf{R}^N)} &\le \|\kappa_t * (f-h)\|_{L^{\Phi}(\mathbf{R}^N)} + \|\kappa_t * h - h\|_{L^{\Phi}(\mathbf{R}^N)} + \|f - h\|_{L^{\Phi}(\mathbf{R}^N)} \\ &\le (C\|\hat{\kappa}\|_{L^1(\mathbf{R}^N)} + 1)\varepsilon + \|\kappa_t * h - h\|_{L^{\Phi}(\mathbf{R}^N)}.\end{aligned}$$

Since $|\kappa_t * h| \le \|h\|_{L^\infty(\mathbf{R}^N)}$, we have

$$\int_{\mathbf{R}^N} \overline{\Phi}(x, |\kappa_t * h(x) - h(x)|\, dx \le C' \int_{\mathbf{R}^N} |\kappa_t * h(x) - h(x)|\, dx \to 0$$

as $t \to 0$ by Lemma 3.1. (Here C' depends on $\|h\|_{L^\infty(\mathbf{R}^N)}$). Hence $\|\kappa_t * h - h\|_{L^{\Phi}(\mathbf{R}^N)} \to 0$ as $t \to 0$ by Lemma 2.2, so that

$$\limsup_{t\to 0} \|\kappa_t * f - f\|_{L^{\Phi}(\mathbf{R}^N)} \le (C\|\hat{\kappa}\|_{L^1(\mathbf{R}^N)} + 1)\varepsilon,$$

which completes the proof. □

4 The case of compact support

We know the following result due to Zo [10]; see also [1, Theorem 2.2].

LEMMA 4.1. *Let* $1 \le p < \infty$, $1/p + 1/p' = 1$ *and* $\{\kappa_t\}_{t>0}$ *be an approximate identity. Suppose that* $\kappa \in L^{p'}(\mathbf{R}^N)$ *and it has compact support. Then for every* $f \in L^p(\mathbf{R}^N)$, $\kappa_t * f$ *converges to* f *pointwise almost everywhere as* $t \to 0$.

In this section, we take $p_0 \ge 1$ as follows. Let P be the set of all $p \ge 1$ such that $t \mapsto t^{-p}\Phi(x,t)$ is uniformly almost increasing, and set $\tilde{p} = \sup P$. Note that $1 \in P$ by $(\Phi3)$, so that $\tilde{p} > 1$ if $\tilde{p} \notin P$. Let $p_0 = \tilde{p}$ if $\tilde{p} \in P$ and $1 < p_0 < \tilde{p}$ otherwise.

EXAMPLE 4.2. For $\Phi(x,t)$ in Example 2.3, $\tilde{p} = \min\{p_1^-, p_2^-\}$, so that $p_0 = 1$ if $p_1^- = 1$ or $p_2^- = 1$; and $1 < p_0 \le \min\{p_1^-, p_2^-\}$ if $p_j^- > 1$, $j = 1, 2$. (Cf. [4]).

Since $t^{-p_0}\Phi(x,t)$ is uniformly almost increasing in t, there exists a constant $A_2' \ge 1$ such that

$$t^{-p_0}\Phi(x,t) \le A_2' s^{-p_0}\Phi(x,s) \quad \text{for all } x \in \mathbf{R}^N \quad \text{whenever } 0 \le t < s.$$

Set

$$\Phi_0(x,t) = \Phi(x,t)^{1/p_0}$$

Then $\Phi_0(x,t)$ also satisfies all the conditions (Φj), $j = 1, 2, \ldots, 6$. In fact, it trivially satisfies (Φj) for $j = 1, 2, 4, 5, 6$ with the same g for $(\Phi 6)$. Since

$$\Phi_0(x,t) = t\phi_0(x,t) \quad \text{with} \quad \phi_0(x,t) = \left[t^{-p_0}\Phi(x,t)\right]^{1/p_0},$$

$\Phi_0(x,t)$ satisfies $(\Phi 3)$ with A_2 replaced by $A_4 = (A_2')^{1/p_0}$.

LEMMA 4.3. *Suppose $\Phi(x,t)$ satisfies $(\Phi 5)$. Let κ have compact support contained in $B(0,R)$ and let $\|\kappa\|_{L^{(p_0)'}(\mathbf{R}^N)} \le 1$. Then there exists a constant $C > 0$, which depends on R, such that*

$$\Phi_0\big(x, |\kappa_t * f(x)|\big) \le C \int_{\mathbf{R}^N} |\kappa_t(x-y)|\Phi_0(y, f(y))\, dy$$

for all $x \in \mathbf{R}^N$, $0 < t \le 1$ and for all nonnegative $f \in L^1_{loc}(\mathbf{R}^N)$ such that $f(y) \ge 1$ or $f(y) = 0$ for each $y \in \mathbf{R}^N$ and $\|f\|_{L^\Phi(\mathbf{R}^N)} \le 1$.

Proof. Given f as in the statement of the lemma, $x \in \mathbf{R}^N$ and $0 < t \le 1$, set

$$F = |\kappa_t * f(x)| \quad \text{and} \quad G = \int_{\mathbf{R}^N} |\kappa_t(x-y)|\Phi_0(y, f(y))\, dy.$$

Note that $\|f\|_{L^\Phi(\mathbf{R}^N)} \le 1$ implies

$$G \le \|\kappa_t\|_{L^{(p_0)'}(\mathbf{R}^N)} \left(\int_{\mathbf{R}^N} \Phi(y, f(y))\, dy\right)^{1/p_0} \le t^{-N/p_0}(2A_3)^{1/p_0} \le (2A_3)^{1/p_0} t^{-N}$$

by Hölder's inequality and (2.1).

By $(\Phi 2)$, $\Phi_0\big(y, f(y)\big) \ge (A_1 A_4)^{-1} f(y)$, since $f(y) \ge 1$ or $f(y) = 0$. Hence $F \le A_1 A_4 G$. Thus, if $G \le 1$, then

$$\Phi_0(x, F) \le (A_1 A_4 G) A_4 (A_1 A_4)^{(1-p_0)/p_0} \phi(x, A_1 A_4)^{1/p_0} \le CG.$$

Next, let $G > 1$. Since $\Phi_0(x,t) \to \infty$ as $t \to \infty$, there exists $K \ge 1$ such that

$$\Phi_0(x, K) = \Phi_0(x, 1) G.$$

Then $K \le A_4 G$, since $\Phi_0(x,K) \ge A_4^{-1} K \Phi_0(x,1)$. With this K, we have

$$F \le K \int_{\mathbf{R}^N} |\kappa_t(x-y)|\,dy + A_4 \int_{\mathbf{R}^N} |\kappa_t(x-y)| f(y) \frac{\phi_0\big(y,f(y)\big)}{\phi_0(y,K)}\,dy.$$

Since

$$1 \le K \le A_4 G \le A_4 (2A_3)^{1/p_0} t^{-N} \le C(tR)^{-N},$$

there is $\beta > 0$, independent of f, x, t, such that

$$\phi_0(x,K) \le \beta \phi_0(y,K) \quad \text{for all } y \in B(x,tR)$$

by (Φ5). Thus, we have

$$\begin{aligned} F &\le K \|\kappa_t\|_{L^1(\mathbf{R}^N)} + \frac{A_4\beta}{\phi_0(x,K)} \int_{\mathbf{R}^N} |\kappa_t(x-y)| f(y) \phi_0\big(y,f(y)\big)\,dy \\ &= K\|\kappa\|_{L^1(\mathbf{R}^N)} + A_4\beta \frac{G}{\phi_0(x,K)} \\ &= K \left(\|\kappa\|_{L^1(\mathbf{R}^N)} + \frac{A_4\beta}{\phi_0(x,1)} \right) \\ &\le K \left(\|\kappa\|_{L^1(\mathbf{R}^N)} + A_1^{1/p_0} A_4 \beta \right) \le CK. \end{aligned}$$

Therefore by (Φ3), (Φ4), the choice of K and (Φ2),

$$\Phi_0(x,F) \le C\Phi_0(x,K) \le CG$$

with constants $C > 0$ independent of f, x, t, as required. □

LEMMA 4.4. *Suppose $\Phi(x,t)$ satisfies (Φ6). Let $M \ge 1$ and assume that $\|\kappa\|_{L^1(\mathbf{R}^N)} \le M$. Then there exists a constant $C > 0$, depending on M, such that*

$$\overline{\Phi}\big(x, |\kappa_t * f(x)|\big) \le C \left\{ \int_{\mathbf{R}^N} |\kappa_t(x-y)| \overline{\Phi}(y,f(y))\,dy + g(x) \right\}$$

for all $x \in \mathbf{R}^N$, $t > 0$ and for all nonnegative $f \in L^1_{loc}(\mathbf{R}^N)$ such that $g(y) \le f(y) \le 1$ or $f(y) = 0$ for each $y \in \mathbf{R}^N$, where g is the function appearing in (Φ6).

Proof. Let f be as in the statement of the lemma, $x \in \mathbf{R}^N$ and $t > 0$. By (Φ4), there is a constant $c_M \ge 1$ such that $\overline{\Phi}(x,Mt) \le c_M \overline{\Phi}(x,t)$ for all $x \in \mathbf{R}^N$ and $t > 0$. By Jensen's inequality, we have

$$\begin{aligned} \overline{\Phi}(x, |\kappa_t * f(x)|) &\le c_M \overline{\Phi}\left(x, \int_{\mathbf{R}^N} (|\kappa_t(x-y)|/M) f(y)\,dy \right) \\ &\le (c_M/M) \int_{\mathbf{R}^N} |\kappa_t(x-y)| \overline{\Phi}\big(x,f(y)\big)\,dy. \end{aligned}$$

If $|x| \ge |y|$, then $\overline{\Phi}\big(x,f(y)\big) \le \overline{B}_\infty \overline{\Phi}\big(y,f(y)\big)$ by (Φ6).

If $|x| < |y|$ and $g(x) < f(y)$, then $\overline{\Phi}\big(x, f(y)\big) \le \overline{B}_\infty \overline{\Phi}\big(y, f(y)\big)$ by (Φ6) again.

If $|x| < |y|$ and $g(x) \ge f(y)$, then

$$\overline{\Phi}\big(x, f(y)\big) \le \overline{\Phi}\big(x, g(x)\big) \le g(x)\overline{\Phi}(x, 1) \le A_1 A_2 g(x)$$

by (2.2).

Hence,

$$\overline{\Phi}\big(x, f(y)\big) \le C\left\{\overline{\Phi}\big(y, f(y)\big) + g(x)\right\}$$

in any case. Therefore, we obtain the required inequality. □

THEOREM 4.5. *Suppose* $\Phi(x,t)$ *satisfies* (Φ5) *and* (Φ6). *Suppose that* $\kappa \in L^{(p_0)'}(\mathbf{R}^N)$ *and it has compact support in* $B(0,R)$. *Then*

$$\|\kappa_t * f\|_{L^\Phi(\mathbf{R}^N)} \le C\|\kappa\|_{L^{(p_0)'}(\mathbf{R}^N)}\|f\|_{L^\Phi(\mathbf{R}^N)}$$

for all $0 < t \le 1$ *and* $f \in L^\Phi(\mathbf{R}^N)$, *where* $C > 0$ *depends on* R.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{L^\Phi(\mathbf{R}^N)} \le 1$ and assume that $\|\kappa\|_{L^{(p_0)'}(\mathbf{R}^N)} = 1$. Note that $\|\kappa\|_{L^1(\mathbf{R}^N)} \le |B(0,R)|^{1/p_0}$ by Hölder's inequality.

Write

$$\begin{aligned} f &= f\chi_{\{y\in\mathbf{R}^N : f(y)\ge 1\}} + f\chi_{\{y\in\mathbf{R}^N : g(y)<f(y)<1\}} + f\chi_{\{y\in\mathbf{R}^N : f(y)\le g(y)\}} \\ &= f_1 + f_2 + f_3, \end{aligned}$$

where g is the function appearing in (Φ6). We have by (2.1) and Lemma 4.3,

$$\overline{\Phi}(x, |\kappa_t * f_1(x)|) \le A_2 \Phi_0(x, |\kappa_t * f_1(x)|)^{p_0} \le C(|\kappa_t| * h(x))^{p_0},$$

where $h(y) = \Phi(y, f(y))^{1/p_0}$. Since $\|h\|^{p_0}_{L^{p_0}(\mathbf{R}^N)} \le 2A_3$, the usual Young's inequality for convolution gives

$$\begin{aligned} \int_{\mathbf{R}^N} \overline{\Phi}(x, |\kappa_t * f_1(x)|)\,dx &\le C\int_{\mathbf{R}^N} (|\kappa_t| * h(x))^{p_0}\,dx \\ &\le C\left(\|\kappa_t\|_{L^1(\mathbf{R}^N)}\|h\|_{L^{p_0}(\mathbf{R}^N)}\right)^{p_0} \le C. \end{aligned}$$

Similarly, applying Lemma 4.4 with $M = |B(0,R)|^{1/p_0}$ and noting that $g \in L^1(\mathbf{R}^N)$, we derive the same result for f_2.

Finally, since $|\kappa_t * f_3(x)| \le \|\kappa_t\|_{L^1(\mathbf{R}^N)} \le M$, we obtain

$$\begin{aligned} \int_{\mathbf{R}^N} \overline{\Phi}(x, |\kappa_t * f_3(x)|)\,dx &\le C\int_{\mathbf{R}^N} |\kappa_t * f_3(x)|\,dx \\ &\le C\|\kappa_t\|_{L^1(\mathbf{R}^N)}\|g\|_{L^1(\mathbf{R}^N)} \le C. \end{aligned}$$

Thus, we have shown that

$$\int_{\mathbf{R}^N} \overline{\Phi}(x, |\kappa_t * f(x)|)\,dx \le C,$$

which implies the required result. □

THEOREM 4.6. *Suppose $\Phi(x,t)$ satisfies* ($\Phi5$) *and* ($\Phi6$). *Let $\{\kappa_t\}_{t>0}$ be an approximate identity such that $\kappa \in L^{(p_0)'}(\mathbf{R}^N)$ and it has compact support. Then $\kappa_t * f$ converges to f in $L^\Phi(\mathbf{R}^N)$:*

$$\lim_{t\to 0} \|\kappa_t * f - f\|_{L^\Phi(\mathbf{R}^N)} = 0$$

for every $f \in L^\Phi(\mathbf{R}^N)$.

Proof. Let $f \in L^\Phi(\mathbf{R}^N)$. Given $\varepsilon > 0$, choose a bounded function h with compact support such that $\|f - h\|_{L^\Phi(\mathbf{R}^N)} < \varepsilon$. As in the proof of Theorem 3.5, using Theorem 4.5 this time, we have

$$\|\kappa_t * f - f\|_{L^\Phi(\mathbf{R}^N)} \le (C\|\kappa\|_{L^{(p_0)'}(\mathbf{R}^N)} + 1)\varepsilon + \|\kappa_t * h - h\|_{L^\Phi(\mathbf{R}^N)}.$$

Obviously, $h \in L^{p_0}(\mathbf{R}^N)$. Hence by Lemma 4.1, $\kappa_t * h \to h$ almost everywhere in $\mathbf{R}^N$, and hence

$$\overline{\Phi}(x, |\kappa_t * h(x) - h(x)|) \to 0$$

almost everywhere in $\mathbf{R}^N$. Since $\{\kappa_t * h - h\}$ is uniformly bounded and there is a compact set S containing all the supports of $\kappa_t * h$, $\{\overline{\Phi}(x, |\kappa_t * h(x) - h(x)|)\}$ is uniformly bounded and S contains all the supports of $\overline{\Phi}(x, |\kappa_t * h(x) - h(x)|)$. Hence the Lebesgue convergence theorem implies

$$\int_{\mathbf{R}^N} \overline{\Phi}(x, |\kappa_t * h(x) - h(x)|)\, dx \to 0$$

as $t \to 0$. Then, by Lemma 2.2, we see that $\|\kappa_t * h - h\|_{L^\Phi(\mathbf{R}^N)} \to 0$ as $t \to 0$, so that

$$\limsup_{t\to 0} \|\kappa_t * f - f\|_{L^\Phi(\mathbf{R}^N)} \le (C\|\kappa\|_{L^{(p_0)'}(\mathbf{R}^N)} + 1)\varepsilon,$$

which completes the proof. □

5 Young type inequality

LEMMA 5.1. *Suppose $\Phi(x,t)$ satisfies* ($\Phi6$). *Let $\kappa \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ with $\|\kappa\|_{L^1(\mathbf{R}^N)} \le 1$. For $f \in L^1_{loc}(\mathbf{R}^N)$, set*

$$I(f;x) = \int_{\mathbf{R}^N \setminus B(0,|x|/2)} |\kappa(x-y) f(y)|\, dy$$

and

$$J(f;x) = \int_{\mathbf{R}^N} |\kappa(x-y)| \overline{\Phi}(y, |f(y)|)\, dy.$$

Then there exists a constant $C > 0$ (depending on $\|\kappa\|_{L^\infty(\mathbf{R}^N)}$) such that

$$\overline{\Phi}(x, I(f;x)) \le C\,\{J(f;x) + g(x/2)\}$$

for all $x \in \mathbf{R}^N$ and $f \in L^\Phi(\mathbf{R}^N)$ with $\|f\|_{L^\Phi(\mathbf{R}^N)} \le 1$, where g is the function appearing in ($\Phi6$).

Proof. Let $k > 0$. Since $t \mapsto \overline{\Phi}(x,t)/t$ is nondecreasing,

$$I(f;x) \le k \int_{\mathbf{R}^N} |\kappa(x-y)|\,dy + k \int_{\mathbf{R}^N \setminus B(0,|x|/2)} \frac{|\kappa(x-y)|\overline{\Phi}(y,|f(y)|)}{\overline{\Phi}(y,k)}\,dy.$$

If $g(x/2) \le k \le 1$, then $\overline{\Phi}(x,k) \le C\overline{\Phi}(y,k)$ for $|y| > |x|/2$ by $(\Phi6)$. Hence

$$I(f;x) \le k\left(1 + \frac{CJ(f;x)}{\overline{\Phi}(x,k)}\right) \quad \text{whenever } g(x/2) \le k \le 1. \tag{5.1}$$

Since $J(f;x) \le \|\kappa\|_{L^\infty(\mathbf{R}^N)}$, there exists $K_x \in [0,1]$ such that

$$\overline{\Phi}(x,K_x) = \frac{J(f;x)}{\|\kappa\|_{L^\infty(\mathbf{R}^N)}}\overline{\Phi}(x,1).$$

If $K_x \ge g(x/2)$, then taking $k = K_x$ in (5.1), we have

$$I(f;x) \le K_x\left(1 + \frac{C\|\kappa\|_{L^\infty(\mathbf{R}^N)}}{\overline{\Phi}(x,1)}\right) \le CK_x,$$

so that

$$\overline{\Phi}(x, I(f;x)) \le C\overline{\Phi}(x,K_x) \le CJ(f;x).$$

If $K_x < g(x/2)$, then

$$J(f;x) = \|\kappa\|_{L^\infty(\mathbf{R}^N)}\frac{\overline{\Phi}(x,K_x)}{\overline{\Phi}(x,1)} \le C\overline{\Phi}(x,g(x/2)).$$

Hence, taking $k = g(x/2)$ in (5.1), we have $I(f;x) \le Cg(x/2)$, so that

$$\overline{\Phi}(x, I(f;x)) \le C\overline{\Phi}(x,g(x/2)) \le Cg(x/2).$$

Hence, we have the assertion of the lemma. □

Here, we recall the following result on the boundedness of the maximal operator M on $L^\Phi(\mathbf{R}^N)$ (see [6, Corollary 4.4]):

LEMMA 5.2. *Suppose $\Phi(x,t)$ satisfies $(\Phi5)$, $(\Phi6)$ and*

$(\Phi3^*)$ *$t \mapsto t^{-\varepsilon_0}\phi(x,t)$ is uniformly almost increasing on $(0,\infty)$ for some $\varepsilon_0 > 0$.*

Then the maximal operator M is bounded from $L^\Phi(\mathbf{R}^N)$ into itself, namely

$$\|Mf\|_{L^\Phi(\mathbf{R}^N)} \le C\|f\|_{L^\Phi(\mathbf{R}^N)}$$

for all $f \in L^\Phi(\mathbf{R}^N)$.

THEOREM 5.3. *Suppose* $\Phi(x,t)$ *satisfies* ($\Phi5$), ($\Phi6$) *and* ($\Phi3^*$). *Let* $p_0 = 1+\varepsilon_0$ (>1) *and* $R>0$. *Assume that* $\kappa \in L^1(\mathbf{R}^N) \cap L^{(p_0)'}(B(0,R))$ *and* $|\kappa(x)| \le c_\kappa |x|^{-N}$ *for* $|x| \ge R$. *Then there is a constant* $C>0$ *such that*

$$\|\kappa * f\|_{L^\Phi(\mathbf{R}^N)} \le C\big(\|\kappa\|_{L^1(\mathbf{R}^N)} + \|\kappa\|_{L^{(p_0)'}(B(0,R))}\big)\|f\|_{L^\Phi(\mathbf{R}^N)}$$

for all $f \in L^\Phi(\mathbf{R}^N)$.

Proof. Let $f \in L^\Phi(\mathbf{R}^N)$ and $f \ge 0$. Assume that $\|f\|_{L^\Phi(\mathbf{R}^N)} \le 1$ and

$$\|\kappa\|_{L^1(\mathbf{R}^N)} + \|\kappa\|_{L^{(p_0)'}(B(0,R))} \le 1.$$

Let $\kappa_0 = \kappa\chi_{B(0,R)}$ and $\kappa_\infty = \kappa\chi_{\mathbf{R}^N\setminus B(0,R)}$.

By Theorem 4.5,

$$\|\kappa_0 * f\|_{L^\Phi(\mathbf{R}^N)} \le C.$$

Hence it is enough to show that

$$\int_{\mathbf{R}^N} \overline{\Phi}(x, |\kappa_\infty| * f(x))\,dx \le C. \tag{5.2}$$

Write

$$\begin{aligned}|\kappa_\infty| * f(x) &= \int_{B(0,|x|/2)} |\kappa_\infty(x-y)| f(y)\,dy + \int_{\mathbf{R}^N\setminus B(0,|x|/2)} |\kappa_\infty(x-y)| f(y)\,dy \\ &= I_1(x) + I_2(x).\end{aligned}$$

Since $|\kappa_\infty(x-y)| \le c_\kappa |x-y|^{-N}$ and $|x-y| \ge |x|/2$ for $|y| \le |x|/2$,

$$I_1(x) \le 2^N c_\kappa |x|^{-N} \int_{B(0,|x|/2)} f(y)\,dy \le 2^N c_\kappa |x|^{-N} \int_{B(x,3|x|/2)} f(y)\,dy \le CMf(x).$$

Hence,

$$\int_{\mathbf{R}^N} \overline{\Phi}(x, I_1(x))\,dx \le C$$

by Lemma 5.2.

On the other hand, by Lemma 5.1,

$$\overline{\Phi}(x, I_2(x)) \le C\big\{|\kappa_\infty| * h(x) + g(x/2)\big\},$$

where $h(y) = \overline{\Phi}(y, f(y))$. Since

$$\||\kappa_\infty| * h\|_{L^1(\mathbf{R}^N)} \le \||\kappa_\infty|\|_{L^1(\mathbf{R}^N)}\|h\|_{L^1(\mathbf{R}^N)} \le 1$$

and $g \in L^1(\mathbf{R}^N)$, it follows that

$$\int_{\mathbf{R}^N} \overline{\Phi}(x, I_2(x))\,dx \le C.$$

Hence we obtain (5.2), and the proof is complete. □

References

[1] D. Cruz-Uribe and A. Fiorenza, Approximate identities in variable L^p spaces, Math. Nachr. **280** (2007), 256-270.

[2] L. Diening, Maximal function on Musielak-Orlicz spaces, Bull. Sci. Math. **129**, 657–700.

[3] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Math. **2017**, Springer-Verlag, Berlin, 2011.

[4] F.-Y. Maeda, Y. Mizuta and T. Ohno, Approximate identities and Young type inequalities in variable Lebesgue-Orlicz spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$, Ann. Acad. Sci. Fenn. Math. **35** (2010), 405-420.

[5] F.-Y. Maeda, Y. Mizuta, T. Ohno and T. Shimomura, Boundedness of maximal operator and Sobolev's inequality on Musielak-Orlicz-Morrey spaces, Research on potential theory 2010, Ohita Univ., 17–30.

[6] F.-Y. Maeda, Y. Mizuta, T. Ohno and T. Shimomura, Boundedness of maximal operators and Sobolev's inequality on Musielak-Orlicz-Morrey spaces, Bull. Sci. Math., in press (DOI:10.1016/j.bulsci.2012.03.008).

[7] F.-Y. Maeda, Y. Mizuta, T. Ohno and T. Shimomura, Mean continuity for potentials of functions in Musielak-Orlicz spaces, submitted.

[8] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. **1034**, Springer-Verlag, Berlin, 1983.

[9] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.

[10] F. Zo, A note on approximation of the identity, Studia. Math. **55** (1976), 111-122.

4-24 Furue-higashi-machi, Nishi-ku, Hiroshima 733-0872, Japan
E-mail : `fymaeda@h6.dion.ne.jp`

Department of Mechanical Systems Engineering,
Hiroshima Institute of Technology
2-1-1 Miyake Saeki-ku Hiroshima 731-5193, Japan
E-mail : `yoshihiromizuta3@gmail.com`

Faculty of Education and Welfare Science, Oita University
Dannoharu Oita-city 870-1192, Japan
E-mail : `t-ohno@oita-u.ac.jp`

Department of Mathematics, Graduate School of Education, Hiroshima University
Higashi-Hiroshima 739-8524, Japan
E-mail : tshimo@hiroshima-u.ac.jp