SOBOLEV'S INEQUALITY FOR RIESZ POTENTIALS IN LORENTZ SPACES OF VARIABLE EXPONENT

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ABSTRACT. In the present paper we discuss the boundedness of the maximal operator in the Lorentz space of variable exponent defined by the symmetric decreasing rearrangement in the sense of Almut [1]. As an application of the boundedness of the maximal operator, we establish the Sobolev inequality by using Hedberg's trick in his paper [10].

1. INTRODUCTION

In this paper we use $B(x,r)$ to denote the open ball centered at x of radius $r>0$. The volume of a measurable set $E \subset \mathbf{R}^n$ is written as $|E|$.

Given a measurable function f on $\mathbf{R}^n$, recall the symmetric decreasing rearrangement of f defined by

$$f^\star(x) = \int_0^\infty \chi_{E_f(t)^\star}(x)dt,$$

where $E^\star = \{x : |B(0,|x|)| < |E|\}$ and $E_f(t) = \{y : |f(y)| > t\}$ (see Almut [1]). Note here that

$$f^*(|B(0,|x|)|) = f^\star(x),$$

where f^* is the usual decreasing rearrangement of f. The fundamental fact of the symmetric decreasing rearrangement of f is that

$$|E_f(t)| = |E_{f^\star}(t)|$$

for all $t \geq 0$. This readily gives the rearrangement preserving L^p-norm property such as

$$\|f\|_{L^p(\mathbf{R}^n)} = \|f^\star\|_{L^p(\mathbf{R}^n)}$$

for $1 \leq p \leq \infty$. For fundamental properties of the symmetric decreasing rearrangement, we refer the reader to the Lecture Notes by Almut [1]; see also his papers [2, 3].

For variable exponents p, q, the Lorentz space $\mathcal{L}^{p,q}(\mathbf{R}^n)$ is defined as the set of all measurable functions f on $\mathbf{R}^n$ with

$$\|f\|_{\mathcal{L}^{p,q}(\mathbf{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbf{R}^n} |f^\star(x)/\lambda|^{q(x)} |x|^{n\left(\frac{q(x)}{p(x)}-1\right)} dx \leq 1\right\} < \infty.$$

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If p and q are radial, then we refer the reader to the paper by Ephremidze, Kokilashvili and Samko [8].

In Lorentz spaces, we establish the Sobolev inequality for the Riesz potential

$$I_\alpha f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)dy$$

of order α; for fundamental properties of Riesz potentials, see e.g. [12]. To do this, we first prepare the boundedness of the maximal operator, and then apply Hedberg's method in [10]. For this purpose, our task is to discuss the Hardy type operator

$$H_\alpha f(x) = \int_{\mathbf{R}^n \setminus B(0,|x|)} |y|^{\alpha-n} f(y)dy.$$

2. Symmetric decreasing rearrangement

Let us recall the Hardy-Littlewood inequality for the symmetric decreasing rearrangement (see Almut [1, Lemma 1.6]).

Lemma 2.1. *For all nonnegative measurable functions f and g on $\mathbf{R}^n$,*

$$\int_{\mathbf{R}^n} f(x)g(x)dx \le \int_{\mathbf{R}^n} f^\star(x)g^\star(x)dx.$$

The (centered) maximal function Mf of a measurable function f on $\mathbf{R}^n$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

Lemma 2.2. *For all measurable functions f on $\mathbf{R}^n$,*

$$\begin{aligned} (Mf)^\star(x) &\le C\frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} f^\star(y)dy \\ &\le CMf^\star(x), \end{aligned}$$

where C is a positive constant independent of f.

Proof. Recall the definition of the symmetric decreasing rearrangement and thus

$$(Mf)^\star(x) = \sup\{r>0 : |B(0,|x|)| < |\{z : Mf(z) \ge r\}|\}.$$

Set $r_0 = (Mf)^\star(x)$. Then, using the covering property (see [12, Theorem 1.10.1]) and Lemma 2.1, we have for $0 < r < r_0$

$$\begin{aligned} |\{z : Mf(z) \ge r\}| &\le Cr^{-1} \int_{\{z:f(z)>r/2\}} f(y)dy \\ &\le Cr^{-1} \int_{\{z:f^\star(z)>r/2\}} f^\star(y)dy. \end{aligned}$$

If $\{z : f^\star(z) > r/2\} \subset B(0,|x|)$, then

$$\begin{aligned} r &\le C\frac{1}{|B(0,|x|)} \int_{B(0,|x|)} f^\star(y)dy \\ &\le CM(f^\star)(x). \end{aligned}$$

If $\{z : f^\star(z) > r/2\} \supset B(0,|x|)$, then

$$|\{z : f^\star(z) > r/2\}| \le \frac{2}{r} \int_{\{z:f^\star(z)>r/2\}} f^\star(y)dy.$$

Noting that

$$\frac{1}{|B(0,t)|}\int_{B(0,t)} f^\star(y)dy \le \frac{1}{|B(0,s)|}\int_{B(0,s)} f^\star(y)dy$$

when $0 < s < t$, we obtain

$$\begin{aligned}
\frac{r}{2} &\le \frac{1}{|\{z : f^\star(z) > r/2\}|}\int_{\{z:f^\star(z)>r/2\}} f^\star(y)dy \\
&\le \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} f^\star(y)dy \\
&\le CMf^\star(x),
\end{aligned}$$

as required. □

The following is known as Riesz' inequality (see Almut [1, §1.3]).

Lemma 2.3. *For all nonnegative measurable functions f, g and h on $\mathbf{R}^n$,*

$$\int_{\mathbf{R}^n} f(x)(g*h)(x)dx \le \int_{\mathbf{R}^n} f^\star(x)(g^\star * h^\star)(x)dx,$$

where

$$g*h(x) = \int_{\mathbf{R}^n} g(x-y)h(y)dy.$$

Lemma 2.4. *For all nonnegative measurable functions f on $\mathbf{R}^n$,*

$$\begin{aligned}
(I_\alpha f)^\star(x) &\le C\int_{\mathbf{R}^n} (|x|+|y|)^{\alpha-n} f^\star(y)dy \\
&\le C(I_\alpha f^\star)(x),
\end{aligned}$$

where C is a positive constant independent of f.

Proof. Set $r_0 = (I_\alpha f)^\star(x)$. For $0 < r < r_0$, write

$$|\{z : I_\alpha f(z) > r\}| = |B(0,t)|.$$

We have

$$\begin{aligned}
|B(0,t)| &= |\{z : I_\alpha f(z) > r\}| \\
&\le r^{-1}\int_{\{z:I_\alpha f(z)>r\}} I_\alpha f(\zeta)d\zeta \\
&\le r^{-1}\int_{\{z:(I_\alpha f)^\star(z)>r\}} I_\alpha f^\star(\zeta)d\zeta \qquad \text{(by Riesz' inequality)} \\
&= r^{-1}\int_{B(0,t)} I_\alpha f^\star(\zeta)d\zeta \\
&= r^{-1}\int_{\mathbf{R}^n}\left(\int_{B(0,t)} |\zeta-y|^{\alpha-n}d\zeta\right) f^\star(y)dy \\
&\le Cr^{-1}t^n\int_{\mathbf{R}^n} (t+|y|)^{\alpha-n} f^\star(y)dy,
\end{aligned}$$

so that

$$r \le C\int_{\mathbf{R}^n} (t+|y|)^{\alpha-n} f^\star(y)dy.$$

Since $t \geq |x|$,

$$r \leq C \int_{\mathbf{R}^n} (|x| + |y|)^{\alpha - n} f^\star(y) dy,$$

which gives the required inequality. $\square$

Remark 2.5. In case $\alpha = 0$, I_α might be replaced by the singular intergal operator (see [4] and [8, Theorem 3.14]).

3. Lorentz spaces of variable exponents

A function p on $\mathbf{R}^n$ is said to be log-Hölder continuous if

(P1) p is locally log-Hölder continuous, namely

$$|p(x) - p(y)| \leq \frac{C_0}{\log(1/|x-y|)} \quad \text{for } |x-y| \leq \frac{1}{e}$$

with a constant $C_0 \geq 0$, and

(P2) p is log-Hölder continuous at infinity, namely

$$|p(x) - p(\infty)| \leq \frac{C_\infty}{\log(e + |x|)}$$

with constants $C_\infty \geq 0$ and $p(\infty)$. Let $\mathcal{P}(\mathbf{R}^n)$ be the class of all log-Hölder continuous functions p on $\mathbf{R}^n$. If in addition p satisfies

(P3) $1 < p^- := \inf_{x \in \mathbf{R}^n} p(x) \leq \sup_{x \in \mathbf{R}^n} p(x) =: p^+ < \infty$,

then we write $p \in \mathcal{P}_1(\mathbf{R}^n)$.

Definition 3.1. *For $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$, $L^{t,q}(\mathbf{R}^n)$ denotes the weighted $L^{q(\cdot)}$ space of all functions f with*

$$\|f\|_{L^{t,q}(\mathbf{R}^n)} = \inf \left\{ \lambda > 0 : \int_{\mathbf{R}^n} |f(x)/\lambda|^{q(x)} |x|^{t(x)} dx \leq 1 \right\} < \infty.$$

We write $L^{0,q}(\mathbf{R}^n) = L^{q(\cdot)}(\mathbf{R}^n)$ and

$$\|f\|_{L^{0,q}(\mathbf{R}^n)} = \|f\|_{L^{q(\cdot)}(\mathbf{R}^n)}.$$

Definition 3.2. *For $p, q \in \mathcal{P}_1(\mathbf{R}^n)$, set $t(x) = n\left(\frac{q(x)}{p(x)} - 1\right)$. Denote by $\mathcal{L}^{p,q}(\mathbf{R}^n)$ as the set of all measurable functions f such that*

$$\|f\|_{\mathcal{L}^{p,q}(\mathbf{R}^n)} = \inf \left\{ \lambda > 0 : \int_{\mathbf{R}^n} |f^\star(x)/\lambda|^{q(x)} |x|^{n\left(\frac{q(x)}{p(x)} - 1\right)} dx \leq 1 \right\} < \infty.$$

We do not know whether $\mathcal{L}^{q,q}(\mathbf{R}^n) = L^{0,q}(\mathbf{R}^n)$ or not, when q is a variable exponent.

4. The boundedness of maximal operator in Lorentz spaces of variable exponents

Throughout this paper, let C denote various constants independent of the variables in question. For functions f, g, we write $f \sim g$ if there is a constant $C > 1$ such that

$$C^{-1} g \leq f \leq Cg.$$

We first show the boundedness of the maximal operator in the weighted $L^{q(\cdot)}$ space along the same lines as in [13], which is an extension of Diening [7] and Cruz-Uribe, Fiorenza and Neugebauer [6].

THEOREM 4.1. *Let* $q \in \mathcal{P}_1(\mathbf{R}^n)$ *and* $t \in \mathcal{P}(\mathbf{R}^n)$*. Suppose*

(T1) $-n < t(0) < n(q(0) - 1)$ *and* $-n < t(\infty) < n(q(\infty) - 1)$.

Then the maximal operator $\mathcal{M} : f \longrightarrow Mf$ *is bounded from* $L^{t,q}(\mathbf{R}^n)$ *into itself, namely, there is a constant* $C > 0$ *such that*

$$\|Mf\|_{L^{t,q}(\mathbf{R}^n)} \le C\|f\|_{L^{t,q}(\mathbf{R}^n)}$$

for all $f \in L^{t,q}(\mathbf{R}^n)$.

With the aid of Lemma 2.2, we obtain the following result.

COROLLARY 4.2. *Let* $p, q \in \mathcal{P}_1(\mathbf{R}^n)$*. Then the maximal operator* $\mathcal{M} : f \longrightarrow Mf$ *is bounded from* $\mathcal{L}^{p,q}(\mathbf{R}^n)$ *into itself, namely, there is a constant* $C > 0$ *such that*

$$\|Mf\|_{\mathcal{L}^{p,q}(\mathbf{R}^n)} \le C\|f\|_{\mathcal{L}^{p,q}(\mathbf{R}^n)}$$

for all $f \in \mathcal{L}^{p,q}(\mathbf{R}^n)$.

Our Theorem 4.1 is a special case of Theorem 1.1 in Hästö and Diening [9] (see also Cruz-Uribe, Diening and Hästö [5]). For the reader's convenience, we give a proof of Theorem 4.1. To do so, we prepare several lemmas in the same way as in [11] and [13].

5. PROOF OF THEOREM 4.1

Let us begin the well-known fact for the boundedness of maximal operator; see Diening [7] and Cruz-Uribe, Fiorenza and Neugebauer [6].

LEMMA 5.1. *Let* $q \in \mathcal{P}_1(\mathbf{R}^n)$*. Then*

$$\int_{\mathbf{R}^n} \{Mf(x)\}^{q(x)} dx \;\; \le \;\; C$$

for all f *with* $\|f\|_{L^{q(\cdot)}(\mathbf{R}^n)} \le 1$.

To show this, we apply Diening's trick with the aid of the following result (see e.g. [11]).

LEMMA 5.2. *Let* $q \in \mathcal{P}_1(\mathbf{R}^n)$*. For a measurable function* f *on* $\mathbf{R}^n$*, set*

$$I = \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

and

$$J = \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} g(y) dy \right)^{1/q(x)},$$

where $g(y) = |f(y)|^{q(y)}$*. Then there exists a constant* $C > 0$ *such that*

$$I \le CJ + C(1 + |x|)^{-n}$$

for all $x \in \mathbf{R}^n$ *and* f *such that* $\|f\|_{L^{q(\cdot)}(\mathbf{R}^n)} \le 1$.

We next consider the weighted case.

Lemma 5.3. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$, $t \in \mathcal{P}(\mathbf{R}^n)$ and $0 < r_0 < 1$. Suppose there exists a constant q_0 with $1 < q_0 < q(0)$ such that*

$$-n < t(0) < n(q_0 - 1).$$

For a measurable function f on $\mathbf{R}^n$, set

$$I = \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

and

$$J_0 = \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} g_0(y) dy \right)^{1/q_0},$$

where $g_0(y) = \{|f(y)||y|^{t(y)/q(y)}\}^{q_0}$. Then

$$I \le C|x|^{-t(x)/q(x)} J_0 + CH(x)$$

for all $x \in B(0, r_0)$ and f such that $f = 0$ outside $B(0, 2r_0)$ and $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$, where

$$H(x) = \int_{\mathbf{R}^n \setminus B(0,|x|)} |f(y)||y|^{-n} dy.$$

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $f = 0$ outside $B(0, 2r_0)$ and $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$. By Hölder's inequality we have

$$I \;\le\; J_0 \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |y|^{-t(y)q_0'/q(y)} dy \right)^{1/q_0'}.$$

If $r \le |x|/2 < r_0/2$, then $|y| \sim |x|$, and moreover

$$|y|^{p(y)-p(x)} \le C|y|^{-C_0/\log(1/|x-y|)} \le C|x|^{-C_0/\log(1/|x|)} \le C,$$

so that

$$|y|^{p(y)} \sim |y|^{p(x)} \sim |x|^{p(x)}$$

for $x \in B(0, r_0)$, $y \in B(x, r)$ and $p \in \mathcal{P}(\mathbf{R}^n)$ by (P1). Hence, in this case,

$$\begin{aligned} \frac{1}{|B(x,r)|} \int_{B(x,r)} |y|^{-t(y)q_0'/q(y)} dy \;&\le\; C \frac{1}{|B(x,r)|} \int_{B(x,r)} |x|^{-t(x)q_0'/q(x)} dy \\ &\le\; C|x|^{-t(x)q_0'/q(x)}, \end{aligned}$$

which gives

$$I \le C|x|^{-t(x)/q(x)} J_0.$$

If $|x|/2 < r \le 2|x| < 2r_0$, then, since $|y|^{p(y)} \sim |y|^{p(0)}$ for $y \in B(0, 3r_0)$ and $p \in \mathcal{P}(\mathbf{R}^n)$ by (P1), we find

$$\begin{aligned} \frac{1}{|B(x,r)|} \int_{B(x,r)} |y|^{-t(y)q_0'/q(y)} dy \;&\le\; C \frac{1}{|B(0,3|x|)|} \int_{B(0,3|x|)} |y|^{-t(0)q_0'/q(0)} dy \\ &\le\; C|x|^{-t(0)q_0'/q(0)} \\ &\le\; C|x|^{-t(x)q_0'/q(x)} \end{aligned}$$

since $t(0) < n(q_0 - 1)$ and $1 < q_0 < q(0)$.

Finally, if $r > 2|x|$ and $|x| < r_0$, then

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy &\le \frac{1}{|B(x,r)|}\int_{B(x,r)\cap B(x,2|x|)}|f(y)|dy \\ &\quad + \frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(x,2|x|)}|f(y)|dy \\ &\le C|x|^{-t(x)/q(x)}J_0 + C\int_{\mathbf{R}^n\setminus B(0,|x|)}|f(y)||y|^{-n}dy.\end{aligned}$$

Now the present lemma is obtained. □

COROLLARY 5.4. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$, $t \in \mathcal{P}(\mathbf{R}^n)$, $0 < r_0 < 1$ and $1 < q_0 < q(0)$ be as in Lemma 5.3. Then there exists a constant $C > 0$ such that*

$$Mf(x) \le C|x|^{-t(x)/q(x)}Mg_0(x)^{1/p_0} + CH(x)$$

for all $x \in B(0, r_0)$ and f such that $f = 0$ outside $B(0, 2r_0)$ and $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

REMARK 5.5. Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$. Suppose

$$-n < t(0) < n(q(0) - 1).$$

Then it is worth to see that

$$\int_{B(0,1)}|f(y)|dy \le C$$

for all f with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

Next we treat the Hardy type operator H along the same manner as in [13].

LEMMA 5.6. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$. For a measurable function f on $\mathbf{R}^n$ and $\beta \ge 0$, set*

$$H_{\beta,1} = H_{\beta,1}(x) = \int_{B(0,1)\setminus B(0,|x|)}|f(y)||y|^{\beta-n}dy$$

and

$$K_{\beta,1} = K_{\beta,1}(x) = \left(|x|^{\varepsilon}\int_{B(0,1)\setminus B(0,|x|)}g(y)|y|^{-\varepsilon}dy\right)^{1/q(x)},$$

where $g(y) = |f(y)|^{q(y)}|y|^{t(y)}$. If $0 < \delta < \varepsilon < (n + t(0))/q(0) - \beta$, then there exists a constant $C > 0$ such that

$$H_{\beta,1} \le C|x|^{\beta-(t(x)+n)/q(x)}K_{\beta,1} + C|x|^{\beta-(t(x)-\delta+n)/q(x)}$$

for all $x \in B(0,1)$ and f with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$. Set

$$E = \{y : f(y) \ge |y|^{-(t(y)+n)/q(y)}\}.$$

Noting that $|y|^{p(y)} \sim |y|^{p(0)}$ when $|y| < 1$ and $p \in \mathcal{P}(\mathbf{R}^n)$ by (P1), we have

$$\begin{aligned}
H_{\beta,1,1} &\equiv \int_{E\cap(B(0,1)\setminus B(0,|x|))} f(y)|y|^{\beta-n}dy \\
&\le \int_{E\cap(B(0,1)\setminus B(0,|x|))} f(y)|y|^{\beta-n}\left(\frac{f(y)}{|y|^{-(t(y)+n)/q(y)}}\right)^{q(y)-1} dy \\
&= \int_{B(0,1)\setminus B(0,|x|)} f(y)^{q(y)}|y|^{t(y)-\varepsilon}|y|^{\beta+\varepsilon-(t(y)+n)/q(y)}dy \\
&\le |x|^{\beta-(t(0)+n)/q(0)}|x|^{\varepsilon}\int_{B(0,1)\setminus B(0,|x|)} g(y)|y|^{-\varepsilon}dy \\
&\le C|x|^{\beta-(t(x)+n)/q(x)}K_{\beta,1}
\end{aligned}$$

since $\varepsilon < (n+t(0))/q(0)-\beta$ and $K_{\beta,1} < 1$.

Noting that $|y|^{p(x)} \sim |y|^{p(y)} \sim |y|^{p(0)}$ for $y \in B(0,1)\setminus B(0,|x|)$ and $p \in \mathcal{P}(\mathbf{R}^n)$ by (P1), we next obtain

$$\begin{aligned}
H_{\beta,1,2} &\equiv \int_{(B(0,1)\setminus B(0,|x|))\setminus E} f(y)|y|^{\beta-n}dy \\
&\le \left(\int_{(B(0,1)\setminus B(0,|x|))\setminus E} |y|^{(\beta-(t(y)-\varepsilon)/q(x)-n)q'(x)}dy\right)^{1/q'(x)} \\
&\quad \times \left(\int_{(B(0,1)\setminus B(0,|x|))\setminus E} f(y)^{q(x)}|y|^{t(y)-\varepsilon}dy\right)^{1/q(x)} \\
&\le C|x|^{\beta-(t(x)-\varepsilon+n)/q(x)}\left(\int_{(B(0,1)\setminus B(0,|x|))\setminus E} f(y)^{q(x)}|y|^{t(y)-\varepsilon}dy\right)^{1/q(x)}.
\end{aligned}$$

Moreover, by taking $0 < \delta < \varepsilon$, we see that

$$\begin{aligned}
&\left(\int_{(B(0,1)\setminus B(0,|x|))\setminus E} f(y)^{q(x)}|y|^{t(y)-\varepsilon}dy\right)^{1/q(x)} \\
&= \left(\int_{(B(0,1)\setminus B(0,|x|))\setminus E} (f(y)|y|^{(t(x)+n)/q(x)})^{q(x)}|y|^{-\varepsilon-n}dy\right)^{1/q(x)} \\
&\le C\left(\int_{(B(0,1)\setminus B(0,|x|))\setminus E} |y|^{\delta}|y|^{-\varepsilon-n}dy\right)^{1/q(x)} \\
&\quad + C\left(\int_{(B(0,1)\setminus B(0,|x|))\setminus E} f(y)^{q(y)}|y|^{t(y)+n}|y|^{-\varepsilon-n}dy\right)^{1/q(x)} \\
&\le C|x|^{(\delta-\varepsilon)/q(x)} + C\left(\int_{(B(0,1)\setminus B(0,|x|))\setminus E} f(y)^{q(y)}|y|^{t(y)-\varepsilon}dy\right)^{1/q(x)},
\end{aligned}$$

so that

$$H_{\beta,1,2} \le C|x|^{\beta-(t(x)-\delta+n)/q(x)} + C|x|^{\beta-(t(x)+n)/q(x)}K_{\beta,1},$$

which completes the proof. □

Corollary 5.7. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$. If $0 < \delta < \varepsilon < (n+t(0))/q(0)$, then there exists a constant $C > 0$ such that*

$$\int_{B(0,1)} H_{0,1}(x)^{q(x)} |x|^{t(x)} dx \quad \le \quad C$$

for all f with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

In fact, letting $0 < \delta < \varepsilon < (n+t(0))/q(0)$, we find from Lemma 5.6

$$\begin{aligned}
\int_{B(0,1)} H_{0,1}(x)^{q(x)} |x|^{t(x)} dx &\le C \int_{\mathbf{R}^n} |x|^{-n} K_{0,1}(x)^{q(x)} dx + C \int_{B(0,1)} |x|^{\delta-n} dx \\
&\le C \int_{B(0,1)} \left(\int_{B(0,|y|)} |x|^{\varepsilon-n} dx \right) g(y)|y|^{-\varepsilon} dy + C \\
&\le C \int_{B(0,1)} g(y) dy + C \\
&\le C
\end{aligned}$$

for all f with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

Collecting Corollaries 5.4, 5.7 and Lemma 5.1, we obtain the following result.

Lemma 5.8. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$. If $-n < t(0) < n(q(0)-1)$, then there exist constants $r_0 > 0$ and $C > 0$ such that*

$$\int_{B(0,r_0)} \{Mf(x)\}^{q(x)} |x|^{t(x)} dx \quad \le \quad C$$

for all f with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$ such that $f = 0$ outside $B(0, 2r_0)$.

To show this, take $r_0 > 0$ such that $1 < q_0 < \inf_{x \in B(0,r_0)} q(x)$ and $-n < t(0) < n(q_0 - 1)$, and apply Corollaries 5.4, 5.7 and Lemma 5.1.

Next we treat the behavior of maximal functions near the infinity. In the same manner as Lemma 5.3, we can prove the following result.

Lemma 5.9. *Let $q \in \mathcal{P}_1(\mathbf{R}^n), t \in \mathcal{P}(\mathbf{R}^n)$ and $R_0 > 1$. Suppose there exists a constant $1 < q_0 < q(\infty)$ such that*

$$-n < t(\infty) < n(q_0 - 1).$$

Then there exists a constant $C > 0$ such that

$$I \le C|x|^{-t(x)/q(x)} J_0 + CH(x)$$

for all $x \in \mathbf{R}^n \setminus B(0, 2R_0)$ and f such that $f = 0$ on $B(0, R_0)$ and $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$, where I and J_0 are given in Lemma 5.3.

Lemma 5.10. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$. For a measurable function f on $\mathbf{R}^n$ and $\beta \ge 0$, set*

$$H_{\beta,2} = H_{\beta,2}(x) = \int_{\mathbf{R}^n \setminus B(0,|x|)} |f(y)||y|^{\beta-n} dy$$

and

$$K_{\beta,2} = K_{\beta,2}(x) = \left(|x|^{\varepsilon} \int_{\mathbf{R}^n \setminus B(0,|x|)} g(y)|y|^{-\varepsilon} dy \right)^{1/q(x)},$$

where $g(y) = |f(y)|^{q(y)}|y|^{t(y)}$. *If* $0 < \varepsilon < \{(n + t(\infty))/q(\infty) - \beta\}/q(\infty)$, *then there exists a constant* $C > 0$ *such that*

$$H_{\beta,2} \le C|x|^{\beta-(t(x)+n)/q(x)}K_{\beta,2} + C|x|^{\beta-\varepsilon-(t(x)+n)/q(x)}$$

for all $x \in \mathbf{R}^n \setminus B(0,1)$ *and* f *with* $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$. Then note that

$$|x|^{\varepsilon}\int_{\mathbf{R}^n\setminus B(0,|x|)} g(y)|y|^{-\varepsilon}dy < 1.$$

If $|x|^{-\varepsilon} < K_{\beta,2} < 1$, then

$$K_{\beta,2}^{p(x)-p(y)} \le K_{\beta,2}^{-C/\log(e+|x|)} \le |x|^{\varepsilon C/\log(e+|x|)} \le C$$

and $|y|^{p(y)} \sim |y|^{p(\infty)}$ when $1 < |x| < |y|$ and $p \in \mathcal{P}(\mathbf{R}^n)$ by (P2), so that

$$\begin{aligned}
H_{\beta,2} &\le \int_{\mathbf{R}^n\setminus B(0,|x|)} K_{\beta,2}|y|^{-(t(y)+n)/q(y)}|y|^{\beta-n}dy \\
&\quad + \int_{\mathbf{R}^n\setminus B(0,|x|)} f(y)|y|^{\beta-n}\left(\frac{f(y)}{K_{\beta,2}|y|^{-(t(y)+n)/q(y)}}\right)^{q(y)-1}dy \\
&\le CK_{\beta,2}|x|^{\beta-(t(\infty)+n)/q(\infty)} \\
&\quad + CK_{\beta,2}^{1-q(x)}\int_{\mathbf{R}^n\setminus B(0,|x|)} g(y)|y|^{\beta-(t(y)+n)/q(y)}dy \\
&\le CK_{\beta,2}|x|^{\beta-(t(x)+n)/q(x)} \\
&\quad + CK_{\beta,2}^{1-q(x)}|x|^{\beta-(t(\infty)+n)/q(\infty)}|x|^{\varepsilon}\int_{B(0,1)\setminus B(0,|x|)} g(y)|y|^{-\varepsilon}dy \\
&\le CK_{\beta,2}|x|^{\beta-(t(x)+n)/q(x)}
\end{aligned}$$

since $\varepsilon < (n + t(\infty))/q(\infty) - \beta$.

Next consider the case $K_{\beta,2} \le |x|^{-\varepsilon}$. Then

$$\begin{aligned}
H_{\beta,2} &\le \int_{\mathbf{R}^n\setminus B(0,|x|)} |y|^{-\varepsilon-(t(y)+n)/q(y)}|y|^{\beta-n}dy \\
&\quad + \int_{\mathbf{R}^n\setminus B(0,|x|)} f(y)|y|^{\beta-n}\left(\frac{f(y)}{|y|^{-\varepsilon-(t(y)+n)/q(y)}}\right)^{q(y)-1}dy \\
&\le C|x|^{\beta-\varepsilon-(t(\infty)+n)/q(\infty)} \\
&\quad + C\int_{\mathbf{R}^n\setminus B(0,|x|)} g(y)|y|^{\beta+\varepsilon(q(y)-1)-(t(y)+n)/q(y)}dy \\
&\le C|x|^{\beta-\varepsilon-(t(x)+n)/q(x)} \\
&\quad + C|x|^{\beta+\varepsilon(q(\infty)-1)-(t(\infty)+n)/q(\infty)}|x|^{\varepsilon}\int_{\mathbf{R}^n\setminus B(0,|x|)} g(y)|y|^{-\varepsilon}dy \\
&\le C|x|^{\beta-\varepsilon-(t(x)+n)/q(x)}
\end{aligned}$$

since $\varepsilon < \{(n + t(\infty))/q(\infty) - \beta\}/q(\infty)$.

Thus the proof is completed. $\square$

Corollary 5.11. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$. If $0 < \varepsilon < (n + t(\infty))/q(\infty)^2$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbf{R}^n \setminus B(0,1)} H_{0,2}(x)^{q(x)} |x|^{t(x)} dx \le C$$

for all f with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

In fact, letting $0 < \varepsilon < (n + t(\infty))/q(\infty)^2$, we find from Lemma 5.10

$$\begin{aligned}
\int_{\mathbf{R}^n \setminus B(0,1)} H_{0,2}(x)^{q(x)} |x|^{t(x)} dx &\le C \int_{\mathbf{R}^n \setminus B(0,1)} |x|^{-n} K_{0,2}(x)^{q(x)} dx \\
&\quad + C \int_{\mathbf{R}^n \setminus B(0,1)} |x|^{-\varepsilon q(x) - n} dx \\
&\le C \int_{\mathbf{R}^n \setminus B(0,1)} \left(\int_{B(0,|y|)} |x|^{\varepsilon - n} dx \right) g(y) |y|^{-\varepsilon} dy + C \\
&\le C \int_{\mathbf{R}^n \setminus B(0,1)} g(y) dy + C \\
&\le C
\end{aligned}$$

for all f with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

Remark 5.12. Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$. Suppose

$$-n < t(\infty) < n(q(\infty) - 1).$$

Then, as in Remark 5.5, it is worth to see that

$$\int_{\mathbf{R}^n \setminus B(0,1)} |f(y)||y|^{-n} dy \le C$$

for all f with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

By Lemma 5.1, Lemma 5.9 and Corollary 5.11, we have the following result.

Lemma 5.13. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$. If $-n < t(\infty) < n(q(\infty) - 1)$, then there exist constants $R_0 > 0$ and $C > 0$ such that*

$$\int_{\mathbf{R}^n \setminus B(0,2R_0)} \{Mf(x)\}^{q(x)} |x|^{t(x)} dx \le C$$

for all f with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$ such that $f = 0$ on $B(0, R_0)$.

Proof of Theorem 4.1. We now show the boundedness of the maximal operator.

To do so, take a nonnegative measurable function f on $\mathbf{R}^n$ with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$. Let r_0 and R_0 be as in Lemma 5.8 and Lemma 5.13.

By Lemma 5.8 we have

$$\int_{B(0,r_0)} \{M[f\chi_{B(0,2r_0)}](x)\}^{q(x)} |x|^{t(x)} dx < C.$$

By Lemmas 5.6 and 5.10 we see that

$$M[f\chi_{\mathbf{R}^n \setminus B(0,2r_0)}](x) \le \int_{\mathbf{R}^n \setminus B(0,2r_0)} |y|^{-n} f(y) dy \le C$$

for $x \in B(0, r_0)$, so that

$$\int_{B(0,r_0)} \{Mf(x)\}^{q(x)} |x|^{t(x)} dx < C.$$

Lemma 5.1 gives

$$\int_{B(0,2R_0)\setminus B(0,r_0)} \{M[f\chi_{B(0,4R_0)\setminus B(0,r_0/2)}](x)\}^{q(x)}dx < C,$$

so that

$$\int_{B(0,2R_0)\setminus B(0,r_0)} \{M[f\chi_{B(0,4R_0)\setminus B(0,r_0/2)}](x)\}^{q(x)}|x|^{t(x)}dx < C.$$

Further, noting from Remarks 5.5 and 5.12 that

$$M[f\chi_{B(0,r_0/2)}](x) \le C|x|^{-n}\int_{B(0,r_0/2)} f(y)dy \le C$$

and

$$M[f\chi_{\mathbf{R}^n\setminus B(0,4R_0)}](x) \le \int_{\mathbf{R}^n\setminus B(0,4R_0)} |y|^{-n}f(y)dy \le C$$

for $x \in B(0,2R_0)\setminus B(0,r_0)$, we have

$$\int_{B(0,2R_0)\setminus B(0,r_0)} \{Mf(x)\}^{q(x)}|x|^{t(x)}dx < C.$$

In view of Lemma 5.13, we find

$$\int_{\mathbf{R}^n\setminus B(0,2R_0)} \{M[f\chi_{\mathbf{R}^n\setminus B(0,R_0)}](x)\}^{q(x)}|x|^{t(x)}dx < C.$$

Noting that $M[f\chi_{B(0,R_0)}](x) \le C|x|^{-n}$ for $x \in \mathbf{R}^n \setminus B(0,2R_0)$ by Lemmas 5.6 and 5.10, we establish

$$\begin{aligned}\int_{\mathbf{R}^n\setminus B(0,2R_0)} \{M[f\chi_{B(0,R_0)}]\}^{q(x)}|x|^{t(x)}dx &\le C\int_{\mathbf{R}^n\setminus B(0,2R_0)} |x|^{-nq(x)+t(x)}dx \\ &\le C\int_{\mathbf{R}^n\setminus B(0,2R_0)} |x|^{-nq(\infty)+t(\infty)}dx \le C\end{aligned}$$

since $t(\infty) < n(q(\infty)-1)$. Now the proof is completed. □

6. Sobolev's inequality in Lorentz spaces

As an application of Theorem 4.1, we establish the Sobolev type inequality for Riesz potentials by use of Hedberg's method in [10].

For $q \in \mathcal{P}_1(\mathbf{R}^n)$, set

$$1/q^\sharp = 1/q - \alpha/n.$$

Theorem 6.1. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$. Suppose $q^+ < n/\alpha$ and*

(T2) $\alpha q(0) - n < t(0) < n\,(q(0)-1)$ *and* $\alpha q(\infty) - n < t(\infty) < n\,(q(\infty)-1)$.

Then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{L^{tq^\sharp/q,q^\sharp}(\mathbf{R}^n)} \le C\|f\|_{L^{t,q}(\mathbf{R}^n)}$$

for all $f \in L^{t,q}(\mathbf{R}^n)$.

With the aid of Lemma 2.4, we find the following result.

COROLLARY 6.2. *For $p, q \in \mathcal{P}_1(\mathbf{R}^n)$, set $t(x) = n\left(\frac{q(x)}{p(x)} - 1\right)$. If $p^+ < n/\alpha$ and $q^+ < n/\alpha$, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\mathcal{L}^{p^\sharp,q^\sharp}(\mathbf{R}^n)} \le C\|f\|_{\mathcal{L}^{p,q}(\mathbf{R}^n)}$$

for all $f \in \mathcal{L}^{p,q}(\mathbf{R}^n)$.

To prove Theorem 6.1, we need several lemmas.

LEMMA 6.3. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$. If $-n < t(0) < n(q(0) - 1)$ and $-n < t(\infty) < n(q(\infty) - 1)$, then there exists a constant $C > 0$ such that*

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \le C|x|^{-t(x)/q(x)}r^{-n/q(x)} + C|x|^{-t(x)/q(x)}(1+|x|)^{-n}$$

for all $x \in \mathbf{R}^n, 0 < r < 2|x|$ and $f \ge 0$ with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$. Write

$$\begin{aligned} f &= f\chi_{B(0,r_0/2)} + f\chi_{B(0,2R_0)\setminus B(0,r_0/2)} + f\chi_{\mathbf{R}^n\setminus B(0,2R_0)} \\ &= f_1 + f_2 + f_3, \end{aligned}$$

where r_0 and R_0, $0 < r_0 < 1 < R_0 < \infty$, will be given soon.

For f_1, take $r_0 > 0$ such that $1 < q_0 < \inf_{x\in B(0,r_0)} q(x)$ with $-n < t(0) < n(q_0 - 1)$. As in the proof of Lemma 5.3, we have

$$\begin{aligned} &\frac{1}{|B(x,r)|}\int_{B(x,r)} f_1(y)dy \\ \le\ & \left(\frac{1}{|B(x,r)|}\int_{B(x,r)\cap B(0,r_0/2)} |y|^{-t(y)q_0'/q(y)}dy\right)^{1/q_0'} \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} g_{1,0}(y)dy\right)^{1/q_0} \\ \le\ & C|x|^{-t(x)/q(x)} \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} g_{1,0}(y)dy\right)^{1/q_0} \end{aligned}$$

for $0 < r < 2|x| < 2r_0$, where $g_{1,0}(y) = \{f_1(y)|y|^{t(y)/q(y)}\}^{q_0}$. By using [11, Lemmas 2.2 and 2.3], we obtain

$$\begin{aligned} \frac{1}{|B(x,r)|}\int_{B(x,r)} f_1(y)dy &\le C|x|^{-t(x)/q(x)} \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} g(y)dy\right)^{1/q(x)} \\ &\quad + C|x|^{-t(x)/q(x)}(1+|x|)^{-n} \\ &\le C|x|^{-t(x)/q(x)}r^{-n/q(x)} + C|x|^{-t(x)/q(x)}(1+|x|)^{-n} \end{aligned}$$

for $0 < r < 2|x| < 2r_0$. If $|x| \ge r_0$, then we see from Remark 5.5 that

$$\begin{aligned} \frac{1}{|B(x,r)|}\int_{B(x,r)} f_1(y)dy &\le C(1+|x|)^{-n} \\ &\le C|x|^{-t(x)/q(x)}r^{-n/q(x)} \end{aligned}$$

when $0 < r < 2|x|$.

For f_3, take $1 < q_0 < \inf_{x\in R^n\setminus B(0,R_0)} q(x)$ with $-n < t(\infty) < n(q(0) - 1)$. We have by the above arguments and the proof of Lemma 5.9

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f_3(y)dy \le C|x|^{-t(x)/q(x)}r^{-n/q(x)} + C|x|^{-t(x)/q(x)}(1+|x|)^{-n}$$

for $0 < r < 2|x|$. Finally, for f_2, taking $1 < q_0 < q^-$, we have by the above arguments

$$\begin{aligned} \frac{1}{|B(x,r)|}\int_{B(x,r)} f_2(y)dy &\le \left(\frac{1}{|B(x,r)|}\int_{B(x,r)\cap(B(0,2R_0)\setminus B(0,r_0/2))} |y|^{-t(y)q_0'/q(y)}dy\right)^{1/q_0'} \\ &\quad \times \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} g_{2,0}(y)dy\right)^{1/q_0} \\ &\le C|x|^{-t(x)/q(x)}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} g(y)dy\right)^{1/q(x)} \\ &\quad + C|x|^{-t(x)/q(x)}(1+|x|)^{-n} \end{aligned}$$

for $0 < r < 2|x| < 8R_0$, where $g_{2,0}(y) = \{f_2(y)|y|^{t(y)/q(y)}\}^{q_0}$. If $|x| \ge 4R_0$, then we see from Remark 5.5 that

$$\begin{aligned} \frac{1}{|B(x,r)|}\int_{B(x,r)} f_2(y)dy &\le C(1+|x|)^{-n} \\ &\le C|x|^{-t(x)/q(x)}r^{-n/q(x)} \end{aligned}$$

when $0 < r < 2|x|$.

Now the lemma is obtained. □

LEMMA 6.4. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$ satisfy $-n < t(0) < n(q(0)-1)$, $-n < t(\infty) < n(q(\infty)-1)$ and $q^+ < n/\alpha$. Then*

$$\begin{aligned} \int_{B(x,2|x|)\setminus B(x,\delta)} |x-y|^{\alpha-n}f(y)dy &\le C|x|^{-t(x)/q(x)}\delta^{\alpha-n/q(x)} \\ &\quad + C|x|^{\alpha-t(x)/q(x)}(1+|x|)^{-n} \end{aligned}$$

for all $x \in \mathbf{R}^n$, $\delta > 0$ and $f \ge 0$ with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$. We have by Lemma 6.3

$$\begin{aligned} &\int_{B(x,2|x|)\setminus B(x,\delta)} |x-y|^{\alpha-n}f(y)dy \\ &\le C\int_\delta^{2|x|}\left(\int_{B(x,r)} f(y)dy\right)r^{\alpha-n-1}dr \\ &\le C|x|^{-t(x)/q(x)}\int_\delta^{2|x|} r^{\alpha-n/q(x)-1}dr + C|x|^{\alpha-t(x)/q(x)}(1+|x|)^{-n} \\ &\le C|x|^{-t(x)/q(x)}\delta^{\alpha-n/q(x)} + C|x|^{\alpha-t(x)/q(x)}(1+|x|)^{-n} \end{aligned}$$

for all $\delta > 0$, as required. □

We are next concerned with the fractional Hardy type operator as in [13], which are treated in the same manner as Corollaries 5.7 and 5.11.

LEMMA 6.5. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$ be as in Theorem 6.1. Then*

$$\int_{B(0,1)} \left\{H_{\alpha,1}(x)|x|^{t(x)/q(x)}\right\}^{q^\sharp(x)} dx \le C$$

for all f with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

In fact, since $K_{\alpha,1} \le 1$,

$$K_{\alpha,1} \le \left(|x|^{\varepsilon} \int_{B(0,1)\setminus B(0,|x|)} g(y)|y|^{-\varepsilon} dy \right)^{1/q^\sharp(x)},$$

so that we have the required result as in the proof of Corollary 5.7.

LEMMA 6.6. *Let $q \in \mathcal{P}_1(\mathbf{R}^n)$ and $t \in \mathcal{P}(\mathbf{R}^n)$ be as in Theorem 6.1. Then*

$$\int_{\mathbf{R}^n\setminus B(0,1)} \left\{ H_{\alpha,2}(x)|x|^{t(x)/q(x)} \right\}^{q^\sharp(x)} dx \quad \le \quad C$$

for all f with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$.

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{L^{t,q}(\mathbf{R}^n)} \le 1$. Then Lemma 6.4 yields

$$\begin{aligned} |x|^{t(x)/q(x)} I_\alpha f(x) \;&=\; |x|^{t(x)/q(x)} \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ &\quad + |x|^{t(x)/q(x)} \int_{\mathbf{R}^n\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ &\le\; C|x|^{t(x)/q(x)} \delta^{\alpha} Mf(x) + C\delta^{\alpha-n/q(x)} \\ &\quad + C|x|^{t(x)/q(x)} H_\alpha(x) + C(1+|x|)^{\alpha-n}, \end{aligned}$$

where

$$H_\alpha(x) = \int_{\mathbf{R}^n\setminus B(0,|x|)} |y|^{\alpha-n} f(y)dy.$$

Letting $\delta = \{Mf(x)|x|^{t(x)/q(x)}\}^{-n/q(x)}$, we find

$$\begin{aligned} |x|^{t(x)/q(x)} I_\alpha f(x) \;&\le\; C\{|x|^{t(x)/q(x)} Mf(x)\}^{q(x)/q^\sharp(x)} + C|x|^{t(x)/q(x)} H_\alpha(x) \\ &\quad + C(1+|x|)^{\alpha-n}. \end{aligned}$$

Now we obtain from Theorem 4.1 and Lemmas 6.5, 6.6,

$$\begin{aligned} \|I_\alpha f\|_{L^{tq^\sharp/q,q^\sharp}(\mathbf{R}^n)} \;&\le\; \|I_\alpha f(x)|x|^{t(x)/q(x)}\|_{L^{q^\sharp(\cdot)}(\mathbf{R}^n)} \\ &\le\; C\|Mf(x)|x|^{t(x)/q(x)}\|_{L^{q(\cdot)}(\mathbf{R}^n)} + C\|H_\alpha(x)|x|^{t(x)/q(x)}\|_{L^{q^\sharp(\cdot)}(\mathbf{R}^n)} \\ &\quad + C\|(1+|x|)^{\alpha-n}\|_{L^{q^\sharp(\cdot)}(\mathbf{R}^n)} \\ &\le\; C \end{aligned}$$

Since $\int_{B(0,1)} |x|^{t(0)q^\sharp(0)/q(0)} dx \le C$ by $\alpha q(0) - n < t(0)$. Hence, we obtain the required result. $\square$

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