Sobolev's inequality for Riesz potentials in Orlicz-Musielak spaces of variable exponent

Yoshihiro Mizuta, Eiichi Nakai, Takao Ohno and Tetsu Shimomura

Abstract. Our aim in this note is to deal with Sobolev's inequality for Riesz potentials of order α for functions in Orlicz-Musielak spaces $L^{p(\cdot)}(\log L)^{p(\cdot)q(\cdot)}(\mathbf{R}^n)$ when $p(\cdot)$ and $q(\cdot)$ are variable exponents satisfying the log-Hölder conditions; the exponent $p(\cdot)$ may approach 1. For this purpose we prepare the boundedness property of the Hardy-Littlewood maximal operator on Orlicz-Musielak spaces.

1 Introduction

In recent years, the generalized Lebesgue spaces have attracted more and more attention, in connection with the study of elasticity, fluid mechanics and, more recently, image restoration; see Růžička [13]. The generalized Lebesgue spaces were first introduced by Orlicz [12] and then by Nakano [10]. After them, these spaces were systematically studied by Musielak [9] and Kováčik and Rákosník [5].

Following Cruz-Uribe and Fiorenza [1], let us consider two variable exponents $p(\cdot)$ and $q(\cdot)$ on $\mathbf{R}^n$ satisfying:

(p1) $1 \le p^- \equiv \inf_{x \in R^n} p(x) \le \sup_{x \in R^n} p(x) \equiv p^+ < \infty$;

(p2) $|p(x) - p(y)| \le \dfrac{C}{\log(e + 1/|x-y|)}$;

(p3) $|p(x) - p(y)| \le \dfrac{C}{\log(e + |x|)}$ whenever $|y| \ge |x|$;

(q1) $0 \le q^- \equiv \inf_{x \in R^n} q(x) \le \sup_{x \in R^n} q(x) \equiv q^+ < \infty$;

2000 *subject classification*: Primary 46E35, Secondary 31B15

Key words and phrases: Variable exponent Lebesgue space, Generalized Lebesgue space, Orlicz–Musielak space, Maximal function, Sobolev's inequality

(q2) $|q(x) - q(y)| \le \dfrac{C}{\log(e + \log(e + 1/|x-y|))}$.

By (p3) one sees that $p(x)$ has a finite limit p_∞ at infinity and

(p3') $|p(x) - p_\infty| \le \dfrac{C}{\log(e + |x|)}$.

For $\gamma \in [0, \infty)$, $x \in \mathbf{R}^n$ and $t \in [0, \infty)$, set

$$\Phi_{p(\cdot),q(\cdot),-\gamma}(x,t) = \{t(\log(e+t))^{q(x)}\}^{p(x)}(\log(e+t+t^{-1}))^{-\gamma};$$

if $\gamma = 0$, then we set $\Phi_{p(\cdot),q(\cdot)}$ for $\Phi_{p(\cdot),q(\cdot),0}$. Note that

(Φ_1) $\Phi_{p(\cdot),q(\cdot)}(x, \cdot)$ is convex on $[0, \infty)$ for fixed $x \in \mathbf{R}^n$,

since $q^- \ge 0$ by our assumption.

For $\Phi = \Phi_{p(\cdot),q(\cdot),\gamma}$, we define the quasi-norm by

$$\|f\|_{\Phi(\mathbf{R}^n)} = \inf \left\{ \lambda > 0 : \int_{\mathbf{R}^n} \Phi\left(x, |f(x)/\lambda|\right) dx \le 1 \right\}$$

and denote by $\Phi(\mathbf{R}^n)$ the space of all measurable functions f on $\mathbf{R}^n$ with $\|f\|_{\Phi(\mathbf{R}^n)} < \infty$; $\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ is sometimes written as $L^{p(\cdot)}(\log L)^{p(\cdot)q(\cdot)}(\mathbf{R}^n)$.

We denote by $B(x,r)$ the ball with center x and of radius $r > 0$, and by $|B(x,r)|$ its Lebesgue measure, i.e. $|B(x,r)| = \sigma_n r^n$, where σ_n is the volume of the unit ball in $\mathbf{R}^n$. For a locally integrable function f on $\mathbf{R}^n$, we define the Hardy-Littlewood maximal function Mf by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

First we prepare the boundedness property of the Hardy-Littlewood maximal operator on Orlicz-Musielak spaces, as an extension of Diening [2].

Theorem 1. *For each $\gamma > 1$, there exists a constant $C > 0$ such that*

$$\|Mf\|_{\Phi_{p(\cdot),q(\cdot),-\gamma}(\mathbf{R}^n)} \le C\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)}$$

for all $f \in \Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$.

If $1/p^*(x) = 1/p(x) - \alpha/n > 0$, then we set

$$\Phi_{p^*(x),q(x),-\gamma}(x,t) \quad = \quad \left\{t(\log(e+t))^{q(x)}\right\}^{p^*(x)} (\log(e+t+t^{-1}))^{-\gamma}.$$

For $0 < \alpha < n$, we define the Riesz potential of order α for a locally integrable function f on $\mathbf{R}^n$ by

$$I_\alpha f(x) := \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y)\, dy.$$

Here it is natural to assume that

$$\int_{\mathbf{R}^n} (1+|y|)^{\alpha-n} |f(y)|\, dy < \infty,$$

which is equivalent to the condition that $I_\alpha |f| \not\equiv \infty$ (see [7, Theorem 1.1, Chapter 2]).

Our main aim in this note is to prove Sobolev's inequality for Riesz potentials of order α for functions in Orlicz–Musielak spaces, through an application of Hedberg's trick [4] and Theorem 1.

Theorem 2. *For each $\gamma > 1$, there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\Phi_{p^*(\cdot),q(\cdot),-\gamma}(\mathbf{R}^n)} \le C \|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)}$$

for all $f \in \Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$.

Remark 1. If $p^- > 1$, then Theorem 1 is true for $\gamma = 0$ by Proposition 2.5 in [6], so that Theorem 2 can be derived by use of the boundedness of the maximal operator. This article treats the case $p^- = 1$, and the results obtained here are considerably weak; for this, we refer to the paper by O'Neil [11].

Remark 2. Let p be an exponent of the form

$$p(x) = 1 + c/\log(e+|x|)$$

with $c > 0$. If $f = 1$ on $B(0,1)$ and $f = 0$ elsewhere, then

$$Mf(x) \ge C|x|^{-n} \qquad \text{for } x \in \mathbf{R}^n \setminus B(0,2).$$

Hence one finds

$$\int_{\mathbf{R}^n\setminus B(0,2)} Mf(x)^{p(x)}(\log(e+Mf(x)^{-1}))^{-1}dx=\infty,$$

so that Theorem 1 fails to hold for $\gamma\le 1$.

Moreover,

$$I_\alpha f(x)\ge C|x|^{\alpha-n} \qquad \text{for } x\in\mathbf{R}^n\setminus B(0,2),$$

so that

$$\int_{\mathbf{R}^n\setminus B(0,2)} I_\alpha f(x)^{p^*(x)}(\log(e+I_\alpha f(x)^{-1}))^{-1}dx=\infty,$$

where $p(x)^*=(1/p(x)-\alpha/n)^{-1}\le n/(n-\alpha)+C/\log(e+|x|)$ for $x\in\mathbf{R}^n\setminus B(0,r_0)$ with large $r_0>2$, say $\alpha(1+c/\log(e+r_0))<n$. This implies that Theorem 2 fails to hold for $\gamma\le 1$.

2 Maximal functions

The next lemma is proved along the same lines as in Stein [14, Chapter 1]; see also [8, Lemma 4.2].

Lemma 3. *Suppose $\gamma>1$. Then there exists a constant $C>0$ such that*

$$\int_{\mathbf{R}^n} Mg(x)(\log(e+Mg(x)+Mg(x)^{-1}))^{-\gamma}dx\le C\|g\|_{L^1(\mathbf{R}^n)}$$

for all $g\in L^1(\mathbf{R}^n)$.

Proof. For $1<\gamma\le 2$, we see that $t(\log(\gamma+t+1/t))^{-\gamma}$ is increasing on $(0,\infty)$ and

$$t(\log(e+t+1/t))^{-\gamma}\le C(\gamma)t(\log(\gamma+t+1/t))^{-\gamma}.$$

Hence

$$\begin{aligned}&\int_{\mathbf{R}^n} Mg(x)(\log(e+Mg(x)+Mg(x)^{-1}))^{-\gamma}dx\\ \le\ & C(\gamma)\int_{\mathbf{R}^n} Mg(x)(\log(\gamma+Mg(x)+Mg(x)^{-1}))^{-\gamma}dx\\ =\ & C(\gamma)\int_0^\infty \lambda(t)d(t(\log(\gamma+t+t^{-1}))^{-\gamma}),\end{aligned}$$

where $\lambda(t) = |\{x \in \mathbf{R}^n : Mg(x) > t\}|$. Here we note from [14, Theorem 1, Chapter 1] that

$$\lambda(t) \le Ct^{-1} \int_{\{x \in \mathbf{R}^n : |g(x)| > t/2\}} |g(x)|\, dx \tag{1}$$

for $t > 0$. Now we obtain by Fubini's Theorem

$$\begin{aligned} & \int_{\mathbf{R}^n} Mg(x)(\log(e + Mg(x) + Mg(x)^{-1}))^{-\gamma} dx \\ \le \ & C \int_{\mathbf{R}^n} |g(x)| \left\{ \int_0^{2|g(x)|} t^{-1} d(t(\log(\gamma + t + t^{-1}))^{-\gamma}) \right\} dx \\ \le \ & C \int_{\mathbf{R}^n} |g(x)|\, dx, \end{aligned}$$

as required. □

Lemma 4. *For $0 \le \gamma < 1$, there exists a constant $C > 0$ such that*

$$\int_{\{x \in \mathbf{R}^n : Mg(x) > 1\}} Mg(x)(\log(e+Mg(x)))^{-\gamma} dx \le C \int_{\mathbf{R}^n} |g(y)|(\log(e+|g(y)|))^{-\gamma+1} dy$$

for all $g \in L^1(\log L)^{-\gamma+1}(\mathbf{R}^n)$.

Proof. In view of (1), we have

$$\begin{aligned} & \int_{\{x \in \mathbf{R}^n : Mg(x) > 1\}} Mg(x)(\log(e + Mg(x)))^{-\gamma} dx \\ & \le \int_1^\infty \lambda(t) d(t(\log(e + t))^{-\gamma}) \\ & \le C \int_{\mathbf{R}^n} |g(x)| \left\{ \int_1^{2|g(x)|} t^{-1} d(t(\log(e + t))^{-\gamma}) \right\} dx \\ & \le C \int_{\mathbf{R}^n} |g(y)|(\log(e + |g(y)|))^{-\gamma+1} dy, \end{aligned}$$

as required.

□

Corollary 5. *For $\gamma > 1$, there exists a constant $C > 0$ such that*

$$\int_{\mathbf{R}^n} Mg(x)(\log(e + Mg(x)^{-1}))^{-\gamma} dx \leq C \int_{\mathbf{R}^n} |g(y)|(\log(e + |g(y)|))dy$$

for all $g \in L^1(\log L)^1(\mathbf{R}^n)$.

3 Proof of Theorem 1

For a proof of Theorem 1, the following result is essential (see [3, Lemmas 2.1–2.3]):

Lemma 6. *If $\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} \leq 1$, then*

$$Mf(x) \leq C \left\{ Mg(x)^{1/p(x)}(\log(e + Mg(x)))^{-q(x)} + (1 + |x|)^{-n/p(x)} \right\}$$

for $x \in \mathbf{R}^n$, where $g(y) = \Phi_{p(\cdot),q(\cdot)}(y, |f(y)|)$.

Proof of Theorem 1: Let f be a nonnegative function in $\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ such that

$$\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} \leq 1.$$

In view of Lemma 6, we find

$$\Phi_{p(\cdot),q(\cdot)}(x, Mf(x)) \leq C\{Mg(x) + (1 + |x|)^{-n}\}$$

for $x \in \mathbf{R}^n$, where $g(y) = \Phi_{p(\cdot),q(\cdot)}(y, f(y))$. Hence, if $\gamma > 1$, then we have by Lemma 3

$$\begin{aligned} & \int_{\mathbf{R}^n} \Phi_{p(\cdot),q(\cdot),\gamma}(x, Mf(x))dx \\ \leq \; & C\left\{ \int_{\mathbf{R}^n} Mg(x)(\log(e + Mg(x) + Mg(x)^{-1}))^{-\gamma} dx \right. \\ & \left. + \int_{\mathbf{R}^n} (1 + |x|)^{-n}(\log(e + |x|))^{-\gamma} dx \right\} \\ \leq \; & C \left\{ \int_{\mathbf{R}^n} g(y)dy + 1 \right\} \\ \leq \; & C, \end{aligned}$$

which proves Theorem 1. □

With the aid of Lemma 4 we have the following result.

Lemma 7. *If G is a bounded open set in $\mathbf{R}^n$ and $q^- > 0$, then there exists a constant $C > 0$ such that*

$$\|Mf\|_{\Phi_{p(\cdot),q(\cdot),-1}(G)} \le C\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)}$$

for all $f \in \Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$.

Lemma 8. (cf. [6, Proposition 2.2]) *If $p^- > 1$, then there exists a constant $C > 0$ such that*

$$\|Mf\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} \le C\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)}$$

for all $f \in \Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$.

Theorem 9. *If $p_\infty > 1$ and $q^- > 0$, then there exists a constant $C > 0$ such that*

$$\|Mf\|_{\Phi_{p(\cdot),q(\cdot),-1}(\mathbf{R}^n)} \le C\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)}$$

for all $f \in \Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$.

Proof. For $f \in \Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} \le 1$, write

$$f = f\chi_{B(0,R)} + f\chi_{\mathbf{R}^n \setminus B(0,R)} = f_1 + f_2.$$

If $p_\infty > 1$, then we can find $R > 1$ such that

$$p_1 = \inf_{x \in \mathbf{R}^n \setminus B(0,R)} p(x) > 1.$$

Consider $\tilde{p}(x) = \max\{p(x), p_1\}$. Then

$$\|f_2\|_{\Phi_{\tilde{p}(\cdot),q(\cdot)}(\mathbf{R}^n)} \le 1,$$

so that Lemma 8 gives

$$\|Mf_2\|_{\Phi_{\tilde{p}(\cdot),q(\cdot)}(\mathbf{R}^n)} \le C,$$

which implies

$$\|Mf_2\|_{\Phi_{p(\cdot),q(\cdot),-1}(\mathbf{R}^n)} \le C.$$

On the other hand, note from Lemma 7 that

$$\|Mf_1\|_{\Phi_{p(\cdot),q(\cdot),-1}(B(0,2R))} \le C\|f_1\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} \le C.$$

Since $Mf_1(x) \le C|x|^{-n}$ for $x \in \mathbf{R}^n \setminus B(0,2R)$, we find

$$\|Mf_1\|_{\Phi_{p(\cdot),q(\cdot),-1}(\mathbf{R}^n)} \le C.$$

Thus Theorem 9 is obtained. □

4 Proof of Theorem 2

To show Theorem 2, we write for $\delta > 0$

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy + \int_{\mathbf{R}^n \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ &= I_1 + I_2. \end{aligned}$$

By Lemmas 3.1 and 3.2 in [3], we have the following result.

Lemma 10. *Suppose $p^+ < n/\alpha$. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)} \le 1$. Then*

$$I_2 \le C\{\delta^{\alpha-n/p(x)}(\log(e+1/\delta))^{-q(x)} + (1+|x|)^{\alpha-n/p(x)}\}$$

for all $x \in \mathbf{R}^n$ and $\delta > 0$.

Proof of Theorem 2: Since $I_1 \le C\delta^\alpha Mf(x)$, we have by Lemma 10

$$I_\alpha f(x) \le C\{\delta^\alpha Mf(x) + \delta^{\alpha-n/p(x)}(\log(e+1/\delta))^{-q(x)}\} + C(1+|x|)^{\alpha-n/p(x)}.$$

Letting $\delta = Mf(x)^{-p(x)/n}(\log(e+Mf(x)))^{-p(x)q(x)/n}$, we find

$$\begin{aligned} I_\alpha f(x) \le\ & C\left\{Mf(x)^{1/p^*(x)}(\log(e+Mf(x)))^{-\alpha q(x)/n}\right\}^{p(x)} \\ & + C(1+|x|)^{-n/p^*(x)}, \end{aligned}$$

so that

$$\begin{aligned} \Phi_{p^*(\cdot),q(\cdot),\gamma}(x, I_\alpha f(x)) \le\ & C\Phi_{p(\cdot),q(\cdot),\gamma}(x, Mf(x)) \\ & + C(1+|x|)^{-n}(\log(e+|x|))^{-\gamma}. \end{aligned}$$

Thus Theorem 1 proves the present theorem. □

Theorem 11. *If $p_\infty > 1$ and $q^- > 0$, then*

$$\|I_\alpha f\|_{\Phi_{p^*(\cdot),q(\cdot),-1}(\mathbf{R}^n)} \le C\|f\|_{\Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)}$$

for all $f \in \Phi_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$.

References

[1] D. Cruz-Uribe and A. Fiorenza, $L \log L$ results for the maximal operator in variable L^p spaces, Trans. Amer. Math. Soc. **361** (2009), 2631–2647.

[2] L. Diening, Maximal functions in generalized $L^{p(\cdot)}$ spaces, Math. Inequal. Appl. **7**(2) (2004), 245–254.

[3] P. Hästö, Y. Mizuta, T. Ohno and T. Shimomura, Sobolev inequalities for Orlicz spaces of two variable exponents, Glasgow Math. J. **52** (2010), 227–240.

[4] L. I. Hedberg, On certain convolution inequalities, Proc. Amer. Math. Soc. **36** (1972), 505–510.

[5] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. **41** (1991), 592–618.

[6] F.-Y. Maeda, Y. Mizuta and T. Ohno, Approximate identities in variable Lebesgue-Orlicz spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$, to appear in Ann. Acad. Sci. Fenn. Math.

[7] Y. Mizuta, *Potential theory in Euclidean spaces*, Gakkōtosho, Tokyo, 1996.

[8] Y. Mizuta, E. Nakai, T. Ohno and T. Shimomura, Hardy's inequality in Orlicz-Sobolev spaces of variable exponent, to appear in Hokkaido Math. J.

[9] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics **1034**, Springer-Verlag, Berlin, 1983.

[10] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen Co., Ltd., Tokyo, 1950.

[11] R. O' Neil, Fractional integration in Orlicz spaces. I, Trans. Amer. Math. Soc. **115** (1965), 300–328.

[12] W. Orlicz, Über konjugierte Exponentenfolgen, Studia Math. **3** (1931), 200–211.

[13] M. Ružička, Electrorheological fluids : modeling and Mathematical theory, Lecture Notes in Math. **1748**, Springer, 2000.

[14] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1970.

Yoshihiro Mizuta
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8521, Japan
E-mail: `yomizuta@hiroshima-u.ac.jp`

Eiichi Nakai
Department of Mathematics
Osaka Kyoiku University
Kashiwara, Osaka 582-8582, Japan
E-mail: `enakai@cc.osaka-kyoiku.ac.jp`

Takao Ohno
Faculty of Education and Welfare Science,
Oita University
Dannoharu Oita-city 870-1192, Japan
E-mail: `t-ohno@oita-u.ac.jp`

Tetsu Shimomura
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524, Japan
E-mail: `tshimo@hiroshima-u.ac.jp`