Herz-Morrey spaces of variable exponent, Riesz potential operator and duality

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Abstract

Our aim in this paper is to deal with the boundedness of the Hardy-Littlewood maximal operator on Herz-Morrey spaces and to establish Sobolev's inequalities for Riesz potentials of functions in Herz-Morrey spaces. Further, we discuss the associate spaces among Herz-Morrey spaces.

1 Introduction

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. In [9, 10], Diening showed that the maximal and Riesz potential operators are bounded on the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{R}^n)$. In the mean time, variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth condition, in connection with the study of elasticity, fluid mechanics; see [28].

The boundedness of the maximal and Riesz potential operators were also studied for variable exponent Morrey spaces (see [4, 23, 26]), Herz spaces with variable exponents (see [3, 20]) and local Morrey type spaces with variable exponents (see [18]); In our paper, local and global Morrey type spaces are referred to as Herz-Morrey spaces. For constant case, we refer the reader to [1, 12, 8, 27, 5, 6] and so on.

Let G be a bounded open set in $\mathbf{R}^n$, whose diameter is denoted by d_G. Let $\omega(\cdot,\cdot) : G\times(0,\infty) \to (0,\infty)$ be a uniformly almost monotone function on $G\times(0,\infty)$ satisfying the uniformly doubling condition. For $x_0 \in G$, $0 < q < \infty$ and a variable exponent $p(\cdot)$, we consider the Herz-Morrey space $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ of variable exponent consisting of all measurable functions f on G satisfying

$$\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r)\setminus B(x_0,r/2))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty;$$

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when $q=\infty$,

$$\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} = \sup_{0<r<d_G} \omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r)\setminus B(x_0,r/2))} < \infty.$$

Set

$$\mathcal{H}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0\in G} \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G),$$

whose norm is defined by

$$\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}(G)} = \sup_{x_0\in G} \|f\|_{\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)}.$$

In connection with $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$, let us consider the families $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ of all functions f on G satisfying

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty$$

and

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(G\setminus B(x_0,r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty,$$

respectively. Note here that

$$\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G) \cup \overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G) \subset \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G).$$

Similarly we consider the spaces

$$\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0\in G} \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G) \quad \text{and} \quad \overline{\mathcal{H}}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0\in G} \overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G),$$

whose norms are defined by

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)} = \sup_{x_0\in G} \|f\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} \quad \text{and} \quad \|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G)} = \sup_{x_0\in G} \|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)},$$

respectively. The spaces $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ are also referred to as the local Morrey type space and the local complementary Morrey type space (see [5, 6, 16, 17, 18]).

In this paper, we study the Herz-Morrey spaces. In Section 3, we establish the boundedness of the Hardy-Littlewood maximal operator on $\mathcal{H}^{p(\cdot),q,\omega}(G)$, $\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$. In Section 4, we prove Sobolev's inequalities for Riesz potentials of functions in those Herz-Morrey spaces; when $q=\infty$, we refer to [24]. In Section 5, Trudinger's exponential integrability is discussed for the borderline case. In Section 6 and 7, following Di Fratta-Fiorenza [13] and Gogatishvili-Mustafayev [14] , we study the associate spaces among those Herz-Morrey spaces. In particular, we show the associate spaces of $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$, which give another characterizations of Morrey spaces by Adams-Xiao [2] (see also [15]). In the final Section, we discuss the relationship between Herz-Morrey spaces and grand Lebesgue spaces.

2 Preliminaries

We denote by $B(x,r)$ the open ball centered at x of radius r and denote by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbf{R}^n$.

Throughout this paper, let C denote various constants independent of the variables in question and $C(a,b,\cdots)$ be a constant that depends on $a,b,\cdots$. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 1$.

Let G be a bounded open set in $\mathbf{R}^n$ whose diameter is denoted by d_G. Consider a function $p(\cdot)$ on G such that

(P1) $1 < p^- := \inf_{x\in G} p(x) \le \sup_{x\in G} p(x) =: p^+ < \infty$

and

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{c_p}{\log(e + /|x-y|)} \quad \text{for } x, y \in G$$

with a constant $c_p \ge 0$; $p(\cdot)$ is referred to as a variable exponent.

We also consider the family $\Omega(G)$ of all positive functions $\omega(\cdot,\cdot) : G \times (0,\infty) \to (0,\infty)$ satisfying the following conditions:

(ω0) $\omega(x,0) = \lim_{r\to+0} \omega(x,r) = 0$ for all $x \in G$ or $\omega(x,0) = \infty$ for all $x \in G$;

(ω1) $\omega(x,\cdot)$ is uniformly almost monotone on $(0,\infty)$, that is, there exists a constant $A_1 > 0$ such that $\omega(x,\cdot)$ is uniformly almost increasing on $(0,\infty)$, that is,

$$\omega(x,r) \le A_1\omega(x,s) \quad \text{for all } x \in G \text{ and } 0 < r < s$$

or $\omega(x,\cdot)$ is uniformly almost decreasing on $(0,\infty)$, that is,

$$\omega(x,s) \le A_1\omega(x,r) \quad \text{for all } x \in G \text{ and } 0 < r < s;$$

(ω2) $\omega(x,\cdot)$ is uniformly doubling on $(0,\infty)$, in the sense that there exists a constant $A_2 > 1$ such that

$$A_2^{-1}\omega(x,r) \le \omega(x,2r) \le A_2\omega(x,r) \quad \text{for all } x \in G \text{ and } r > 0;$$

and

(ω3) there exists a constant $A_3 > 1$ such that

$$A_3^{-1} \le \omega(x,1) \le A_3 \quad \text{for all } x \in G.$$

Then one can find constants $a, b > 0$ and $C > 1$ such that

$$C^{-1}r^a \le \omega(x,r) \le Cr^{-b} \tag{2.1}$$

for all $x \in G$ and $0 < r \le d_G$.

For later use, it is convenient to note the following result, which is proved by (P1), (P2) and (2.1).

LEMMA 2.1. *Let $x_0 \in G$. Then there exists a constant $C > 0$ such that*

$$\omega(x_0, r)^{p(x_0)} \le C\omega(x_0, r)^{p(y)}$$

whenever $|x_0 - y| < r \le d_G$.

For a locally integrable function f on G, set

$$\|f\|_{L^{p(\cdot)}(G)} = \inf\left\{\lambda > 0 : \int_G \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} dy \le 1\right\};$$

in what follows, set $f = 0$ outside G. We denote by $L^{p(\cdot)}(G)$ the family of locally integrable functions f on G satisfying $\|f\|_{L^{p(\cdot)}(G)} < \infty$.

Set $A(x, r) = B(x, r) \setminus B(x, r/2)$.

LEMMA 2.2 ([24, Lemma 2.2]). *Let $0 < q < \infty$. Then*

(1) $$\int_0^{2d_G} \left(\omega(x_0, r)\|f\|_{L^{p(\cdot)}(A(x_0,r))}\right)^q \frac{dr}{r} \sim \sum_{j=1}^{\infty} \left(\omega(x_0, 2^{-j+1}d_G)\|f\|_{L^{p(\cdot)}(A(x_0,2^{-j+1}d_G))}\right)^q;$$

(2) $$\int_0^{2d_G} \left(\omega(x_0, r)\|f\|_{L^{p(\cdot)}(B(x_0,r))}\right)^q \frac{dr}{r} \sim \sum_{j=1}^{\infty} \left(\omega(x_0, 2^{-j+1}d_G)\|f\|_{L^{p(\cdot)}(B(x_0,2^{-j+1}d_G))}\right)^q;$$
and

(3) $$\int_0^{d_G} \left(\omega(x_0, r)\|f\|_{L^{p(\cdot)}(G\setminus B(x_0,r))}\right)^q \frac{dr}{r} \sim \sum_{j=1}^{\infty} \left(\omega(x_0, 2^{-j}d_G)\|f\|_{L^{p(\cdot)}(G\setminus B(x_0,2^{-j}d_G))}\right)^q$$

when $x_0 \in G$ and f is a measurable function on G.

Further, we obtain the next result.

LEMMA 2.3. *Let $x_0 \in G$. Suppose $0 < q \le \infty$. If $\|f\|_{h^{p(\cdot),q,\omega}(G)} \le 1$, then there exists a constant $C > 0$ such that $\|f\|_{h^{p(\cdot),\infty,\omega}(G)} \le C$, for $h = \mathcal{H}_{\{x_0\}}, \underline{\mathcal{H}}_{\{x_0\}}, \overline{\mathcal{H}}_{\{x_0\}}, \mathcal{H}, \underline{\mathcal{H}}, \overline{\mathcal{H}}$.*

By Lemma 2.1, we have the following result.

LEMMA 2.4. *There is a constant $C > 0$ such that*

$$\int_{B(x_0,r)} |f(y)|^{p(y)} dy \le C\omega(x_0, r)^{-p(x_0)}$$

when $x_0 \in G$, $0 < r < d_G$ and $\omega(x_0, r)\|f\|_{L^{p(\cdot)}(B(x_0,r))} \le 1$.

By Hölder's inequality one easily obtains the following result (cf. [24, Lemma 2.5]).

LEMMA 2.5. *There is a constant $C > 0$ such that*

$$\frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} |f(y)| dy \le Cr^{-n/p(x_0)}\omega(x_0, r)^{-1}$$

when $x_0 \in G$, $0 < r < d_G$ and $\omega(x_0, r)\|f\|_{L^{p(\cdot)}(B(x_0,r))} \le 1$.

Corollary 2.6. *There is a constant* $C > 0$ *such that*

$$\frac{1}{|A(x_0,r)|}\int_{A(x_0,r)} |f(y)|dy \le Cr^{-n/p(x_0)}\omega(x_0,r)^{-1}$$

when $x_0 \in G$, $0 < r < d_G$ *and* $\omega(x_0,r)\|f\|_{L^{p(\cdot)}(A(x_0,r))} \le 1$.

Next we show the following result.

Lemma 2.7. *There is a constant* $C > 0$ *such that*

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |f(y)|dy \le C\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |f(y)|^{p(y)}dy\right)^{1/p(x_0)} + 1$$

when $x_0 \in G$, $0 < r < d_G$ *and* $\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r))} \le 1$.

Proof. Fix $x_0 \in G$ and $0 < r < d_G$. Let f be a nonnegative measurable function on G satisfying $\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r))} \le 1$. Set $J = \left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |f(y)|^{p(y)}dy\right)^{1/p(x_0)}$. If $J > 1$, then

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f(y)dy \le J + \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f(y)\left(\frac{f(y)}{J}\right)^{p(y)-1} dy.$$

Since $J^{-p(y)} \le CJ^{-p(x_0)}$ when $|x_0 - y| < r \le d_G$ by Lemma 2.4 and (2.1), we obtain

$$\begin{aligned}\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f(y)dy &\le J + CJ^{-p(x_0)+1}\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f(y)^{p(y)}dy \\ &\le CJ.\end{aligned}$$

If $J \le 1$, then

$$\begin{aligned}\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f(y)dy &\le 1 + \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f(y)\left(\frac{f(y)}{1}\right)^{p(y)-1} dy \\ &\le 1 + J^{p(x_0)} \\ &\le 1 + J,\end{aligned}$$

which proves the result. □

Corollary 2.8. *There is a constant* $C > 0$ *such that*

$$\frac{1}{|A(x_0,r)|}\int_{A(x_0,r)} |f(y)|dy \le C\left(\frac{1}{|A(x_0,r)|}\int_{A(x_0,r)} |f(y)|^{p(y)}dy\right)^{1/p(x_0)} + 1$$

when $x_0 \in G$, $0 < r < d_G$ *and* $\omega(x_0,r)\|f\|_{L^{p(\cdot)}(A(x_0,r))} \le 1$.

Lemma 2.9. *For* $\varepsilon > 0$, *there is a constant* $C(\varepsilon) > 0$ *such that*

$$\left(\int_{B(x_0,r)} |f(y)|^{p(y)}dy\right)^{1/p(x_0)} \le C(\varepsilon)\left(\|f\|_{L^{p(\cdot)}(B(x_0,r))} + r^{\varepsilon}\omega(x_0,r)^{-1}\right)$$

when $x_0 \in G$, $0 < r < d_G$ *and* $\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r))} \le 1$.

Proof. Fix $x_0 \in G$ and $0 < r < d_G$. Let f be a nonnegative measurable function on G satisfying $\omega(x_0, r)\|f\|_{L^{p(\cdot)}(B(x_0,r))} \le 1$.

If $r^\varepsilon < \|\omega(x_0, r)f\|_{L^{p(\cdot)}(B(x_0,r))} \le 1$, we have by Lemma 2.1

$$1 = \int_{B(x_0,r)} \left(\frac{\omega(x_0, r)f(y)}{\|\omega(x_0, r)f\|_{L^{p(\cdot)}(B(x_0,r))}} \right)^{p(y)} dy \ge C(\varepsilon)\|f\|^{-p(x_0)}_{L^{p(\cdot)}(B(x_0,r))} \int_{B(x_0,r)} f(y)^{p(y)} dy$$

since $\|\omega(x_0, r)f\|^{-p(y)}_{L^{p(\cdot)}(B(x_0,r))} \ge C(\varepsilon)\|\omega(x_0, r)f\|^{-p(x_0)}_{L^{p(\cdot)}(B(x_0,r))}$ by (P2), so that

$$\left(\int_{B(x_0,r)} f(y)^{p(y)} dy \right)^{1/p(x_0)} \le C(\varepsilon)\|f\|_{L^{p(\cdot)}(B(x_0,r))}.$$

If $\|\omega(x_0, r)f\|_{L^{p(\cdot)}(B(x_0,r))} \le r^\varepsilon$, we see from Lemma 2.1 and (P2) that

$$\begin{aligned} 1 &= \int_{B(x_0,r)} \left(\frac{f(y)}{\|f\|_{L^{p(\cdot)}(B(x_0,r))}} \right)^{p(y)} dy \\ &\ge \int_{B(x_0,r)} \left(\frac{f(y)}{r^\varepsilon \omega(x_0, r)^{-1}} \right)^{p(y)} dy \\ &\ge C(\varepsilon) r^{-\varepsilon p(x_0)} \omega(x_0, r)^{p(x_0)} \int_{B(x_0,r)} f(y)^{p(y)} dy, \end{aligned}$$

so that

$$\left(\int_{B(x_0,r)} f(y)^{p(y)} dy \right)^{1/p(x_0)} \le C(\varepsilon) r^\varepsilon \omega(x_0, r)^{-1}.$$

Thus we obtain the required result. □

COROLLARY 2.10. *For $\varepsilon > 0$, there is a constant $C(\varepsilon) > 0$ such that*

$$\left(\int_{A(x_0,r)} |f(y)|^{p(y)} dy \right)^{1/p(x_0)} \le C(\varepsilon) \left(\|f\|_{L^{p(\cdot)}(A(x_0,r))} + r^\varepsilon \omega(x_0, r)^{-1} \right)$$

when $x_0 \in G$, $0 < r < d_G$ and $\omega(x_0, r)\|f\|_{L^{p(\cdot)}(A(x_0,r))} \le 1$.

3 Boundedness of the maximal operator for $0 < q < \infty$

In this section we investigate the boundedness of the maximal operator in $\mathcal{H}^{p(\cdot),q,\omega}(G)$, $\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$. For this purpose we consider the following conditions:

($\omega\sigma 0$) there exist constants σ_0 and $A > 0$ such that

$$\int_0^r \left(\omega(x_0, t)t^{\sigma_0}\right)^q \frac{dt}{t} \le A \left(\omega(x_0, r)r^{\sigma_0}\right)^q$$

for all $x_0 \in G$ and $0 < r < d_G$; and

($\omega\sigma 1$) there exist constants $\sigma_1 > 0$ and $A > 0$ such that

$$\int_r^{d_G} \left(\omega(x_0,t)t^{-\sigma_1}\right)^q \frac{dt}{t} \le A\left(\omega(x_0,r)r^{-\sigma_1}\right)^q$$

for all $x_0 \in G$ and $0 < r < d_G$.

In fact, conditions ($\omega\sigma 0$) and ($\omega\sigma 1$) are only required to hold for some specific $q > 0$.

For a locally integrable function f on G, the Hardy-Littlewood maximal function is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy;$$

recall that $f = 0$ outside G. Now we state the cerebrated result by Diening [9].

LEMMA 3.1. *The maximal operator $\mathcal{M} : f \longrightarrow Mf$ is bounded in $L^{p(\cdot)}(G)$, that is, there exists a constant $C > 0$ such that*

$$\|Mf\|_{L^{p(\cdot)}(G)} \le C\|f\|_{L^{p(\cdot)}(G)}.$$

THEOREM 3.2. *Let $0 < q < \infty$.*

1. *If ($\omega\sigma 0$) with $\sigma_0 < n/p^+$ and ($\omega\sigma 1$) with $\sigma_1 < n/(p^-)'$ hold, then the maximal operator $\mathcal{M}$ is bounded in $\mathcal{H}^{p(\cdot),q,\omega}(G)$.*
2. *If ($\omega\sigma 0$) with $\sigma_1 < n/p^+$ holds, then the maximal operator $\mathcal{M}$ is bounded in $\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$.*
3. *If ($\omega\sigma 0$) with $\sigma_0 \le n/p^+$ and ($\omega\sigma 1$) with $\sigma_1 < n/(p^-)'$ hold, then the maximal operator $\mathcal{M}$ is bounded in $\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$.*

Proof. We show only assertion (1) when $q \ge 1$. Let f be a nonnegative measurable function on G such that $\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}(G)} \le 1$ and let $x_0 \in G$. For $r > 0$, write

$$\begin{aligned} f &= f\chi_{B(x_0,r/4)} + f\chi_{B(x_0,2r)\setminus B(x_0,r/4)} + f\chi_{G\setminus B(x_0,2r)} \\ &= f_{1,r} + f_{2,r} + f_{3,r}. \end{aligned}$$

If $x \in A(x_0,r) = B(x_0,r) \setminus B(x_0,r/2)$, then

$$Mf_{1,r}(x) \le Cr^{-n} \int_{B(x_0,r/4)} f(y)dy$$

and

$$Mf_{3,r}(x) \le C \int_{G\setminus B(x_0,2r)} f(y)|x_0 - y|^{-n}dy.$$

Since $\|1\|_{L^{p(\cdot)}(A(x_0,r))} \sim r^{n/p(x_0)}$, we note from Lemma 3.1 that

$$\int_0^{2d_G} \left(\omega(x_0,r)\|Mf\|_{L^{p(\cdot)}(A(x_0,r))}\right)^q \frac{dr}{r}$$
$$\le C\left\{\int_0^{2d_G} \left(\omega(x_0,r) r^{-n+n/p(x_0)} \int_{B(x_0,r/4)} f(y)dy\right)^q \frac{dr}{r}\right.$$
$$+ \int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,2r)\setminus B(x_0,r/4))}\right)^q \frac{dr}{r}$$
$$\left. + \int_0^{2d_G} \left(\omega(x_0,r) r^{n/p(x_0)} \left(\int_{G\setminus B(x_0,2r)} f(y)|x_0-y|^{-n}dy\right)\right)^q \frac{dr}{r}\right\}$$
$$= C\left(I_1 + \int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(A(x_0,r))}\right)^q \frac{dr}{r} + I_2\right).$$

For I_1, take $\varepsilon_1 > 0$ such that $\sigma_1 p(x_0) < \varepsilon_1 < n(p(x_0)-1)$. We have by $(\omega 1)$ and $(\omega 2)$

$$\begin{aligned} I_1 &\le C\sum_{k=1}^{\infty} \left(\omega(x_0,2^{-k}d_G)(2^{-k}d_G)^{-n+n/p(x_0)} \int_{B(x_0,2^{-k}d_G)} f(y)dy\right)^q \\ &= C\sum_{k=1}^{\infty} \left(\omega(x_0,2^{-k}d_G)(2^{-k}d_G)^{-n+n/p(x_0)} \left(\sum_{j\ge k}\int_{A_j} f(y)dy\right)\right)^q, \end{aligned} \tag{3.1}$$

where $A_j = A(x_0, 2^{-j}d_G)$. By Corollary 2.8, we find

$$\int_{A_j} f(y)dy \le C(2^{-j}d_G)^{n-(n+\varepsilon_1)/p(x_0)} \left(\int_{A_j} f(y)^{p(y)}|y|^{\varepsilon_1}dy + (2^{-j}d_G)^{n+\varepsilon_1}\right)^{1/p(x_0)},$$

so that we see from Hölder's inequality that

$$\sum_{j\ge k}\int_{A_j} f(y)dy \le C(2^{-k}d_G)^{n-(n+\varepsilon_1)/p(x_0)} \left(\sum_{j\ge k}\left(\int_{A_j} f(y)^{p(y)}|y|^{\varepsilon_1}dy + (2^{-j}d_G)^{n+\varepsilon_1}\right)^{q/p(x_0)}\right)^{1/q}$$

since $n-(n+\varepsilon_1)/p(x_0) > 0$. Hence we obtain by Fubini's theorem, Lemma 2.2, Corollary 2.10, $(\omega\sigma 1)$, $(\omega 1)$, $(\omega 2)$ and $(\omega\sigma 0)$

$$\begin{aligned} I_1 &\le C\sum_{j=1}^{\infty} \left(\int_{A_j} f(y)^{p(y)}|y|^{\varepsilon_1}dy + (2^{-j}d_G)^{n+\varepsilon_1}\right)^{q/p(x_0)} \left(\sum_{k\le j}\omega(x_0,2^{-k}d_G)^q(2^{-k}d_G)^{-q\varepsilon_1/p(x_0)}\right) \\ &\le C\sum_{j=1}^{\infty} \left(\omega(x_0,2^{-j}d_G)\left(\int_{A_j}|f(y)|^{p(y)}dy + (2^{-j}d_G)^n\right)^{1/p(x_0)}\right)^q \\ &\le C\left\{\int_0^{2d_G}\left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(A(x_0,r))}\right)^q\frac{dr}{r} + \int_0^{2d_G}\left(\omega(x_0,r)r^{n/p(x_0)}\right)^q\frac{dr}{r} + 1\right\} \\ &\le C. \end{aligned}$$

For I_2, let $\sigma_0 p(x_0) < \varepsilon_0 < n$. We find by $(\omega 1)$ and $(\omega 2)$

$$I_2 \le C \sum_{k=1}^{\infty} \left(\omega(x_0, 2^{-k} d_G)(2^{-k} d_G)^{n/p(x_0)} \left(\sum_{j \le k} (2^{-j} d_G)^{-n} \int_{A_j} f(y) dy \right) \right)^q .$$

Here we have by Corollary 2.8 and Hölder's inequality

$$\begin{aligned}
& \sum_{j \le k} (2^{-j} d_G)^{-n} \int_{A_j} f(y) dy \\
\le\ & C \sum_{j \le k} (2^{-j} d_G)^{-(n-\varepsilon_0)/p(x_0)} \left(\int_{A_j} f(y)^{p(y)} |y|^{-\varepsilon_0} dy + (2^{-j} d_G)^{n-\varepsilon_0} \right)^{1/p(x_0)} \\
\le\ & C (2^{-k} d_G)^{-(n-\varepsilon_0)/p(x_0)} \left(\sum_{j \le k} \left(\int_{A_j} f(y)^{p(y)} |y|^{-\varepsilon_0} dy + (2^{-j} r)^{n-\varepsilon_0} \right)^{q/p(x_0)} \right)^{1/q}
\end{aligned}$$

since $n - \varepsilon_0 > 0$. Hence we obtain by Fubini's theorem, Lemma 2.2, Corollary 2.10, $(\omega\sigma 0)$, $(\omega 1)$ and $(\omega 2)$

$$\begin{aligned}
I_2 \ \le\ & C \sum_{j=1}^{\infty} \left(\int_{A_j} f(y)^{p(y)} |y|^{-\varepsilon_0} dy + (2^{-j} r)^{n-\varepsilon_0} \right)^{q/p(x_0)} \left(\sum_{k \ge j} \omega(x_0, 2^{-k} d_G)^q (2^{-k} d_G)^{q\varepsilon_0/p(x_0)} \right) \\
\le\ & C \sum_{j=1}^{\infty} \left(\omega(x_0, 2^{-j} d_G) \left(\int_{A_j} |f(y)|^{p(y)} dy + (2^{-j} d_G)^n \right)^{1/p(x_0)} \right)^q \\
\le\ & C \left\{ \int_0^{2d_G} \left(\omega(x_0, r) \|f\|_{L^{p(\cdot)}(A(x_0,r))} \right)^q \frac{dr}{r} + \int_0^{2d_G} \left(\omega(x_0, r) r^{n/p(x_0)} \right)^q \frac{dr}{r} + 1 \right\} \\
\le\ & C,
\end{aligned}$$

which completes the proof. □

Remark 3.3. Let $0 < q < \infty$. If the conditions on ω hold at $x_0 \in G$ only, then one can see that $\mathcal{M}$ is bounded from $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$, $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ to $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$, $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$, respectively.

Corollary 3.4. *Let $0 < q < \infty$. For bounded functions $\nu(\cdot) : G \to (-\infty, \infty)$ and $\beta(\cdot) : G \to (-\infty, \infty)$, set $\omega(x, r) = r^{\nu(x)} (\log(2d_G/r))^{\beta(x)}$.*

1. *When $-n/p^+ < \nu^- \le \nu^+ < n/(p^-)'$, the maximal operator $\mathcal{M}$ is bounded in $\mathcal{H}^{p(\cdot),q,\omega}(G)$.*
2. *When $-n/p^+ < \nu^-$, the maximal operator $\mathcal{M}$ is bounded in $\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$.*
3. *When $-n/p^+ < \nu^- \le \nu^+ < n/(p^-)'$, the maximal operator $\mathcal{M}$ is bounded in $\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$.*

4 Sobolev's inequality for $0 < q < \infty$

In this section we give Sobolev's inequality for Riesz potentials of functions in $\mathcal{H}^{p(\cdot),q,\omega}(G)$, $\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$, when $0 < \alpha < n/p^+$.

For $0 < \alpha < n$, the Riesz potential $I_\alpha f$ is defined by

$$I_\alpha f(x) = I_\alpha * f(x) = \int_G |x-y|^{\alpha-n} f(y)dy$$

for measurable functions f on G; and define

$$\frac{1}{p^\sharp(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}.$$

Let us begin with Sobolev's inequality proved by Diening [10, Theorem 5.2]:

LEMMA 4.1. *If $0 < \alpha < n/p^+$, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L^{p^\sharp(\cdot)}(G)} \le C\|f\|_{L^{p(\cdot)}(G)}$$

for all $f \in L^{p(\cdot)}(G)$.

Our result is stated in the following:

THEOREM 4.2. *Let $0 < q < \infty$ and $0 < \alpha < n/p^+$.*

(1) *If $(\omega\sigma0)$ with $\sigma_0 < n/p^+ - \alpha$ and $(\omega\sigma1)$ with $\sigma_1 < n/(p^-)'$ hold, then the operator I_α is bounded from $\mathcal{H}^{p(\cdot),q,\omega}(G)$ to $\mathcal{H}^{p^\sharp(\cdot),q,\omega}(G)$.*

(2) *If $(\omega\sigma0)$ with $\sigma_0 < n/p^+ - \alpha$ holds, then the operator I_α is bounded from $\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$ to $\underline{\mathcal{H}}^{p^\sharp(\cdot),q,\omega}(G)$.*

(3) *If $(\omega\sigma0)$ with $\sigma_0 \le n/p^+$ and $(\omega\sigma1)$ with $\sigma_1 < n/(p^-)'$ hold, then the operator I_α is bounded from $\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$ to $\overline{\mathcal{H}}^{p^\sharp(\cdot),q,\omega}(G)$.*

Proof. We show only assertion (1) with $q \ge 1$. Let f be a nonnegative measurable function on G such that $\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}(G)} \le 1$ and let $x_0 \in G$. We note from Lemma 4.1 that

$$\begin{aligned}
&\int_0^{2d_G} \left(\omega(x_0,r)\|I_\alpha f\|_{L^{p^\sharp(\cdot)}(A(x_0,r))}\right)^q \frac{dr}{r} \\
&= \int_0^{2d_G} \left(\omega(x_0,r)\|I_\alpha(f\chi_{B(x_0,r/4)} + f\chi_{B(x_0,2r)\setminus B(x_0,r/4)} + f\chi_{G\setminus B(x_0,2r)})\|_{L^{p^\sharp(\cdot)}(A(x_0,r))}\right)^q \frac{dr}{r} \\
&\le C\Bigg\{\int_0^{2d_G} \left(\omega(x_0,r)r^{-n+n/p(x_0)} \int_{B(x_0,r/4)} f(y)dy\right)^q \frac{dr}{r} \\
&\qquad + \int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,2r)\setminus B(x_0,r/4))}\right)^q \frac{dr}{r} \\
&\qquad\qquad + \int_0^{2d_G} \left(\omega(x_0,r)r^{n/p^\sharp(x_0)} \left(\int_{G\setminus B(x_0,2r)} f(y)|x_0-y|^{\alpha-n}dy\right)\right)^q \frac{dr}{r}\Bigg\} \\
&\le C\left(I_1 + \int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(A(x_0,r))}\right)^q \frac{dr}{r} + I_2\right).
\end{aligned}$$

For I_1, we have by (ω1) and (ω2)

$$I_1 \le C \sum_{k=1}^{\infty} \left(\omega(x_0, 2^{-k} d_G)(2^{-k} d_G)^{-n+n/p(x_0)} \left(\sum_{j \ge k} \int_{A_j} f(y) dy \right) \right)^q,$$

which is nothing but (3.1), where $A_j = A(x_0, 2^{-j} d_G)$. Thus $I_1 \le C$ by the proof of Theorem 3.2.

For I_2, let $\sigma_0 p(x_0) < \varepsilon_0 < n - \alpha p(x_0)$. We see from ($\omega$1) and ($\omega$2) that

$$I_2 \le C \sum_{k=1}^{\infty} \left(\omega(x_0, 2^{-k} d_G)(2^{-k} d_G)^{n/p^\sharp(x_0)} \left(\sum_{j \le k} (2^{-j} d_G)^{\alpha-n} \int_{A_j} f(y) dy \right) \right)^q.$$

Here we have by Corollary 2.8 and Hölder's inequality

$$\begin{aligned} & \sum_{j \le k} (2^{-j} d_G)^{\alpha-n} \int_{A_j} f(y) dy \\ \le \; & C \sum_{j \le k} (2^{-j} d_G)^{\alpha-(n-\varepsilon_0)/p(x_0)} \left(\int_{A_j} f(y)^{p(y)} |y|^{-\varepsilon_0} dy + (2^{-j} d_G)^{n-\varepsilon_0} \right)^{1/p(x_0)} \\ \le \; & C(2^{-k} d_G)^{\alpha-(n-\varepsilon_0)/p(x_0)} \left(\sum_{j \le k} \left(\int_{A_j} f(y)^{p(y)} |y|^{-\varepsilon_0} dy + (2^{-j} r)^{n-\varepsilon_0} \right)^{q/p(x_0)} \right)^{1/q} \end{aligned}$$

since $\varepsilon_0 < n - \alpha p(x_0)$. Now the proof is completed as in the proof of Theorem 3.2. □

Remark 4.3. Let $0 < q < \infty$. If the conditions on ω hold at $x_0 \in G$ only, then one can see that I_α is bounded from $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$, $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ to $\mathcal{H}^{p^\sharp(\cdot),q,\omega}_{\{x_0\}}(G)$, $\underline{\mathcal{H}}^{p^\sharp(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{p^\sharp(\cdot),q,\omega}_{\{x_0\}}(G)$, respectively.

Corollary 4.4. *Let $0 < q < \infty$ and $0 < \alpha < n/p^+$. Let ν, β and ω be as in Corollary 3.4.*

(1) *When $\alpha - n/p^+ < \nu^- \le \nu^+ < n/(p^-)'$, the operator I_α is bounded from $\mathcal{H}^{p(\cdot),q,\omega}(G)$ to $\mathcal{H}^{p^\sharp(\cdot),q,\omega}(G)$.*

(2) *When $\alpha - n/p^+ < \nu^-$, the operator I_α is bounded from $\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$ to $\underline{\mathcal{H}}^{p^\sharp(\cdot),q,\omega}(G)$.*

(3) *When $-n/p^+ < \nu^- \le \nu^+ < n/(p^-)'$, the operator I_α is bounded from $\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$ to $\overline{\mathcal{H}}^{p^\sharp(\cdot),q,\omega}(G)$.*

5 Exponential integrability for $0 < q < \infty$

Set

$$E_1(x,t) = \exp\left(t^{p'(x)}\right) - 1,$$

where $1/p(x)+1/p'(x)=1$. For a locally integrable function f on G, set

$$\|f\|_{L^{E_1}(G)} = \inf\left\{\lambda>0 : \int_G E_1\left(y, \frac{|f(y)|}{\lambda}\right) dy \le 1\right\}.$$

We denote by $L^{E_1}(G)$ the class of locally integrable functions f on G satisfying $\|f\|_{L^{E_1}(G)}<\infty$.

In connection with $\mathcal{H}^{p(\cdot),q,\omega}(G)$, let us consider $\mathcal{H}^{E_1,q,\omega}(G)$ of all functions f satisfying

$$\|f\|_{\mathcal{H}^{E_1,q,\omega}(G)} = \sup_{x_0\in G}\left(\int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{E_1}(A(x_0,r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty.$$

Similarly, we define $\mathcal{H}^{E_1,q,\omega}_{\{x_0\}}(G), \underline{\mathcal{H}}^{E_1,q,\omega}_{\{x_0\}}(G), \underline{\mathcal{H}}^{E_1,q,\omega}(G), \overline{\mathcal{H}}^{E_1,q,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{E_1,q,\omega}(G)$.

LEMMA 5.1.

$$\|1\|_{L^{E_1}(B(x_0,r))} \sim (\log(1+1/r))^{-1/p'(x_0)}$$

for all $x_0\in G$ and $0<r<d_G$.

LEMMA 5.2 (cf. [25, Theorem 4.1, Corollary 4.2]). *If $\alpha\ge n/p^-$, then there exists a constant $C>0$ such that*

$$\|I_\alpha f\|_{L^{E_1}(G)} \le C\|f\|_{L^{p(\cdot)}(G)}$$

for all $f\in L^{p(\cdot)}(G)$.

THEOREM 5.3. *Let $0<q<\infty$ and $\alpha\ge n/p^-$.*

1. *If $(\omega\sigma 0)$ with $\sigma_0<0$ and $(\omega\sigma 1)$ with $\sigma_1<n/(p^-)'$ hold, then the operator I_α is bounded from $\mathcal{H}^{p(\cdot),q,\omega}(G)$ to $\mathcal{H}^{E_1,q,\omega}(G)$.*
2. *If $(\omega\sigma 0)$ with $\sigma_0<0$ holds, then the operator I_α is bounded from $\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$ to $\underline{\mathcal{H}}^{E_1,q,\omega}(G)$.*
3. *If $(\omega\sigma 0)$ with $\sigma_0\le n/p^+$ and $(\omega\sigma 1)$ with $\sigma_1<n/(p^-)'$ hold, then the operator I_α is bounded from $\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$ to $\overline{\mathcal{H}}^{E_1,q,\omega}(G)$.*

Proof. We show only assertion (1) with $q\ge 1$. Let f be a nonnegative measurable function on G such that $\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}(G)}\le 1$ and let $x_0\in G$. We note from Lemmas 5.1 and 5.2 that

$$\begin{aligned}
&\int_0^{2d_G} \left(\omega(x_0,r)\|I_\alpha f\|_{L^{E_1}(A(x_0,r))}\right)^q \frac{dr}{r}\\
&\le C\Bigg\{\int_0^{2d_G}\left(\omega(x_0,r)r^{-n+n/p(x_0)}\int_{B(x_0,r/4)} f(y)dy\right)^q \frac{dr}{r}\\
&\quad+\int_0^{2d_G}\left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,2r)\setminus B(x_0,r/4))}\right)^q\frac{dr}{r}\\
&\qquad+\int_0^{2d_G}\left(\omega(x_0,r)\left(\int_{G\setminus B(x_0,2r)} f(y)|x_0-y|^{\alpha-n}dy\right)\right)^q\frac{dr}{r}\Bigg\}\\
&\le C\left(I_1+\int_0^{2d_G}\left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(A(x_0,r))}\right)^q\frac{dr}{r}+I_2\right)
\end{aligned}$$

since $\alpha \geq n/p^-$.

For I_1, we have by (3.1)

$$I_1 \leq C \sum_{k=1}^{\infty} \left(\omega(x_0, 2^{-k} d_G)(2^{-k} d_G)^{-n+n/p(x_0)} \left(\sum_{j \geq k} \int_{A_j} f(y) dy \right) \right)^q \leq C,$$

where $A_j = A(x_0, 2^{-j} d_G)$. For I_2, let $0 < \varepsilon_0 < -\sigma_0 p(x_0)$. We see from ($\omega$1) and ($\omega$2) that

$$I_2 \leq C \sum_{k=1}^{\infty} \left(\omega(x_0, 2^{-k} d_G) \left(\sum_{j \leq k} (2^{-j} d_G)^{\alpha - n} \int_{A_j} f(y) dy \right) \right)^q.$$

Here we have by Corollary 2.8 and Hölder's inequality

$$\begin{aligned} & \sum_{j \leq k} (2^{-j} d_G)^{\alpha - n} \int_{A_j} f(y) dy \\ \leq \quad & C (2^{-k} d_G)^{-\varepsilon_0/p(x_0)} \left(\sum_{j \leq k} \left(\int_{A_j} f(y)^{p(y)} |y|^{\varepsilon_0} dy + (2^{-j} r)^{n+\varepsilon_0} \right)^{q/p(x_0)} \right)^{1/q} \end{aligned}$$

since $\varepsilon_0 > 0$ and $\alpha \geq n/p^-$. Now the proof is completed as in the proof of Theorem 3.2. □

Remark 5.4. Let $0 < q < \infty$. If the conditions on ω hold at $x_0 \in G$ only, then one can see that I_α is bounded from $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$, $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ to $\mathcal{H}^{E_1,q,\omega}_{\{x_0\}}(G)$, $\underline{\mathcal{H}}^{E_1,q,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{E_1,q,\omega}_{\{x_0\}}(G)$, respectively.

Corollary 5.5. *Let $0 < q < \infty$ and $\alpha \geq n/p^-$. Let ν, β and ω be as in Corollary 3.4.*

(1) *If $0 < \nu^- \leq \nu^+ < n/(p^-)'$, then the operator I_α is bounded from $\mathcal{H}^{p(\cdot),q,\omega}(G)$ to $\mathcal{H}^{E_1,q,\omega}(G)$.*

(2) *If $\nu^- > 0$, then the operator I_α is bounded from $\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$ to $\underline{\mathcal{H}}^{E_1,q,\omega}(G)$.*

(3) *If $-n/p^+ < \nu^- \leq \nu^+ < n/(p^-)'$, then the operator I_α is bounded from $\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$ to $\overline{\mathcal{H}}^{E_1,q,\omega}(G)$.*

6 Associate space of $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$

From now on, let $\eta \in \Omega(G)$.

Let X be a family of measurable functions on G with a norm $\|\cdot\|_X$. Then the associate space X' of X is defined as the family of all measurable functions f on G such that

$$\|f\|_{X'} = \sup_{g \in X: \|g\|_X \leq 1} \int_G |f(x) g(x)| dx < \infty.$$

Further, we denote by X^* the dual space of X.

Our first aim in this section is to discuss the associate space of $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$.

Theorem 6.1. *Let $x_0 \in G$ and $1 < q \le \infty$. Suppose there exist constants $a > 0$, $b \ge 0$ and $Q > 0$ such that*

$$(\omega 6.1.0) \qquad \int_0^{2d_G} \eta(x_0, s)^{q'} \frac{ds}{s} < Q;$$

$$(\omega 6.1.1) \qquad \int_t^{2d_G} s^{-a}\omega(x_0, s)^{-q'} \frac{ds}{s} \le Q t^{-a} \eta(x_0, t)^{q'} \text{ for all } 0 < t < d_G \text{ ; and}$$

$$(\omega 6.1.2) \qquad \int_0^{t} s^{-b}\eta(x_0, s)^{q'} \frac{ds}{s} \le Q t^{-b} \omega(x_0, t)^{-q'} \text{ for all } 0 < t < d_G.$$

Then

$$\left(\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)' = \overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G).$$

Remark 6.2. (1) Note that (ω6.1.0) follows from (ω6.1.2).

(2) If (ω6.1.0) does not hold, then

$$\overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G) = \{0\}.$$

(3) If (ω6.1.1) and (ω6.1.2) hold, then the doubling property gives

$$\omega(x_0, t) \sim \eta(x_0, t)^{-1}.$$

Example 6.3. Typical examples of ω and η are

$$\omega(x_0, t) = \eta(x_0, t)^{-1} = t^{-\varepsilon}$$

for $\varepsilon > 0$.

Corollary 6.4. *Let $x_0 \in G$ and $1 < q < \infty$. If (ω6.1.1) and (ω6.1.2) hold, then*

$$\left(\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)^* = \overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G).$$

To show Theorem 6.1, we prepare the following two lemmas.

Lemma 6.5. *Let $1 < q \le \infty$ and $x_0 \in G$. If (ω6.1.1) holds, then there exists a constant $C > 0$ such that*

$$\int_G |f(x)g(x)|dx \le C\|f\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} \|g\|_{\overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)}$$

for all measurable functions f and g on G.

Proof. Let f and g be nonnegative measurable functions on G such that $\|f\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} \le 1$ and $\|g\|_{\overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)} \le 1$. We only prove the case $1 < q < \infty$. For $\gamma > 0$ we have by Fubini's theorem and Hölder's inequality

$$
\begin{aligned}
&\int_G f(x)g(x)dx \le C\int_G f(x)g(x)|x-x_0|^\gamma\left(\int_{|x-x_0|}^{2d_G} t^{-\gamma}\frac{dt}{t}\right)dx \\
\le\ & C\int_0^{2d_G}\left(\int_{B(x_0,t)} f(x)g(x)|x-x_0|^\gamma dx\right)t^{-\gamma}\frac{dt}{t} \\
\le\ & C\int_0^{2d_G}\left(\int_{B(x_0,t)} f(x)g(x)\left(\int_0^{|x-x_0|} r^\gamma\frac{dr}{r}\right)dx\right)t^{-\gamma}\frac{dt}{t} \\
\le\ & C\int_0^{2d_G}\left(\int_0^t r^\gamma\left(\int_{B(x_0,t)\setminus B(x_0,r)} f(x)g(x)dx\right)\frac{dr}{r}\right)t^{-\gamma}\frac{dt}{t} \\
\le\ & C\int_0^{2d_G}\left(\int_0^t r^\gamma\|f\|_{L^{p(\cdot)}(B(x_0,t)\setminus B(x_0,r))}\|g\|_{L^{p'(\cdot)}(B(x_0,t)\setminus(B(x_0,r))}\frac{dr}{r}\right)t^{-\gamma}\frac{dt}{t} \\
\le\ & C\int_0^{2d_G} t^{-\gamma}\|f\|_{L^{p(\cdot)}(B(x_0,t))}\left(\int_0^t r^\gamma\|g\|_{L^{p'(\cdot)}(G\setminus B(x_0,r))}\frac{dr}{r}\right)\frac{dt}{t} \\
\le\ & C\left\{\int_0^{2d_G}\left(\omega(x_0,t)\|f\|_{L^{p(\cdot)}(B(x_0,t))}\right)^q\frac{dt}{t}\right\}^{1/q} \\
& \times\left\{\int_0^{2d_G} t^{-\gamma q'}\omega(x_0,t)^{-q'}\left(\int_0^t r^\gamma\|g\|_{L^{p'(\cdot)}(G\setminus B(x_0,r))}\frac{dr}{r}\right)^{q'}\frac{dt}{t}\right\}^{1/q'} \\
\le\ & C\left\{\int_0^{2d_G} t^{-\gamma q'}\omega(x_0,t)^{-q'}\left(\int_0^t r^\gamma\|g\|_{L^{p'(\cdot)}(G\setminus B(x_0,r))}\frac{dr}{r}\right)^{q'}\frac{dt}{t}\right\}^{1/q'}.
\end{aligned}
$$

Here we see that for $\gamma > a > 0$

$$
\begin{aligned}
&\int_0^t r^\gamma\|g\|_{L^{p'(\cdot)}(G\setminus B(x_0,r))}\frac{dr}{r} \\
\le\ & \left(\int_0^t r^{(\gamma-a/q')q}\frac{dr}{r}\right)^{1/q}\left(\int_0^t r^a\|g\|^{q'}_{L^{p'(\cdot)}(G\setminus B(x_0,r))}\frac{dr}{r}\right)^{1/q'} \\
\le\ & Ct^{\gamma-a/q'}\left(\int_0^t r^a\|g\|^{q'}_{L^{p'(\cdot)}(G\setminus B(x_0,r))}\frac{dr}{r}\right)^{1/q'},
\end{aligned}
$$

so that we find by (ω6.1.1)

$$
\begin{aligned}
&\int_G f(x)g(x)dx \\
\le\ & C\left\{\int_0^{2d_G} t^{-a}\omega(x_0,t)^{-q'}\left(\int_0^t r^a\|g\|^{q'}_{L^{p'(\cdot)}(G\setminus B(x_0,r))}\frac{dr}{r}\right)\frac{dt}{t}\right\}^{1/q'} \\
\le\ & C\left\{\int_0^{2d_G} r^a\|g\|^{q'}_{L^{p'(\cdot)}(G\setminus B(x_0,r))}\left(\int_r^{2d_G} t^{-a}\omega(x_0,t)^{-q'}\frac{dt}{t}\right)\frac{dr}{r}\right\}^{1/q'} \\
\le\ & C\left\{\int_0^{2d_G}\left(\eta(x_0,r)\|g\|_{L^{p'(\cdot)}(G\setminus B(x_0,r))}\right)^{q'}\frac{dr}{r}\right\}^{1/q'} \le C,
\end{aligned}
$$

as required. □

Lemma 6.6. *Let $x_0 \in G$ and $1 < q \le \infty$. Set $X = \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$. If ($\omega$6.1.2) holds, then there exists a constant $C > 0$ such that*

$$\|g\|_{\overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)} \le C \sup_f \int_G |f(x)g(x)|dx = C\|g\|_{X'}$$

for all measurable functions g on G, where the supremum is taken over all measurable functions f on G such that $\|f\|_X \le 1$.

Proof. We may assume that g is a nonnegative measurable function on G and

$$\sup_{f \in X: \|f\|_X \le 1} \int_G |f(x)g(x)|dx \le 1.$$

We only prove the case $1 < q < \infty$; for $q = \infty$, see [24, Theorem 7.4]. Let K be a compact set in $G \setminus \{x_0\}$. Then we see that $L^{p(\cdot)}(K) = \{h\chi_K : h \in L^{p(\cdot)}(G)\} \subset X$, so that we have by ($\omega$6.1.0) and [11, Theorem 3.2.13]

$$\sum_{j \in N_0} \eta(x_0, 2^{-j}d_G)^{q'} G_j^{q'} \sim \|g\chi_K\|^{q'}_{\overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)},$$

where $G_j = \|g_j\|_{L^{p'(\cdot)}(G)}$ with $g_j = g\chi_{K \setminus B(x_0, 2^{-j}d_G)}$ and N_0 is the set of positive integers j such that $G_j > 0$. Consider

$$f(x) = \sum_{j \in N_0} \eta(x_0, 2^{-j}d_G)^{q'} G_j^{q'-2} (g_j(x)/G_j)^{p'(x)-2} g_j(x).$$

Then we obtain

$$\begin{aligned} \|f\|_{L^{p(\cdot)}(B(x_0,r))} &\le \sum_{j \in N_0, 2^{-j}d_G < r} \eta(x_0, 2^{-j}d_G)^{q'} G_j^{q'-1} \|(g_j/G_j)^{p'(\cdot)-1}\|_{L^{p(\cdot)}(G)} \\ &\le \sum_{2^{-j}d_G < r} \eta(x_0, 2^{-j}d_G)^{q'} G_j^{q'-1}, \end{aligned}$$

so that it follows from Lemma 2.2, (ω2), Hölder's inequality and condition (ω6.1.2) that

$$\begin{aligned}
\|f\|_X &\le C\left(\sum_{k=1}^{\infty}\left(\omega(x_0,2^{-k+1}d_G)\|f\|_{L^{p(\cdot)}(B(x_0,2^{-k+1}d_G))}\right)^q\right)^{1/q}\\
&\le C\left(\sum_{k=1}^{\infty}\left(\omega(x_0,2^{-k}d_G)\sum_{j\ge k}\eta(x_0,2^{-j}d_G)^{q'}G_j^{q'-1}\right)^q\right)^{1/q}\\
&\le C\left(\sum_{k=1}^{\infty}\omega(x_0,2^{-k}d_G)^q\left(\sum_{j\ge k}(2^{-j}d_G)^{-b}\eta(x_0,2^{-j}d_G)^{q'}\right)^{q/q'}\right.\\
&\quad\times\left.\left(\sum_{j\ge k}(2^{-j}d_G)^{bq/q'}\eta(x_0,2^{-j}d_G)^{(q'-1)q}G_j^{(q'-1)q}\right)\right)^{1/q}\\
&\le C\left(\sum_{k=1}^{\infty}(2^{-k}d_G)^{-bq/q'}\left(\sum_{j\ge k}(2^{-j}d_G)^{bq/q'}\eta(x_0,2^{-j}d_G)^{q'}G_j^{q'}\right)\right)^{1/q}\\
&\le C\left(\sum_{j=1}^{\infty}(2^{-j}d_G)^{bq/q'}\eta(x_0,2^{-j}d_G)^{q'}G_j^{q'}\left(\sum_{k\le j}(2^{-k}d_G)^{-bq/q'}\right)\right)^{1/q}\\
&\le C\left(\sum_{j\in N_0,}\eta(x_0,2^{-j}d_G)^{q'}G_j^{q'}\right)^{1/q}\\
&\le C.
\end{aligned}$$

On the other hand, we find

$$\begin{aligned}
\int_G f(x)g(x)dx &= \sum_{j\in N_0}\eta(x_0,2^{-j}d_G)^{q'}G_j^{q'-2}\int_G |g_j(x)/G_j|^{p'(x)-2}g_j(x)g(x)dx\\
&= \sum_{j\in N_0}\eta(x_0,2^{-j+1}d_G)^{q'}G_j^{q'}\\
&\ge C\|g\chi_K\|^{q'}_{\overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)}.
\end{aligned}$$

Hence, by the monotone convergence theorem, we have

$$\|g\|^{q'}_{\overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)} \le C\sup_{f\in X:\|f\|_X\le 1}\int_G f(x)g(x)dx \le C,$$

which gives the required inequality. □

Proof of Theorem 6.1. Let $T\in\left(\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)^*$. Take a compact set K in $G\setminus\{x_0\}$. Then, noting that $\left\{g\chi_K : g\in\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right\} = \left\{g\chi_K : g\in L^{p(\cdot)}(G)\right\}$, we can find $f_K\in L^{p'(\cdot)}(G)$ such that

$$T(g\chi_K)=\int_K f_K(x)g(x)dx$$

for all $g \in \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$.

Construct f such that $f\chi_K = f_K$ for each compact set K in $G \setminus \{x_0\}$. Then

$$T(g\chi_K) = \int_K f(x)g(x)dx$$

for all $g \in \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$. Take a nonnegative $g \in \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$. Then we have

$$\int_G f(x)\{g(x)\chi_K(x)\mathrm{sgn} f(x)\}dx = T(g\chi_K \mathrm{sgn} f) \le C\|g\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)}.$$

Letting $K \to G$, we obtain

$$\int_G |f(x)|g(x)dx \le C\|g\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)}.$$

By Lemma 6.6 we see that $f \in \overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)$. Hence the standard argument yield the required result. □

For $0 < q \le \infty$, we may consider

$$\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G) = \sum_{x_0 \in G} \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G),$$

whose quasi-norm is defined by

$$\|f\|_{\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G)} = \inf_{|f| = \sum_j |f_j|, \{x_j\} \subset G} \sum_j \|f_j\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_j\}}(G)}.$$

Remark 6.7. Let $1 < q < \infty$. If (ω6.1.0) holds for all $x_0 \in G$ with the same constants $Q > 0$, then $\overline{\mathcal{H}}^{p'(\cdot),q',\eta}(G) = L^{p(\cdot)}(G)$ and hence

$$\left(\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G)\right)^* = \left(L^{p(\cdot)}(G)\right)^* = L^{p'(\cdot)}(G) = \overline{\mathcal{H}}^{p'(\cdot),q',\eta}(G).$$

For this, we only show the inclusion $L^{p(\cdot)}(G) \subset \underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G)$. Take $f \in L^{p(\cdot)}(G)$ and $x_1, x_2 \in G$ ($x_1 \ne x_2$). Write

$$f = f\chi_{B(x_2,|x_1-x_2|/2)} + f\chi_{G\setminus B(x_2,|x_1-x_2|/2)} = f_1 + f_2.$$

Then

$$\begin{aligned}
\|f_1\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_1\}}(G)} &\le \left(\int_{|x_1-x_2|/2}^{2d_G} \left(\omega(x_1,r)\|f_1\|_{L^{p(\cdot)}(B(x_1,r))}\right)^q \frac{dr}{r}\right)^{1/q} \\
&\le \|f_1\|_{L^{p(\cdot)}(G)} \left(\int_{|x_1-x_2|/2}^{2d_G} \omega(x_1,r)^q \frac{dr}{r}\right)^{1/q} \\
&= A\|f_1\|_{L^{p(\cdot)}(G)}
\end{aligned}$$

and

$$\begin{aligned}
\|f_2\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_2\}}(G)} &\le \left(\int_{|x_1-x_2|/2}^{2d_G} \left(\omega(x_2,r)\|f_2\|_{L^{p(\cdot)}(B(x_2,r))}\right)^q \frac{dr}{r}\right)^{1/q} \\
&\le \|f_2\|_{L^{p(\cdot)}(G)} \left(\int_{|x_1-x_2|/2}^{2d_G} \omega(x_2,r)^q \frac{dr}{r}\right)^{1/q} \\
&= B\|f_2\|_{L^{p(\cdot)}(G)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|f\|_{\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G)} &\le \|f_1\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_1\}}(G)} + \|f_2\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_2\}}(G)} \\
&\le A\|f_1\|_{L^{p(\cdot)}(G)} + B\|f_2\|_{L^{p(\cdot)}(G)} \\
&\le (A+B)\|f\|_{L^{p(\cdot)}(G)} < \infty,
\end{aligned}$$

as required.

For logarithmic weights, we modify Theorem 6.1 in the following manner.

THEOREM 6.8. *Let $x_0 \in G$ and $1 < q \le \infty$. Suppose there exist constants $a > 0$, $b \ge 0$ and $Q > 0$ such that*

(ω6.2.1) $\displaystyle\int_t^{2d_G} \left(\log\frac{2d_G}{s}\right)^{a-q'-1} \omega(x_0,s)^{-q'}\frac{ds}{s} \le Q\left(\log\frac{2d_G}{t}\right)^a \eta(x_0,t)^{q'}$ *for all* $0 < t < d_G$*; and*

(ω6.2.2) $\displaystyle\int_0^t \left(\log\frac{2d_G}{s}\right)^b \eta(x_0,s)^{q'}\frac{ds}{s} \le Q\left(\log\frac{2d_G}{t}\right)^{b-q'+1} \omega(x_0,t)^{-q'}$ *for all* $0 < t < d_G$.

Then

$$\left(\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)' = \overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G).$$

EXAMPLE 6.9. Typical examples are

$$\omega(x_0,t) = \left(\log\frac{2d_G}{t}\right)^{\varepsilon} \qquad \text{and} \qquad \eta(x_0,t) = \left(\log\frac{2d_G}{t}\right)^{-1-\varepsilon}$$

for $\varepsilon > -1/q$.

The conditions in Theorems 6.1 and 6.8 are also written in another manner; for $q = 1$, see [24, Theorem 8.1] :

THEOREM 6.10. *Let $x_0 \in G$ and $1 \le q < \infty$. Suppose there exist constants $a > 0$, $b \ge 0$ and $Q > 0$ such that*

(ω6.3.0) $\displaystyle\int_0^{2d_G} \eta(x_0,s)^{q'}\frac{ds}{s} < Q$ *when* $1 < q < \infty$ *and* $\eta(x_0,0) = 0$ *when* $q = 1$*;*

(ω6.3.1) $\displaystyle\int_0^t s^a \eta(x_0,s)^{-q}\frac{ds}{s} \le Qt^a\omega(x_0,t)^q$ *for all* $0 < t < d_G$*; and*

$$(\omega 6.3.2) \qquad \int_t^{2d_G} s^b \omega(x_0,s)^q \frac{ds}{s} \le Q t^b \eta(x_0,t)^{-q} \text{ for all } 0<t<d_G.$$

Then

$$\left(\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)^* = \overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G).$$

THEOREM 6.11. *Let $x_0 \in G$ and $1 \le q < \infty$. Suppose there exist constants $a > 0$, $b \ge 0$ and $Q > 0$ such that*

$$(\omega 6.4.0) \qquad \int_0^{2d_G} \eta(x_0,s)^{q'} \frac{ds}{s} < Q \text{ when } 1<q<\infty \text{ and } \eta(x_0,0)=0 \text{ when } q=1;$$

$$(\omega 6.4.1) \qquad \int_0^t \left(\log \frac{2d_G}{t}\right)^{-a-q-1} \eta(x_0,s)^{-q} \frac{ds}{s} \le Q \left(\log \frac{2d_G}{t}\right)^{-a} \omega(x_0,t)^q$$

for all $0<t<d_G$; and

$$(\omega 6.4.2) \qquad \int_t^{2d_G} \left(\log \frac{2d_G}{t}\right)^{-b} \omega(x_0,s)^q \frac{ds}{s} \le Q \left(\log \frac{2d_G}{t}\right)^{-b-q+1} \eta(x_0,t)^{-q}$$

for all $0<t<d_G$.

Then

$$\left(\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)^* = \overline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G).$$

7 Associate space of $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$

In this section, we discuss the associate space of $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$.

THEOREM 7.1. *Let $x_0 \in G$ and $1 < q \le \infty$. Suppose there exist constants $a > 0$, $b \ge 0$ and $Q > 0$ such that*

$$(\omega 7.1.0) \qquad \int_0^{2d_G} \omega(x_0,s)^q \frac{ds}{s} < Q \text{ when } 1<q<\infty \text{ and } \omega(x_0,0)=0 \text{ when } q=\infty;$$

$$(\omega 7.1.1) \qquad \int_0^t s^a \omega(x_0,s)^{-q'} \frac{ds}{s} \le Q t^a \eta(x_0,t)^{q'} \text{ for all } 0<t<d_G; \text{ and}$$

$$(\omega 7.1.2) \qquad \int_t^{2d_G} s^b \eta(x_0,s)^{q'} \frac{ds}{s} \le Q t^b \omega(x_0,t)^{-q'} \text{ for all } 0<t<d_G.$$

Then

$$\left(\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)' = \underline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G).$$

EXAMPLE 7.2. Typical examples are

$$\omega(x_0,t) = \eta(x_0,t)^{-1} = t^\varepsilon$$

for $\varepsilon > 0$.

COROLLARY 7.3. *Let $1 < q < \infty$. If $(\omega 7.1.0)$, $(\omega 7.1.1)$ and $(\omega 7.1.2)$ hold for $x_0 \in G$, then*

$$\left(\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)^* = \underline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G).$$

For $0 < q \leq \infty$, set

$$\widetilde{\mathcal{H}}^{p(\cdot),q,\omega}(G) = \sum_{x_0 \in G} \overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G),$$

whose quasi-norm is defined by

$$\|f\|_{\widetilde{\mathcal{H}}^{p(\cdot),q,\omega}(G)} = \inf_{|f|=\sum_j |f_j|, \{x_j\}\subset G} \sum_j \|f_j\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_j\}}(G)}.$$

Corollary 7.4. *Let $1 < q < \infty$. If (ω7.1.0), (ω7.1.1) and (ω7.1.2) hold for all $x_0 \in G$ with the same constants $a > 0, b \geq 0$ and $Q > 0$, then*

$$\left(\widetilde{\mathcal{H}}^{p(\cdot),q,\omega}(G)\right)^* = \underline{\mathcal{H}}^{p'(\cdot),q',\eta}(G).$$

To prove Theorem 7.1, we need the following two lemmas.

Lemma 7.5. *Let $1 < q \leq \infty$ and $x_0 \in G$. Suppose (ω7.1.1) holds. Then there exists a constant $C > 0$ such that*

$$\int_G |f(x)g(x)|dx \leq C\|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)}\|g\|_{\underline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)}$$

for all measurable functions f and g on G.

Lemma 7.6. *Let $x_0 \in G$ and $1 < q \leq \infty$. Suppose (ω7.1.0) and (ω7.1.2) hold. Set $Y = \overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$. Then there exists a constant $C > 0$ such that*

$$\|f\|_{\underline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)} \leq C \sup_g \int_G |f(x)g(x)|dx$$

for all measurable functions g on G, where the supremum is taken over all measurable functions g on G such that $\|g\|_Y \leq 1$.

Proof. We may assume that f is a nonnegative measurable function on G and

$$\sup_{g\in Y:\|g\|_Y\leq 1} \int_G |f(x)g(x)|dx \leq 1.$$

We only prove the case $1 < q < \infty$ (For $q = \infty$, see [24, Theorem 6.4]). Let K be a compact set in $G \setminus \{x_0\}$. Then, by (ω7.1.0), we see that $L^{p(\cdot)}(K) = \{h\chi_K : h \in L^{p(\cdot)}(G)\} \subset Y$ and

$$\sum_{j\in N_0} \eta(x_0, 2^{-j+1}d_G)^{q'} F_j^{q'} \sim \|f\chi_K\|^{q'}_{\underline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)},$$

where $F_j = \|f_j\|_{L^{p'(\cdot)}(G)}$, $f_j = f\chi_{K\cap B(x_0,2^{-j+1}d_G)}$ and N_0 is the set of positive integers j such that $F_j > 0$.

Consider

$$g(x) = \sum_{j\in N_0} \eta(x_0, 2^{-j+1}d_G)^{q'} F_j^{q'-2}(f_j(x)/F_j)^{p'(x)-2} f_j(x).$$

Then we obtain

$$\begin{aligned}
&\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,r))}\\
\le\ & \sum_{j\in N_0, 2^{-j+1}d_G>r} \eta(x_0,2^{-j+1}d_G)^{q'}F_j^{q'-1}\|(f_j/F_j)^{p'(\cdot)-1}\|_{L^{p(\cdot)}(G)}\\
\le\ & \sum_{2^{-j+1}d_G>r} \eta(x_0,2^{-j+1}d_G)^{q'}F_j^{q'-1},
\end{aligned}$$

so that

$$\begin{aligned}
\|g\|_Y &\le C\left(\sum_{k=1}^{\infty}\left(\omega(x_0,2^{-k}d_G)\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,2^{-k}d_G))}\right)^q\right)^{1/q}\\
&\le C\left(\sum_{k=1}^{\infty}\left(\omega(x_0,2^{-k}d_G)\sum_{j\le k}\eta(x_0,2^{-j}d_G)^{q'}F_j^{q'-1}\right)^q\right)^{1/q}\\
&\le C\left(\sum_{k=1}^{\infty}\omega(x_0,2^{-k}d_G)^q\left(\sum_{j\le k}(2^{-j}d_G)^b\eta(x_0,2^{-j}d_G)^{q'}\right)^{q/q'}\right.\\
&\qquad\left.\times\left(\sum_{j\le k}(2^{-j}d_G)^{-bq/q'}\eta(x_0,2^{-j}d_G)^{(q'-1)q}F_j^{(q'-1)q}\right)\right)^{1/q}\\
&\le C\left(\sum_{k=1}^{\infty}(2^{-k}d_G)^{bq/q'}\left(\sum_{j\le k}(2^{-j}d_G)^{-bq/q'}\eta(x_0,2^{-j}d_G)^{q'}F_j^{q'}\right)\right)^{1/q}\\
&\le C\left(\sum_{j=1}^{\infty}(2^{-j}d_G)^{-bq/q'}\eta(x_0,2^{-j}d_G)^{q'}F_j^{q'}\left(\sum_{k\ge j}(2^{-k}d_G)^{bq/q'}\right)\right)^{1/q}\\
&\le C\left(\sum_{j\in N_0}\eta(x_0,2^{-j}d_G)^{q'}F_j^{q'}\right)^{1/q}\\
&\le C
\end{aligned}$$

by condition (ω7.1.2). On the other hand, we find

$$\begin{aligned}
\int_G f(x)g(x)dx &= \sum_{j\in N_0}\eta(x_0,2^{-j+1}d_G)^{q'}F_j^{q'-2}\int_G f(x)|f_j(x)/F_j|^{p'(x)-2}f_j(x)dx\\
&\ge C\|f\chi_K\|^{q'}_{\underline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)}.
\end{aligned}$$

Hence, by the monotone convergence theorem, we have

$$\|f\|^{q'}_{\underline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G)} \le C\sup_{g\in Y:\|g\|_Y\le 1}\int_G f(x)g(x)dx \le C,$$

which gives the required inequality. □

Logarithmic weights can be treated as follows:

THEOREM 7.7. *Let $x_0 \in G$ and $1 < q \le \infty$. Suppose there exist constants $a > 0$, $b \ge 0$ and $Q > 0$ such that*

$$(\omega 7.2.0) \qquad \int_0^{2d_G} \omega(x_0, s)^q \frac{ds}{s} < \infty \text{ when } 1 < q < \infty \text{ and } \omega(x_0, 0) = 0 \text{ when } q = \infty;$$

$$(\omega 7.2.1) \qquad \int_0^t \left(\log \frac{2d_G}{s}\right)^{-a-q'-1} \omega(x_0, s)^{-q'} \frac{ds}{s} \le Q \left(\log \frac{2d_G}{s}\right)^{-a} \eta(x_0, t)^{q'}$$

for all $0 < t < d_G$; and

$$(\omega 7.2.2) \qquad \int_t^{2d_G} \left(\log \frac{2d_G}{r}\right)^{-b} \eta(x_0, s)^{q'} \frac{ds}{s} \le Q \left(\log \frac{2d_G}{t}\right)^{-b-q'+1} \omega(x_0, t)^{-q'}$$

for all $0 < t < d_G$.

Then

$$\left(\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)' = \underline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G).$$

EXAMPLE 7.8. Typical examples are

$$\omega(x_0, t) = \left(\log \frac{2d_G}{r}\right)^{-\varepsilon} \qquad \text{and} \qquad \eta(x_0, t) = \left(\log \frac{2d_G}{r}\right)^{-1+\varepsilon}$$

for $\varepsilon > 1/q$.

COROLLARY 7.9. *Let $1 < q < \infty$. If (ω7.2.0), (ω7.2.1) and (ω7.2.2) hold for $x_0 \in G$, then*

$$\left(\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)^* = \underline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G).$$

COROLLARY 7.10. *Let $1 < q < \infty$. If (ω7.2.0), (ω7.2.1) and (ω7.2.2) hold for all $x_0 \in G$ with the same constants $a > 0, b \ge 0$ and $Q > 0$, then*

$$\left(\widetilde{\mathcal{H}}^{p(\cdot),q,\omega}(G)\right)^* = \underline{\mathcal{H}}^{p'(\cdot),q',\eta}(G).$$

Theorems 7.1 and 7.7 are also written as follows; for $q = \infty$, see [24, Theorem 9.1]:

THEOREM 7.11. *Let $x_0 \in G$ and $1 \le q < \infty$. Suppose there exist constants $a > 0$, $b \ge 0$ and $Q > 0$ such that*

$$(\omega 7.3.1) \qquad \int_t^{2d_G} s^{-a} \eta(x_0, s)^{-q} \frac{ds}{s} \le Q t^{-a} \omega(x_0, t)^q \text{ for all } 0 < t < d_G; \text{ and}$$

$$(\omega 7.3.2) \qquad \int_0^t s^{-b} \omega(x_0, s)^q \frac{ds}{s} \le Q t^{-b} \eta(x_0, t)^{-q} \text{ for all } 0 < t < d_G.$$

Then

$$\left(\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)^* = \underline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G).$$

THEOREM 7.12. *Let $x_0 \in G$ and $1 \le q < \infty$. Suppose there exist constants $a > 0$, $b \ge 0$ and $Q > 0$ such that*

(ω7.4.1) $\displaystyle\int_t^{2d_G}\left(\log\frac{2d_G}{s}\right)^{a-q-1}\eta(x_0,s)^{-q}\frac{ds}{s} \le Q\left(\log\frac{2d_G}{s}\right)^{a}\omega(x_0,t)^q$ *for all* $0<t<d_G$*; and*

(ω7.4.2) $\displaystyle\int_0^{t}\left(\log\frac{2d_G}{s}\right)^{b}\omega(x_0,s)^{q}\frac{ds}{s} \le Q\left(\log\frac{2d_G}{s}\right)^{b-q+1}\eta(x_0,t)^{-q}$ *for all* $0<t<d_G$.

Then

$$\left(\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\right)^* = \underline{\mathcal{H}}^{p'(\cdot),q',\eta}_{\{x_0\}}(G).$$

8 Grand Lebesgue spaces

Following Capone-Fiorenza [7], for $\theta>0$, $1<q<\infty$ and a measurable function f on the unit ball $\mathbf{B}=B(0,1)$, we define the norm

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\theta}_{\{0\}}(\mathbf{B})} = \left(\int_0^1\left(\left(\log\frac{2}{t}\right)^{-\theta}\|f\|_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}\right)^q\frac{dt}{t}\right)^{1/q}$$

and

$$\|f\|_{L^{p(\cdot)-0,q,\theta}(\mathbf{B})} = \left(\int_0^1\left(\varepsilon^{\theta}\|f\|_{L^{p(\cdot)-\varepsilon}(\mathbf{B})}\right)^q\frac{d\varepsilon}{\varepsilon}\right)^{1/q}.$$

THEOREM 8.1. *Let* $0<\gamma<\theta+\min\{1/q,1/p(0)\}$. *Then there exists a constant* $C>0$ *such that*

$$\|f\|_{L^{p(\cdot)-0,q,\theta}(\mathbf{B})} \le C\|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\gamma}_{\{0\}}(\mathbf{B})}$$

for all measurable functions f *on* $\mathbf{B}$.

Proof. Let f be a nonnegative measurable function on $\mathbf{B}$ such that $\|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\gamma}_{\{0\}}(\mathbf{B})}\le 1$. Then we have by Lemma 2.3

$$\left(\log\frac{2}{t}\right)^{-\theta}\|f\|_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}\le C \tag{8.1}$$

for all $0<t<1$. For $0<\varepsilon<1$ and $n/p(0)<a<\theta+n/p(0)$, we have by Fubini's theorem

$$\begin{aligned}\int_{\mathbf{B}} f(x)^{p(x)-\varepsilon}dx &= \int_{\mathbf{B}} f(x)^{p(x)-\varepsilon}\left(a\varepsilon|x|^{-a\varepsilon}\int_0^{|x|}t^{a\varepsilon}\frac{dt}{t}\right)dx\\ &= a\varepsilon\int_0^1\left(\int_{\mathbf{B}\setminus B(0,t)} f(x)^{p(x)-\varepsilon}|x|^{-a\varepsilon}dx\right)t^{a\varepsilon}\frac{dt}{t}.\end{aligned}$$

Note from (8.1) that

$$\begin{aligned}&\int_{\mathbf{B}\setminus B(0,t)} f(x)^{p(x)-\varepsilon}|x|^{-a\varepsilon}dx\\ \le\ &\int_{\mathbf{B}\setminus B(0,t)}\left(f(x)^{p(x)-\varepsilon}\right)^{p(x)/(p(x)-\varepsilon)}dx+\int_{\mathbf{B}\setminus B(0,t)}\left(|x|^{-a\varepsilon}\right)^{p(x)/\varepsilon}dx\\ \le\ &C\left\{\left(\log\frac{2}{t}\right)^{\theta p^+}+t^{-ap(0)+n}\right\}\end{aligned}$$

for all $0 < t < 1$, so that

$$\int_{\mathbf{B}\setminus B(0,(\log 2)/\log(2/t))} f(x)^{p(x)-\varepsilon}|x|^{-a\varepsilon}dx \le C\left(\log\frac{2}{t}\right)^{A}$$

for some $ap(0)-n < A < \theta p(0)$ and all $0 < t < 1$. By (P2), we find

$$\begin{aligned}
&\int_{\mathbf{B}\setminus B(0,t)}\left(\frac{|x|^{-a\varepsilon}}{t^{(-ap(0)+n)\varepsilon/p(0)}}\right)^{p(x)/\varepsilon}dx \\
\le\ &\int_{\mathbf{B}\setminus B(0,t^{p^-/p(0)})}\left(\frac{|x|^{-a\varepsilon}}{t^{(-ap(0)+n)\varepsilon/p(0)}}\right)^{p(x)/\varepsilon}dx + \int_{B(0,t^{p^-/p(0)})\setminus B(0,t)}\left(\frac{|x|^{-a\varepsilon}}{t^{(-ap(0)+n)\varepsilon/p(0)}}\right)^{p(x)/\varepsilon}dx \\
\le\ & t^{(ap(0)-n)p^-/p(0)}\int_{\mathbf{B}\setminus B(0,t^{p^-/p(0)})}|x|^{-ap(0)}dx + Ct^{ap(0)-n}\int_{B(0,t^{p^-/p(0)})\setminus B(0,t)}|x|^{-ap(0)}dx \\
\le\ & C\left(t^{(ap(0)-n)p^-/p(0)}t^{(-ap(0)+n)p^-/p(0)} + t^{ap(0)-n}t^{-ap(0)+n}\right) \\
\le\ & C
\end{aligned}$$

for all $0 < t < 1$, which implies

$$\||\cdot|^{-a\varepsilon}\|_{L^{p(\cdot)/\varepsilon}(\mathbf{B}\setminus B(0,t))} \le Ct^{(-ap(0)+n)\varepsilon/p(0)}.$$

If $\|f\|_{L^{p(\cdot)}(B(0,(\log 2)/\log(2/t))\setminus B(0,t))} \le 1$, then

$$\|f(\cdot)^{p(\cdot)-\varepsilon}\|_{L^{p(\cdot)/(p(\cdot)-\varepsilon)}(B(0,(\log 2)/\log(2/t))\setminus B(0,t))} \le C\|f\|^{p^- -\varepsilon}_{L^{p(\cdot)}(B(0,(\log 2)/\log(2/t))\setminus B(0,t))} \le C$$

and if $1 \le J = \|f\|_{L^{p(\cdot)}(B(0,(\log 2)/\log(2/t))\setminus B(0,t))} \le C\left(\log(2/t)\right)^{\theta}$, then

$$\begin{aligned}
&\int_{B(0,(\log 2)/\log(2/t))\setminus B(0,t)}\left(\frac{f(x)^{p(x)-\varepsilon}}{J^{p(0)-\varepsilon}}\right)^{p(x)/(p(x)-\varepsilon)}dx \\
\le\ & C\int_{B(0,(\log 2)/\log(2/t))\setminus B(0,t)}\left(\frac{f(x)}{J}\right)^{p(x)}dx \\
\le\ & C
\end{aligned}$$

by (P2), and thus

$$\|f(\cdot)^{p(\cdot)-\varepsilon}\|_{L^{p(\cdot)/(p(\cdot)-\varepsilon)}(B(0,(\log 2)/\log(2/t))\setminus B(0,t))} \le CJ^{p(0)-\varepsilon}.$$

Now it follows that

$$\begin{aligned}
&\int_{\mathbf{B}\setminus B(0,t)} f(x)^{p(x)-\varepsilon}|x|^{-a\varepsilon}dx \\
\le\ &\int_{\mathbf{B}\setminus B(0,(\log 2)/\log(2/t))} f(x)^{p(x)-\varepsilon}|x|^{-a\varepsilon}dx + \int_{B(0,(\log 2)/\log(2/t))\setminus B(0,t)} f(x)^{p(x)-\varepsilon}|x|^{-a\varepsilon}dx \\
\le\ & C\left(\log\frac{2}{t}\right)^{A} + 2\|f(\cdot)^{p(\cdot)-\varepsilon}\|_{L^{p(\cdot)/(p(\cdot)-\varepsilon)}(B(0,(\log 2)/\log(2/t))\setminus B(0,t))}\||\cdot|^{-a\varepsilon}\|_{L^{p(\cdot)/\varepsilon}(\mathbf{B}\setminus B(0,t))} \\
\le\ & C\left\{\left(\log\frac{2}{t}\right)^{A} + t^{(-ap(0)+n)\varepsilon/p(0)} + \|f\|^{p(0)-\varepsilon}_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}t^{(-ap(0)+n)\varepsilon/p(0)}\right\}
\end{aligned}$$

for all $0 < t < 1$. Here, for $\sigma > 0$ and $\tau > 0$, we find

$$t^{\sigma}(\log(2/t))^{\tau} \le 2^{\sigma+\tau}\tau^{\tau}\sigma^{-\tau}e^{-\tau} \tag{8.2}$$

for all $0 < t < 1$, so that

$$\int_0^1 t^{\sigma}(\log(2/t))^{\tau}\frac{dt}{t} \le 2^{\sigma+\tau+1}\tau^{\tau}\sigma^{-\tau-1}e^{-\tau}. \tag{8.3}$$

Hence we have by (8.3)

$$\begin{aligned}\int_{\mathbf{B}} f(x)^{p(x)-\varepsilon}dx &\le C\varepsilon\Bigg(\int_0^1\left(\log\frac{2}{t}\right)^{A} t^{a\varepsilon}\frac{dt}{t} + \int_0^1 t^{(-ap(0)+n)\varepsilon/p(0)}t^{a\varepsilon}\frac{dt}{t} \\ &\qquad + \int_0^1\left(\|f\|^{p(0)-\varepsilon}_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}t^{(-ap(0)+n)\varepsilon/p(0)}\right)t^{a\varepsilon}\frac{dt}{t}\Bigg) \\ &\le C\left(\varepsilon^{-A} + \varepsilon\int_0^1\left(\|f\|^{p(0)-\varepsilon}_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}t^{n\varepsilon/p(0)}\right)\frac{dt}{t}\right). \qquad (8.4)\end{aligned}$$

We show

$$\|f\|_{L^{p(\cdot)-\varepsilon}(\mathbf{B})} \le C\left\{\varepsilon^{-A/p(0)} + \varepsilon^{\kappa}\left(\int_0^1\left(\|f\|_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}t^{(n-b)\varepsilon/(p(0)(p(0)-\varepsilon))}\right)^{q}\ \frac{dt}{t}\right)^{1/q}\right\}, \tag{8.5}$$

where $\kappa = \min\{1/q, 1/p(0)\}$. First we consider the case $p(0)-\varepsilon \le q$. Taking b with $0 < b < n$, we have by Hölder's inequality again

$$\begin{aligned}&\int_0^1\ \|f\|^{(p(0)-\varepsilon)}_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}t^{n\varepsilon/p(0)}\ \frac{dt}{t} \\ &\le \left(\int_0^1\left(\|f\|^{(p(0)-\varepsilon)}_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}t^{(n-b)\varepsilon/p(0)}\right)^{q/(p(0)-\varepsilon)}\ \frac{dt}{t}\right)^{(p(0)-\varepsilon)/q} \\ &\quad\times\left(\int_0^1\left(t^{b\varepsilon/p(0)}\right)^{q/(q-(p(0)-\varepsilon))}\ \frac{dt}{t}\right)^{1-(p(0)-\varepsilon)/q} \\ &\le \left(\int_0^1\left(\|f\|_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}t^{(n-b)\varepsilon/(p(0)(p(0)-\varepsilon))}\right)^{q}\ \frac{dt}{t}\right)^{(p(0)-\varepsilon)/q}\left(C\varepsilon^{-1}\right)^{1-(p(0)-\varepsilon)/q} \\ &\le C\varepsilon^{-1+(p(0)-\varepsilon)/q}\left(\int_0^1\left(\|f\|_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}t^{(n-b)\varepsilon/(p(0)(p(0)-\varepsilon))}\right)^{q}\ \frac{dt}{t}\right)^{(p(0)-\varepsilon)/q},\end{aligned}$$

and hence

$$\begin{aligned}\int_{\mathbf{B}} f(x)^{p(x)-\varepsilon}dx \ &\le\ C\Bigg\{\varepsilon^{-A} + \varepsilon^{(p(0)-\varepsilon)/q} \\ &\qquad\times\left(\int_0^1\left(\|f\|_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}t^{(n-b)\varepsilon/(p(0)(p(0)-\varepsilon))}\right)^{q}\ \frac{dt}{t}\right)^{(p(0)-\varepsilon)/q}\Bigg\}.\end{aligned}$$

We derive from (8.1)

$$\int_{\mathbf{B}\setminus B(0,\varepsilon)}\left(\frac{f(x)}{(\log(2/\varepsilon))^{\theta}}\right)^{p(x)-\varepsilon}dx \le \int_{\mathbf{B}\setminus B(0,\varepsilon)}\left\{\left(\frac{f(x)}{(\log(2/\varepsilon))^{\theta}}\right)^{p(x)}+1\right\}dx \le C.$$

Set $H = \varepsilon^{-A/p(0)} + K$ with $K = \varepsilon^{1/q} \left(\int_0^1 \left(\|f\|_{L^{p(\cdot)}(\mathbf{B} \setminus B(0,t))} t^{(n-b)\varepsilon/(p(0)(p(0)-\varepsilon))} \right)^q \frac{dt}{t} \right)^{1/q}$.
If $\varepsilon^{-A} \le K^{(p(0)-\varepsilon)}$, we find by (P2)

$$\int_{B(0,\varepsilon)} \left(\frac{f(x)}{K} \right)^{p(x)-\varepsilon} dx \le C K^{-p(0)+\varepsilon} \int_{B(0,\varepsilon)} f(x)^{p(x)-\varepsilon} dx \le C$$

since $K \le C\varepsilon^{-\theta}$ by (8.3). If $\varepsilon^{-A} > K^{(p(0)-\varepsilon)}$, we see from (P2) that

$$\int_{B(0,\varepsilon)} \left(\frac{f(x)}{\varepsilon^{-A/p(0)}} \right)^{p(x)-\varepsilon} dx \le \varepsilon^{A} \int_{B(0,\varepsilon)} f(x)^{p(x)-\varepsilon} dx \le C.$$

Consequently it follows that

$$\int_{\mathbf{B}} \left(\frac{f(x)}{H} \right)^{p(x)-\varepsilon} dx = \int_{\mathbf{B} \setminus B(0,\varepsilon)} \left(\frac{f(x)}{H} \right)^{p(x)-\varepsilon} dx + \int_{B(0,\varepsilon)} \left(\frac{f(x)}{H} \right)^{p(x)-\varepsilon} dx \le C,$$

which gives (8.5).

Next we show (8.5) in case $q < p(0) - \varepsilon$. In this case (8.4) gives

$$\int_{\mathbf{B}} f(x)^{p(x)-\varepsilon} dx \le C \left\{ \varepsilon^{-A} + \varepsilon \int_0^1 (\|f\|_{L^{p(\cdot)}(\mathbf{B} \setminus B(0,t))} t^{(n-b)\varepsilon/(p(0)(p(0)-\varepsilon))})^q \frac{dt}{t} \right\}^{(p(0)-\varepsilon)/q},$$

so that we obtain by the above considerations

$$\|f\|_{L^{p(\cdot)-\varepsilon}(\mathbf{B})} \le C \left\{ \varepsilon^{-A/p(0)} + \varepsilon^{1/p(0)} \left(\int_0^1 \left(\|f\|_{L^{p(\cdot)}(\mathbf{B} \setminus B(0,t))} t^{(n-b)\varepsilon/(p(0)(p(0)-\varepsilon))} \right)^q \frac{dt}{t} \right)^{1/q} \right\},$$

which proves (8.5).

Now we find

$$\begin{aligned}
\int_0^1 \left(\varepsilon^{\theta} \|f\|_{L^{p(\cdot)-\varepsilon}(\mathbf{B})} \right)^q \frac{d\varepsilon}{\varepsilon} &\le C \Bigg\{ \int_0^1 \varepsilon^{(\theta - A/p(0))q} \frac{d\varepsilon}{\varepsilon} \\
&\quad + \int_0^1 \varepsilon^{(\theta+\kappa)q} \left(\int_0^1 \left(\|f\|_{L^{p(\cdot)}(\mathbf{B} \setminus B(0,t))} t^{(n-b)\varepsilon/(p(0)(p(0)-\varepsilon)} \right)^q \frac{dt}{t} \right) \frac{d\varepsilon}{\varepsilon} \Bigg\} \\
&\le C \left\{ 1 + \int_0^1 \|f\|^q_{L^{p(\cdot)}(\mathbf{B} \setminus B(0,t))} \left(\int_0^1 \varepsilon^{(\theta+\kappa)q} t^{(n-b)\varepsilon q/(p(0)(p(0)-\varepsilon))} \frac{d\varepsilon}{\varepsilon} \right) \frac{dt}{t} \right\}
\end{aligned}$$

Here note from (8.2) that

$$t^{(n-b)\varepsilon q/(p(0)(p(0)-\varepsilon))} \le C \left(\log \frac{2}{t} \right)^{-\gamma q} \varepsilon^{-\gamma q}$$

for all $0 < \varepsilon < 1$ and $0 < t < 1$. Hence we find

$$\begin{aligned}
&\int_0^1 \left(\varepsilon^{\theta} \|f\|_{L^{p(\cdot)-\varepsilon}(\mathbf{B})} \right)^q \frac{d\varepsilon}{\varepsilon} \\
&\le C \left\{ 1 + \int_0^1 \left(\log \frac{2}{t} \right)^{-\gamma q} \|f\|^q_{L^{p(\cdot)}(\mathbf{B} \setminus B(0,t))} \left(\int_0^1 \varepsilon^{(\theta+\kappa-\gamma)q} \frac{d\varepsilon}{\varepsilon} \right) \frac{dt}{t} \right\} \\
&\le C \left\{ 1 + \int_0^1 \left(\left(\log \frac{2}{t} \right)^{-\gamma} \|f\|_{L^{p(\cdot)}(\mathbf{B} \setminus B(0,t))} \right)^q \frac{dt}{t} \right\} \\
&\le C,
\end{aligned}$$

which completes the proof. □

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