Sobolev's theorem and duality for Herz-Morrey spaces of variable exponent

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Abstract

In this paper, we consider the Herz-Morrey space $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ of variable exponent consisting of all measurable functions f on a bounded open set $G \subset \mathbf{R}^n$ satisfying

$$\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{2d_G} \left(\omega(x_0, r) \|f\|_{L^{p(\cdot)}(B(x_0,r)\setminus B(x_0,r/2))} \right)^q dr/r \right)^{1/q} < \infty,$$

and set $\mathcal{H}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0 \in G} \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$.

Our first aim in this paper is to give the boundedness of the maximal and Riesz potential operators in $\mathcal{H}^{p(\cdot),q,\omega}(G)$ when $q = \infty$.

In connection with $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\mathcal{H}^{p(\cdot),q,\omega}(G)$, let us consider the families $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$, $\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$, $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\tilde{\mathcal{H}}^{p(\cdot),q,\omega}(G)$. Following Fiorenza-Rakotoson [18], Di Fratta-Fiorenza [17] and Gogatishvili-Mustafayev [19], we next discuss the duality properties among these Herz-Morrey spaces.

1 Introduction

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. We denote by $B(x, r)$ the open ball centered at x of radius r, and by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbf{R}^n$.

It is well known that the maximal operator is bounded in the Lebesgue space $L^p(\mathbf{R}^n)$ if $p > 1$ (see [34]). In [12], the boundedness of the maximal operator is still valid by replacing the Lebesgue space by several Morrey spaces; the original one was introduced by Morrey [30] to estimate solutions of partial differential equations; for Morrey spaces, we also refer to Peetre [32] and Nakai [31].

One of important applications of the boundedness of the maximal operator is Sobolev's inequality; in the classical case,

$$\|I_\alpha * f\|_{L^{p^\sharp}(\mathbf{R}^n)} \le C\|f\|_{L^p(\mathbf{R}^n)}$$

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for $f \in L^p(\mathbf{R}^n)$, $0 < \alpha < n$ and $1 < p < n/\alpha$, where I_α is the Riesz kernel of order α and $1/p^\sharp = 1/p - \alpha/n$ (see, e.g. [2, Theorem 3.1.4]). Sobolev's inequality for Morrey spaces was given by Adams [1] (also [12]). Further, Sobolev's inequality was also studied on generalized Morrey spaces (see [31]). This result was extended to local and global Morrey type spaces by Burenkov, Gogatishvili, Guliyev and Mustafayev [8] (see also [7, 9, 10]). The local Morrey type spaces are also called Herz spaces introduced by Herz [23]. In our paper, those Morrey type spaces are referred to as Herz-Morrey spaces.

In [13], Diening showed that the maximal operator is bounded on the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{R}^n)$ if the variable exponent $p(\cdot)$, which is a constant outside a ball, satisfies the locally log-Hölder condition and $\inf p(x) > 1$ (see condition (P2) in Section 2). In the mean time, variable exponent Lebesgue spaces were used to discuss nonlinear partial differential equations with non-standard growth condition. These spaces have attracted more and more attention, in connection with the study of elasticity and fluid mechanics; see [16], [33]. On the other hand, variable exponent Morrey or Herz versions were discussed in [4, 5, 24, 26, 29].

Let G be a bounded open set in $\mathbf{R}^n$, whose diameter is denoted by d_G. Let $\omega(\cdot,\cdot) : G\times(0,\infty) \to (0,\infty)$ be a uniformly almost monotone function on $G\times(0,\infty)$ satisfying the uniformly doubling condition. For $x_0 \in G$, $0 < q \le \infty$ and a variable exponent $p(\cdot)$, we consider the Herz-Morrey space $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ of variable exponent consisting of all measurable functions f on G satisfying

$$\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r)\setminus B(x_0,r/2))}\right)^q dr/r \right)^{1/q} < \infty;$$

when $q = \infty$,

$$\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} = \sup_{0<r<d_G} \omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r)\setminus B(x_0,r/2))} < \infty.$$

Set

$$\mathcal{H}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0\in G} \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G),$$

whose norm is defined by

$$\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}(G)} = \sup_{x_0\in G} \|f\|_{\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)}.$$

In connection with $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$, let us consider the families $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ of all functions f on G satisfying

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r))}\right)^q \frac{dr}{r} \right)^{1/q} < \infty$$

and

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(G\setminus B(x_0,r))}\right)^q \frac{dr}{r} \right)^{1/q} < \infty,$$

respectively. In the paper by Fiorenza and Rakotoson [18], the Herz-Morrey space $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ is referred to as the generalized Lorentz space denoted by $G\Gamma(p,q,\omega)$.

Note here that

$$\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G) \cup \overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G) \subset \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G).$$

Similarly we consider the space

$$\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0 \in G} \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G),$$

whose norm is defined by

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)} = \sup_{x_0 \in G} \|f\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)}.$$

Our first aim in this paper is to establish the boundedness of the maximal operator and the Riesz potential operator in $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$; when $q < \infty$, we refer to [27]. In the borderline case, Trudinger's exponential integrability is discussed.

Next, following Di Fratta-Fiorenza [17] and Gogatishvili-Mustafayev [19] , we study the duality properties among those Herz-Morrey spaces. In particular, we show the associate spaces of $\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$, which give another characterizations of Morrey spaces by Adams-Xiao [3] (see also [20]).

2 Preliminaries

Throughout this paper, let C denote various constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 1$. Set $A(x,r) = B(x,r) \setminus B(x,r/2)$.

Consider a function $p(\cdot)$ on G such that

(P1) $1 < p^- := \inf_{x\in G} p(x) \le \sup_{x \in G} p(x) =: p^+ < \infty$

and

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{c_p}{\log(2d_G/|x-y|)} \quad \text{for } x, y \in G$$

with a constant $c_p \ge 0$; $p(\cdot)$ is referred to as a variable exponent.

We also consider the family $\Omega(G)$ of all positive functions $\omega(\cdot,\cdot) : G \times (0,\infty) \to (0,\infty)$ satisfying the following conditions:

($\omega 0$) $\omega(x,0) = \lim_{r\to+0} \omega(x,r) = 0$ for all $x \in G$ or $\omega(x,0) = \infty$ for all $x \in G$;

($\omega 1$) $\omega(x,\cdot)$ is uniformly almost monotone on $(0,\infty)$, that is, there exists a constant $Q_1 > 0$ such that $\omega(x,\cdot)$ is uniformly almost increasing on $(0,\infty)$, that is,

$$\omega(x,r) \le Q_1 \omega(x,s) \quad \text{for all } x \in G \text{ and } 0 < r < s$$

or $\omega(x,\cdot)$ is uniformly almost decreasing on $(0,\infty)$, that is,

$$\omega(x,s) \le Q_1 \omega(x,r) \quad \text{for all } x \in G \text{ and } 0 < r < s;$$

(ω2) $\omega(x,\cdot)$ is uniformly doubling on $(0,\infty)$, that is, there exists a constant $Q_2>0$ such that

$$Q_2^{-1}\omega(x,r) \le \omega(x,2r) \le Q_2\omega(x,r) \quad \text{for all } x\in G \text{ and } r>0;$$

and

(ω3) there exists a constant $Q_3>0$ such that

$$Q_3^{-1} \le \omega(x,1) \le Q_3 \quad \text{for all } x\in G.$$

Then one can find constants $a,b>0$ and $C>1$ such that

$$C^{-1}r^a \le \omega(x,r) \le Cr^{-b} \tag{2.1}$$

for all $x\in G$ and $0<r\le d_G$.

For later use, it is convenient to note the following result, which is proved by (P1), (P2) and (2.1).

LEMMA 2.1. *There exists a constant $C>0$ such that*

$$\omega(x,r)^{p(x)} \le C\omega(x,r)^{p(y)}$$

whenever $|x-y|<r\le d_G$.

For a locally integrable function f on G, set

$$\|f\|_{L^{p(\cdot)}(G)} = \inf\left\{\lambda>0 : \int_G \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} dy \le 1\right\};$$

in what follows, set $f=0$ outside G. We denote by $L^{p(\cdot)}(G)$ the family of locally integrable functions f on G satisfying $\|f\|_{L^{p(\cdot)}(G)}<\infty$.

LEMMA 2.2. *Let $0<q<\infty$. Then*

(1) $$\int_0^{2d_G} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r \sim \sum_{j=1}^{\infty} \left(\omega(x,2^{-j+1}d_G)\|f\|_{L^{p(\cdot)}(A(x,2^{-j+1}d_G))}\right)^q;$$

(2) $$\int_0^{2d_G} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(B(x,r))}\right)^q dr/r \sim \sum_{j=1}^{\infty} \left(\omega(x,2^{-j+1}d_G)\|f\|_{L^{p(\cdot)}(B(x,2^{-j+1}d_G))}\right)^q;$$

and

(3) $$\int_0^{2d_G} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(G\setminus B(x,r))}\right)^q dr/r \sim \sum_{j=1}^{\infty} \left(\omega(x,2^{-j}d_G)\|f\|_{L^{p(\cdot)}(G\setminus B(x,2^{-j}d_G))}\right)^q$$

for all $x\in G$ and measurable functions f on G.

Proof. We only prove (1), since the remaining assertions can be proved similarly. Since $A(x,r) \supset B(x,3t/2) \setminus B(x,t)$ when $3t/2 < r < 2t \le 2d_G$, we have by (ω1) and (ω2)

$$\int_{3t/2}^{2t} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r \ge C\left(\omega(x,t)\|f\|_{L^{p(\cdot)}(B(x,3t/2)\setminus B(x,t))}\right)^q$$

and similarly, we have

$$\int_{t}^{3t/2} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r \ge C\left(\omega(x,t)\|f\|_{L^{p(\cdot)}(B(x,t)\setminus B(x,3t/4))}\right)^q .$$

Thus

$$\int_{t}^{2t} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r \ge C\left(\omega(x,t)\|f\|_{L^{p(\cdot)}(B(x,3t/2)\setminus B(x,3t/4))}\right)^q .$$

Therefore, letting $3t/2 = 2^{-j+1}d_G$ for a positive integer j, we see that

$$\int_{2^{-j}d_G}^{2^{-j+2}d_G} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r \ge C\left(\omega(x,2^{-j+1}d_G)\|f\|_{L^{p(\cdot)}(A(x,2^{-j+1}d_G))}\right)^q ,$$

so that

$$\begin{aligned}\int_{0}^{2d_G} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r &\ge \frac{1}{2}\sum_{j=1}^{\infty}\int_{2^{-j}d_G}^{2^{-j+2}d_G} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r \\ &\ge C\sum_{j=1}^{\infty}\left(\omega(x,2^{-j+1}d_G)\|f\|_{L^{p(\cdot)}(A(x,2^{-j+1}d_G))}\right)^q .\end{aligned}$$

The converse inequality is easily obtained. □

Further, we obtain the next result.

LEMMA 2.3. *Suppose* $0 < q \le \infty$. *If* $\|f\|_{h^{p(\cdot),q,\omega}(G)} \le 1$, *then there exists a constant* $C > 0$ *such that* $\|f\|_{h^{p(\cdot),\infty,\omega}(G)} \le C$, *for* $h = \mathcal{H}_{\{x_0\}}, \underline{\mathcal{H}}_{\{x_0\}}, \overline{\mathcal{H}}_{\{x_0\}}, \mathcal{H}, \underline{\mathcal{H}}$.

By Lemma 2.1, we have the following result.

LEMMA 2.4. *There is a constant* $C > 0$ *such that*

$$\int_{B(x_0,r)} |f(y)|^{p(y)} dy \le C\omega(x_0,r)^{-p(x_0)}$$

when $x_0 \in G$, $0 < r < d_G$ *and* $\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r))} \le 1$.

LEMMA 2.5. *There is a constant* $C > 0$ *such that*

$$\frac{1}{|A(x_0,r)|}\int_{A(x_0,r)} |f(y)| dy \le C r^{-n/p(x_0)}\omega(x_0,r)^{-1}$$

when $x_0 \in G$, $0 < r < d_G$ *and* $\omega(x_0,r)\|f\|_{L^{p(\cdot)}(A(x_0,r))} \le 1$.

Proof. Fix $x_0 \in G$ and $0 < r < d_G$. Let f be a nonnegative measurable function on G satisfying $\omega(x_0,r)\|f\|_{L^{p(\cdot)}(A(x_0,r))} \le 1$. Then we have by (P2) and Lemmas 2.1 and 2.4,

$$\begin{aligned}
&\frac{1}{|A(x_0,r)|}\int_{A(x_0,r)} f(y)dy \\
&\le r^{-n/p(x_0)}\omega(x_0,r)^{-1} + \frac{1}{|A(x_0,r)|}\int_{A(x_0,r)} f(y)\left(\frac{f(y)}{r^{-n/p(x_0)}\omega(x_0,r)^{-1}}\right)^{p(y)-1} dy \\
&\le r^{-n/p(x_0)}\omega(x_0,r)^{-1} + C\left(r^{-n/p(x_0)}\omega(x_0,r)^{-1}\right)^{1-p(x_0)} \frac{1}{|A(x_0,r)|}\int_{A(x_0,r)} f(y)^{p(y)}dy \\
&\le Cr^{-n/p(x_0)}\omega(x_0,r)^{-1},
\end{aligned}$$

as required. □

3 Boundedness of the maximal operator for $q = \infty$

Let us consider the following conditions: let $\eta \in \Omega(G)$ and $x_0 \in G$.

(ω3.1) There exists a constant $Q > 0$ such that

$$\int_0^r t^{n-n/p(x_0)}\omega(x_0,t)^{-1}\frac{dt}{t} \le Qr^{n-n/p(x_0)}\eta(x_0,r)^{-1}$$

for all $0 < r \le d_G$; and

(ω3.2) there exists a constant $Q > 0$ such that

$$\int_r^{2d_G} t^{-n/p(x_0)}\omega(x_0,t)^{-1}\frac{dt}{t} \le Qr^{-n/p(x_0)}\eta(x_0,r)^{-1}$$

for all $0 < r \le d_G$.

By the doubling condition on ω, one notes from (ω3.1) or (ω3.2) that

$$\omega(x_0,r)^{-1} \le C\eta(x_0,r)^{-1}.$$

LEMMA 3.1. *If (ω3.1) and (ω3.2) hold for all $x_0 \in G$ with the same constant Q, then there is a constant $C > 0$ such that*

$$\int_{B(x,r)} |f(y)|dy \le Cr^{n-n/p(x)}\eta(x,r)^{-1}$$

and

$$\int_{G\setminus B(x,r)} |f(y)||x-y|^{-n}dy \le Cr^{-n/p(x)}\eta(x,r)^{-1}$$

for all $x \in G$, $0 < r \le d_G$ and f with $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$.

Proof. Let f be a nonnegative measurable function on G satisfying $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$. By Lemma 2.5 and ($\omega$3.1), we have

$$\begin{aligned}\int_{B(x,r)} f(y)dy &= \sum_{j=1}^{\infty} \int_{A(x,2^{-j+1}r)} f(y)dy \\ &\le C\sum_{j=1}^{\infty} (2^{-j}r)^{n-n/p(x)}\omega(x,2^{-j}r)^{-1} \\ &\le Cr^{n-n/p(x)}\eta(x,r)^{-1}.\end{aligned}$$

Similarly, we obtain by use of Lemma 2.5 and (ω3.2)

$$\begin{aligned}\int_{G\setminus B(x,r)} |f(y)||x-y|^{-n}dy &\le C \sum_{j\ge 1, 2^{j-1}r\le d_G} (2^j r)^{-n} \int_{A(x,2^j r)} f(y)dy \\ &\le C \sum_{j\ge 1, 2^{j-1}r\le d_G} (2^j r)^{-n/p(x)}\omega(x,2^j r)^{-1} \\ &\le Cr^{-n/p(x)}\eta(x,r)^{-1},\end{aligned}$$

as required. □

For a locally integrable function f on G, the Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy;$$

recall that $f = 0$ outside G. Now we state the celebrated result by Diening [13].

LEMMA 3.2. *The maximal operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}(G)$, that is, there exists a constant $C > 0$ such that*

$$\|\mathcal{M}f\|_{L^{p(\cdot)}(G)} \le C\|f\|_{L^{p(\cdot)}(G)}.$$

THEOREM 3.3. *If (ω3.1) and (ω3.2) hold for all $x_0 \in G$ with the same constant Q, then the maximal operator $\mathcal{M}$ is bounded from $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$ to $\mathcal{H}^{p(\cdot),\infty,\eta}(G)$.*

Guliyev, Hasanov and Samko [21, 22] proved that if (ω3.2) holds for all $x_0 \in G$ with the same constant Q, then the maximal operator $\mathcal{M}$ is bounded from $\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)$ to $\underline{\mathcal{H}}^{p(\cdot),\infty,\eta}(G)$ and if (ω3.1) holds for $x_0 \in G$, then the maximal operator $\mathcal{M}$ is bounded from $\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$ to $\overline{\mathcal{H}}^{p(\cdot),\infty,\eta}_{\{x_0\}}(G)$.

Proof of Theorem 3.3. Let f be a nonnegative measurable function on G such that $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$. For $x \in G$ and $0 < r < d_G$, it suffices to show that

$$\|\mathcal{M}f\|_{L^{p(\cdot)}(A(x,r))} \le C\eta(x,r)^{-1}.$$

For this purpose, set

$$f = f\chi_{G\setminus B(x,2r)} + f\chi_{B(x,2r)\setminus B(x,r/4)} + f\chi_{B(x,r/4)} = f_1 + f_2 + f_3,$$

where χ_E denotes the characteristic function of E. We note from Lemma 3.2 that

$$\begin{aligned}
\|\mathcal{M} f_2\|_{L^{p(\cdot)}(A(x,r))} &\le C\|f_2\|_{L^{p(\cdot)}(G)} \\
&\le C\|f_2\|_{L^{p(\cdot)}(B(x,2r)\setminus B(x,r/4))} \\
&\le C\{\|f_2\|_{L^{p(\cdot)}(B(x,2r)\setminus B(x,r))} + \|f_2\|_{L^{p(\cdot)}(B(x,r)\setminus B(x,r/2))} \\
&\quad + \|f_2\|_{L^{p(\cdot)}(B(x,r/2)\setminus B(x,r/4))}\} \\
&\le C\omega(x,r)^{-1} \\
&\le C\eta(x,r)^{-1}.
\end{aligned}$$

For $z \in A(x,r)$, Lemma 3.1 gives

$$\mathcal{M} f_3(z) \le Cr^{-n} \int_{B(x,r/4)} f(y)dy \le Cr^{-n/p(x)}\eta(x,r)^{-1},$$

so that

$$\|\mathcal{M} f_3\|_{L^{p(\cdot)}(A(x,r))} \le Cr^{-n/p(x)}\eta(x,r)^{-1}\|1\|_{L^{p(\cdot)}(A(x,r))} \le C\eta(x,r)^{-1}.$$

Moreover, Lemma 3.1 again gives

$$\mathcal{M} f_1(z) \le C \int_{G\setminus B(x,2r)} f(y)|x-y|^{-n}dy \le Cr^{-n/p(x)}\eta(x,r)^{-1}$$

and hence

$$\|\mathcal{M} f_1\|_{L^{p(\cdot)}(A(x,r))} \le Cr^{-n/p(x)}\eta(x,r)^{-1}\|1\|_{L^{p(\cdot)}(A(x,r))} \le C\eta(x,r)^{-1},$$

as required. □

Remark 3.4. If the conditions on ω hold at $x_0 \in G$ only, then one can see that $\mathcal{M}$ is bounded from $\mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$ to $\mathcal{H}^{p(\cdot),\infty,\eta}_{\{x_0\}}(G)$.

Corollary 3.5. *For bounded functions $\nu(\cdot) : G \to (-\infty,\infty)$ and $\beta(\cdot) : G \to (-\infty,\infty)$, set $\omega(x,r) = r^{\nu(x)}(\log(2d_G/r))^{\beta(x)}$. If $-n/p^+ < \nu^- \le \nu^+ < n(1-1/p^-)$, then the maximal operator $\mathcal{M}$ is bounded in $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$.*

Define

$$\omega_*(x,r) = \left(\int_0^r \omega(x,t)^{-1}\,\frac{dt}{t}\right)^{-1}$$

and

$$\omega^*(x,r) = \left(\int_r^{2d_G} \omega(x,t)^{-1}\,\frac{dt}{t}\right)^{-1}$$

for $x \in G$ and $0 < r \le d_G$.

Theorem 3.6. (1) *If $\omega_*(\cdot, d_G)$ is bounded in G, then $\mathcal{H}^{p(\cdot),\infty,\omega}(G) \subset \underline{\mathcal{H}}^{p(\cdot)\infty,\omega_*}(G)$.*

(2) *For each $x_0 \in G$, $\mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G) \subset \overline{\mathcal{H}}^{p(\cdot)\infty,\omega^*}_{\{x_0\}}(G)$.*

Proof. Let f be a measurable function on G such that $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$. We show only (1), because (2) can be proved similarly.

For (1) , we see that

$$\|f\|_{L^{p(\cdot)}(B(x,r))} \le \sum_{j=1}^{\infty} \|f\|_{L^{p(\cdot)}(A(x,2^{-j+1}r))} \le \sum_{j=1}^{\infty} \omega(x, 2^{-j}r)^{-1} \le C\omega_*(x,r)^{-1}$$

for all $x \in G$ and $0 < r \le d_G$, as required. $\square$

REMARK 3.7. Let $\omega(x,r) = (\log(2d_G/r))^{\beta(x)+1}$ for a bounded function $\beta(\cdot) : G \to (-\infty, \infty)$.

(1) If $\operatorname{ess\,inf}_{x\in G} \beta(x) > 0$, then

$$\omega_*(x,r) \sim \left(\log \frac{2d_G}{r}\right)^{\beta(x)}$$

for all $x \in G$ and $0 < r < d_G$; and

(2) if $\beta(x_0) < 0$ for $x_0 \in G$, then

$$\omega^*(x_0,r) \sim \left(\log \frac{2d_G}{r}\right)^{\beta(x_0)}$$

for all $0 < r < d_G$.

REMARK 3.8. Let $\omega(x,r) = r^{\nu(x)}$ for a bounded function $\nu(\cdot) : G \to (-\infty, \infty)$.

(1) If $\operatorname{ess\,sup}_{x\in G} \nu(x) < 0$, then

$$\omega_*(x,r) \sim \omega(x,r)$$

for all $x \in G$ and $0 < r < d_G$; and

(2) if $\nu(x_0) > 0$ for $x_0 \in G$, then

$$\omega^*(x_0,r) \sim \omega(x_0,r).$$

for all $0 < r < d_G$.

COROLLARY 3.9. (1) *Suppose* (ω3.1) *and* (ω3.2) *hold for all* $x_0 \in G$ *with the same constant* Q. *If* $\omega_*(\cdot, d_G)$ *is bounded in* G, *then the maximal operator* $\mathcal{M}$ *is bounded from* $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$ *to* $\underline{\mathcal{H}}^{p(\cdot),\infty,\omega_*}(G)$.

(2) *If* (ω3.1) *and* (ω3.2) *hold for* $x_0 \in G$, *then the maximal operator* $\mathcal{M}$ *is bounded from* $\mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$ *to* $\overline{\mathcal{H}}^{p(\cdot),\infty,\omega^*}_{\{x_0\}}(G)$.

REMARK 3.10. Let us consider a singular integral operator T associated with a standard kernel $k(x,y)$ in [15, Section 6.3] such that

$$|k(x,y)| \le K_1|x-y|^{-n}$$

for all $x, y \in \mathbf{R}^n$ and

$$\|Tf\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le K_2\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)}$$

for all $f \in L^{p(\cdot)}(\mathbf{R}^n)$.

If (ω3.1) and (ω3.2) hold for all $x_0 \in G$ with the same constant Q, then every singular integral operator T is bounded from $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$ to $\mathcal{H}^{p(\cdot),\infty,\eta}(G)$.

4 Sobolev's inequality for $q = \infty$

We consider the following condition: let $\eta \in \Omega(G)$ and $x_0 \in G$.

(ω4.1) For $0 < \alpha < n$, there exists a constant $Q > 0$ such that

$$\int_r^{2d_G} t^{\alpha - n/p(x)} \omega(x,t)^{-1} \frac{dt}{t} \le Q r^{\alpha - n/p(x)} \eta(x,r)^{-1}$$

for all $0 < r < d_G$.

As in the proof of Lemma 3.1, we have the following result.

LEMMA 4.1. *If (ω4.1) holds for all $x_0 \in G$ with the same constant Q, then there is a constant $C > 0$ such that*

$$\int_{G \setminus B(x,r)} |x-y|^{\alpha - n} |f(y)| dy \le C r^{\alpha - n/p(x)} \eta(x,r)^{-1}$$

for all $x \in G$, $0 < r < d_G$ and f with $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$.

For $0 < \alpha < n$, the Riesz potential $I_\alpha f$ is defined by

$$I_\alpha f(x) = I_\alpha * f(x) = \int_G |x-y|^{\alpha - n} f(y) dy$$

for measurable functions f on G; and define

$$\frac{1}{p^\sharp(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}.$$

Let us begin with Sobolev's inequality proved by Diening [14, Theorem 5.2]:

LEMMA 4.2. *If $0 < \alpha < n/p^+$, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L^{p^\sharp(\cdot)}(G)} \le C \|f\|_{L^{p(\cdot)}(G)}$$

for all $f \in L^{p(\cdot)}(G)$.

Our result is stated in the following:

THEOREM 4.3. *Let $0 < \alpha < n/p^+$. If (ω3.1) and (ω4.1) hold for all $x_0 \in G$ with the same constant Q, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\mathcal{H}^{p^\sharp(\cdot),\infty,\eta}(G)} \le C \|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \mathcal{H}^{p(\cdot),\infty,\omega}(G)$.

In view of Guliyev, Hasanov and Samko [21, 22], if (ω4.1) holds for all $x_0 \in G$ with the same constant Q, then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{\underline{\mathcal{H}}^{p^\sharp(\cdot),\infty,\eta}(G)} \le C \|f\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)$ and if (ω3.1) holds for $x_0 \in G$, then there exists a constant $C > 0$ (which may depend on x_0) such that

$$\|I_\alpha f\|_{\overline{\mathcal{H}}^{p^\sharp(\cdot),\infty,\eta}_{\{x_0\}}(G)} \le C \|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all $f \in \overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$.

Proof of Theorem 4.3. Let f be a nonnegative measurable function on G such that $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$. For $x \in G$ and $0 < r < d_G$, we have only to show the inequality

$$\|I_\alpha f\|_{L^{p^\sharp(\cdot)}(A(x,r))} \le C\eta(x,r)^{-1}.$$

Set

$$f = f\chi_{G\setminus B(x,2r)} + f\chi_{B(x,2r)\setminus B(x,r/4)} + f\chi_{B(x,r/4)} = f_1 + f_2 + f_3,$$

as before. We note from Lemma 4.2 that

$$\begin{aligned} \|I_\alpha f_2\|_{L^{p^\sharp(\cdot)}(A(x,r))} &\le C\|f_2\|_{L^{p(\cdot)}(G)} \\ &\le C\|f_2\|_{L^{p(\cdot)}(B(x,2r)\setminus B(x,r/4))} \\ &\le C\omega(x,r)^{-1} \\ &\le C\eta(x,r)^{-1}. \end{aligned}$$

If $z \in A(x,r)$, then Lemma 3.1 gives

$$I_\alpha f_3(z) \le Cr^{\alpha-n}\int_{B(x,r/4)} f(y)dy \le Cr^{\alpha-n/p(x)}\eta(x,r)^{-1},$$

so that

$$\|I_\alpha f_3\|_{L^{p^\sharp(\cdot)}(A(x,r))} \le Cr^{\alpha-n/p(x)}\eta(x,r)^{-1}\|1\|_{L^{p^\sharp(\cdot)}(A(x,r))} \le C\eta(x,r)^{-1}.$$

Moreover, Lemma 4.1 gives

$$I_\alpha f_1(z) \le \int_{G\setminus B(x,2r)} |x-y|^{\alpha-n} f(y)dy \le Cr^{\alpha-n/p(x)}\eta(x,r)^{-1},$$

so that

$$\|I_\alpha f_1\|_{L^{p^\sharp(\cdot)}(A(x,r))} \le Cr^{\alpha-n/p(x)}\eta(x,r)^{-1}\|1\|_{L^{p^\sharp(\cdot)}(A(x,r))} \le C\eta(x,r)^{-1},$$

as required. □

COROLLARY 4.4. *Let $0 < \alpha < n/p^+$ and let ν, β and ω be as in Corollary 3.5. If $\alpha - n/p^+ < \nu^- \le \nu^+ < n(1-1/p^-)$, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\mathcal{H}^{p^\sharp(\cdot),\infty,\omega}(G)} \le C\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \mathcal{H}^{p(\cdot),\infty,\omega}(G)$.

COROLLARY 4.5. *Assume that $0 < \alpha < n/p^+$.*

(1) *Suppose (ω3.1) and (ω4.1) hold for all $x_0 \in G$ with the same constant Q. If $\omega_*(\cdot, d_G)$ is bounded in G, then the operator I_α is bounded from $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$ to $\overline{\mathcal{H}}^{p^\sharp(\cdot),\infty,\omega_*}(G)$.*

(2) *If (ω3.1) and (ω4.1) hold for $x_0 \in G$, then the operator I_α is bounded from $\mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$ to $\overline{\mathcal{H}}^{p^\sharp(\cdot),\infty,\omega^*}_{\{x_0\}}(G)$.*

5 Exponential integrability for $q = \infty$

Set

$$E_1(x,t) = \exp\left(t^{q(x)}\right) - 1,$$

where $1/p(x) + 1/q(x) = 1$. For a locally integrable function f on G, set

$$\|f\|_{L^{E_1}(G)} = \inf\left\{\lambda > 0 : \int_G E_1\left(x, \frac{|f(y)|}{\lambda}\right) dy \le 1\right\}.$$

We denote by $L^{E_1}(G)$ the class of locally integrable functions f on G satisfying $\|f\|_{L^{E_1}(G)} < \infty$.

In connection with $\mathcal{H}^{p(\cdot),q,\omega}(G)$, let us consider $\mathcal{H}^{E_1,q,\omega}(G)$ of all functions f satisfying

$$\|f\|_{\mathcal{H}^{E_1,q,\omega}(G)} \quad = \quad \sup_{x_0 \in G}\left(\int_0^{2d_G}\left(\omega(x_0,r)\|f\|_{L^{E_1}(A(x_0,r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty.$$

Similarly, we define $\underline{\mathcal{H}}^{E_1,q,\omega}(G)$ and $\overline{\mathcal{H}}^{E_1,q,\omega}_{\{x_0\}}(G)$.

Lemma 5.1.

$$\|1\|_{L^{E_1}(B(x,r))} \sim (\log(1+1/r))^{-1/q(x)}$$

for all $x \in G$ *and* $0 < r < d_G$.

Lemma 5.2 ([28, Theorem 4.1, Corollary 4.2]). *If* $\alpha \ge n/p^-$, *then there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{L^{E_1}(G)} \le C\|f\|_{L^{p(\cdot)}(G)}$$

for all $f \in L^{p(\cdot)}(G)$.

Our result is stated in the following:

Theorem 5.3. *Let* $\alpha \ge n/p^-$.

(1) *If* (ω3.1) *and* (ω4.1) *hold for all* $x_0 \in G$ *with the same constant* Q, *then there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{\mathcal{H}^{E_1,\infty,\eta}(G)} \le C\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \mathcal{H}^{p(\cdot),\infty,\omega}(G)$.

(2) *If* (ω4.1) *holds for all* $x_0 \in G$ *with the same constant* Q, *then there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{\underline{\mathcal{H}}^{E_1,\infty,\eta}(G)} \le C\|f\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)$.

(3) *If* (ω3.1) *holds for* $x_0 \in G$, *then there exists a constant* $C > 0$ (*which may depend on* x_0) *such that*

$$\|I_\alpha f\|_{\overline{\mathcal{H}}^{E_1,\infty,\eta}_{\{x_0\}}(G)} \le C\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all $f \in \overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$.

Proof. We give only a proof of assertion (1). Let f be a nonnegative measurable function on G such that $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$. We have only to show the inequality

$$\|I_\alpha f\|_{L^{E_1}(A(x,r))} \le C\eta(x,r)^{-1}$$

for all $x \in G$ and $0 < r < d_G$. Set

$$f = f\chi_{G\setminus B(x,2r)} + f\chi_{B(x,2r)\setminus B(x,r/4)} + f\chi_{B(x,r/4)} = f_1 + f_2 + f_3,$$

as before. We note from Lemma 5.2 that

$$\|I_\alpha f_2\|_{L^{E_1}(A(x,r))} \le C\|f_2\|_{L^{p(\cdot)}(B(x,2r)\setminus B(x,r/4))} \le C\eta(x,r)^{-1}.$$

If $z \in A(x,r)$, then Lemma 3.1 gives

$$I_\alpha f_3(z) \le Cr^{\alpha-n}\int_{B(x,r/4)} f(y)dy \le C\eta(x,r)^{-1}$$

since $\alpha \ge n/p^-$, so that

$$\|I_\alpha f_3\|_{L^{E_1}(A(x,r))} \le C\eta(x,r)^{-1}\|1\|_{L^{E_1}(A(x,r))} \le C\eta(x,r)^{-1}$$

by Lemma 5.1. Moreover, Lemma 4.1 gives

$$I_\alpha f_1(z) \le C\int_{G\setminus B(x,2r)} |x-y|^{\alpha-n} f(y)dy \le C\eta(x,r)^{-1}$$

since $\alpha \ge n/p^-$, so that

$$\|I_\alpha f_1\|_{L^{E_1}(A(x,r))} \le C\eta(x,r)^{-1}\|1\|_{L^{E_1}(A(x,r))} \le C\eta(x,r)^{-1},$$

as required. □

COROLLARY 5.4. *Let* $\alpha \ge n/p^-$ *and let* ν, β *and* ω *be as in Corollary 3.5.*

(1) *When* $\alpha - n/p^+ < \nu^- \le \nu^+ < n(1-1/p^-)$, *there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{\mathcal{H}^{E_1,\infty,\omega}(G)} \le C\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \mathcal{H}^{p(\cdot),\infty,\omega}(G)$.

(2) *When* $\alpha - n/p^+ < \nu^-$, *there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{\underline{\mathcal{H}}^{E_1,\infty,\omega}(G)} \le C\|f\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)$.

(3) *When* $\nu(x_0) < n(1-1/p(x_0))$ *for* $x_0 \in G$, *there exists a constant* $C > 0$ (*which may depend on* x_0) *such that*

$$\|I_\alpha f\|_{\overline{\mathcal{H}}^{E_1,\infty,\omega}_{\{x_0\}}(G)} \le C\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all $f \in \overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$.

6 Associate spaces of $\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$

Recall that for $x_0 \in G$ and measurable functions f on G,

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} = \sup_{0<t<d_G} \omega(x_0,t)\|f\|_{L^{p(\cdot)}(G\setminus B(x_0,t))}$$

and

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)} = \int_0^{2d_G} \omega(x_0,t)\|f\|_{L^{p(\cdot)}(B(x_0,t))}\frac{dt}{t}.$$

Remark 6.1. Let $x_0 \in G$. Note here that if $\omega(x_0,0)=\infty$, then $\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} < \infty$ if and only if $f=0$ a.e.. Hence we may assume that $\omega(x_0,0)=0$ and then $\omega(x_0,\cdot)$ is uniformly almost increasing on $(0,\infty)$ when $\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} < \infty$.

By the above remark, in this section, suppose

$$\omega(x,0)=0 \qquad \text{for all } x \in G.$$

For $x \in G$ and $0<t<d_G$, we set

$$p^+(B(x,t)) = \sup_{y\in B(x,t)} p(y),$$

as before. We define $1/q(x) = 1-1/p(x)$.

Following Di Fratta and Fiorenza [17], we have the following Hölder type inequality for log-type weights.

Theorem 6.2. *For $x_0 \in G$, suppose*

(ω6.1) *there exist constants $b, Q>0$ such that*

$$\int_0^t \left(\log\frac{2d_G}{r}\right)^{-bp(x_0)-1} \omega(x_0,r)^{-p^+(B(x_0,t))}\frac{dr}{r} \le Q\left(\log\frac{2d_G}{t}\right)^{-bp(x_0)} \omega(x_0,t)^{-p(x_0)}$$

for all $0<t<d_G$.

Then there exists a constant $C>0$ such that

$$\int_G |f(x)g(x)|dx \le C\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)}\|g\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all measurable functions f and g on G, where

$$\eta(x_0,r) = \left(\log\frac{2d_G}{r}\right)^{-1}\omega(x_0,r)^{-1}.$$

Proof. Let $x_0 \in G$. Let f and g be nonnegative measurable functions on G such that $\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)} \le 1$ and $\|g\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} \le 1$. We have by Fubini's theorem and Hölder's inequality

$$\begin{aligned}
&\int_G f(x)g(x)dx \\
= \ & \int_G f(x)g(x)\left(b\left(\log\frac{2d_G}{|x-x_0|}\right)^{-b}\int_{|x-x_0|}^{2d_G}\left(\log\frac{2d_G}{t}\right)^{b-1}\frac{dt}{t}\right)dx \\
= \ & b\int_0^{2d_G}\left(\int_{B(x_0,t)} f(x)g(x)\left(\log\frac{2d_G}{|x-x_0|}\right)^{-b}dx\right)\left(\log\frac{2d_G}{t}\right)^{b-1}\frac{dt}{t} \\
\le \ & C\int_0^{2d_G}\|f\|_{L^{q(\cdot)}(B(x_0,t))}\left\|g\left(\log\frac{2d_G}{|\cdot-x_0|}\right)^{-b}\right\|_{L^{p(\cdot)}(B(x_0,t))}\left(\log\frac{2d_G}{t}\right)^{b-1}\frac{dt}{t}.
\end{aligned}$$

Here it suffices to show

$$\left\|g\left(\log\frac{2d_G}{|\cdot-x_0|}\right)^{-b}\right\|_{L^{p(\cdot)}(B(x_0,t))} \le C\left(\log\frac{2d_G}{t}\right)^{-b}\omega(x_0,t)^{-1} = C\left(\log\frac{2d_G}{t}\right)^{-b+1}\eta(x_0,t)$$

for $0<t<d_G$. In fact, we obtain

$$\begin{aligned}
&\int_{B(x_0,t)}\left(\frac{g(x)}{(\log(2d_G/t))^{-b}\,\omega(x_0,t)^{-1}}\right)^{p(x)}\left(\log\frac{2d_G}{|x-x_0|}\right)^{-bp(x)}dx \\
\le \ & C\int_{B(x_0,t)}\left(\frac{g(x)}{(\log(2d_G/t))^{-b}\,\omega(x_0,t)^{-1}}\right)^{p(x)}\left(\log\frac{2d_G}{|x-x_0|}\right)^{-bp(x_0)}dx \\
\le \ & C\int_{B(x_0,t)}\left(\frac{g(x)}{(\log(2d_G/t))^{-b}\,\omega(x_0,t)^{-1}}\right)^{p(x)}\left(\int_0^{|x-x_0|}\left(\log\frac{2d_G}{r}\right)^{-bp(x_0)-1}\frac{dr}{r}\right)dx \\
\le \ & C\int_0^t\left(\int_{B(x_0,t)\setminus B(x_0,r)} g(x)^{p(x)}\left(\log\frac{2d_G}{t}\right)^{bp(x)}\omega(x_0,t)^{p(x)}\left(\log\frac{2d_G}{r}\right)^{-bp(x_0)-1}dx\right)\frac{dr}{r} \\
\le \ & C\left(\log\frac{2d_G}{t}\right)^{bp(x_0)}\omega(x_0,t)^{p(x_0)}\int_0^t\left(\log\frac{2d_G}{r}\right)^{-bp(x_0)-1} \\
&\times\left(\int_{B(x_0,t)\setminus B(x_0,r)}\left(\frac{g(x)}{\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,r))}}\right)^{p(x)}\|g\|^{p(x)}_{L^{p(\cdot)}(G\setminus B(x_0,r))}dx\right)\frac{dr}{r} \\
\le \ & C\left(\log\frac{2d_G}{t}\right)^{bp(x_0)}\omega(x_0,t)^{p(x_0)}\int_0^t\left(\log\frac{2d_G}{r}\right)^{-bp(x_0)-1}\omega(x_0,r)^{-p^+(B(x_0,t))}\frac{dr}{r} \\
\le \ & C
\end{aligned}$$

by (P2), Lemma 2.1 and (ω6.1). □

Power weights can be treated simpler than Theorem 6.2 in the following manner.

THEOREM 6.3. *For $x_0 \in G$, suppose*

(ω6.2) *there exist constants $b, Q > 0$ such that*

$$\int_0^t r^b \omega(x_0, r)^{-1} \frac{dr}{r} \le Q t^b \omega(x_0, t)^{-1}$$

for all $0 < t < d_G$.

Then there exists a constant $C > 0$ such that

$$\int_G |f(x)g(x)|dx \le C\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)} \|g\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all measurable functions f and g on G, where $\eta(x_0, r) = \omega(x_0, r)^{-1}$.

Proof. Let $x_0 \in G$. Let f and g be nonnegative measurable functions on G such that $\|f\|_{\overline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)} \le 1$ and $\|g\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} \le 1$. For $b > 0$, we have by Fubini's theorem and Hölder's inequality

$$\begin{aligned}
\int_G f(x)g(x)dx &\le C\int_0^{2d_G} \left(\int_{B(x_0,t)} f(x)g(x)|x-x_0|^b dx \right) t^{-b}\frac{dt}{t} \\
&\le C\int_0^{2d_G} \|f\|_{L^{q(\cdot)}(B(x_0,t))} \left\| g|\cdot - x_0|^b \right\|_{L^{p(\cdot)}(B(x_0,t))} t^{-b}\frac{dt}{t}.
\end{aligned}$$

First, we show that

$$\left\| g|\cdot - x_0|^b \right\|_{L^{p(\cdot)}(B(x_0,2s)\setminus B(x_0,s))} \le Cs^b\omega(x_0,s)^{-1} \le Cs^b\eta(x_0,s)$$

for all $0 < s < d_G$. In fact, we obtain

$$\begin{aligned}
&\int_{B(x_0,2s)\setminus B(x_0,s)} \left(\frac{g(x)}{s^b\omega(x_0,s)^{-1}} \right)^{p(x)} |x-x_0|^{bp(x)} dx \\
\le\ & C\int_{B(x_0,2s)\setminus B(x_0,s)} \left(\frac{g(x)}{\|g\|_{L^{p(\cdot)}(B(x_0,2s)\setminus B(x_0,s))}} \right)^{p(x)} \left(\omega(x_0,s)\|g\|_{L^{p(\cdot)}(B(x_0,2s)\setminus B(x_0,s))} \right)^{p(x)} dx \\
\le\ & C
\end{aligned}$$

by (P2) and Lemma 2.1, which gives

$$\begin{aligned}
\left\| g|\cdot - x_0|^b \right\|_{L^{p(\cdot)}(B(x_0,t))} &\le \sum_{j=1}^{\infty} \left\| g|\cdot - x_0|^b \right\|_{L^{p(\cdot)}(B(x_0,2^{-j+1}t)\setminus B(x_0,2^{-j}t))} \\
&\le C\int_0^t r^b\omega(x_0,r)^{-1}\frac{dr}{r} \\
&\le Ct^b\omega(x_0,t)^{-1}
\end{aligned}$$

by (ω6.2). Thus we obtain the required result. □

THEOREM 6.4. *Let $\eta(\cdot,\cdot) \in \Omega(G)$. For $x_0 \in G$, suppose*

(ω6.3) *there exists a constant $Q > 0$ such that*

$$\int_t^{2d_G} \eta(x_0, r)\frac{dr}{r} \le Q\omega(x_0, t)^{-1}$$

for all $0 < t < d_G$.

Then there exists a constant $C > 0$ such that

$$\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)} \le C \sup_g \int_G |f(x)g(x)|dx$$

for all measurable functions f on G, where the supremum is taken over all measurable functions g on G such that $\|g\|_X \le 1$ with $X = \overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$.

Proof. Let $x_0 \in G$. Let f be a nonnegative measurable function on G. To show the claim, we may assume that

$$\sup_g \int_G |f(x)g(x)|dx \le 1,$$

where the supremum is taken over all measurable functions g on G such that $\|g\|_X \le 1$. Take a compact set $K \subset G \setminus \{x_0\}$. Since $L^{p(\cdot)}(K) = \{g\chi_K : g \in L^{p(\cdot)}(G)\} \subset X$, $f\chi_K \in L^{q(\cdot)}(G)$, in view of [25] or [16, Theorem 3.2.13]. By (ω6.3), we find

$$\|f\chi_K\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)} < \infty$$

and, moreover, we have by Lemma 2.2

$$\sum_{j\in N_0} \eta(x_0, 2^{-j+1}d_G)F_j \sim \|f\chi_K\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)},$$

where $F_j = \|f_j\|_{L^{q(\cdot)}(G)}$, $f_j = f\chi_{K\cap B(x_0,2^{-j+1}d_G)}$ and N_0 is the set of positive integers j such that $F_j > 0$. Set

$$g(x) = \sum_{j\in N_0} \eta(x_0, 2^{-j+1}d_G)|f_j(x)/F_j|^{q(x)-2}f_j(x)/F_j.$$

Then we see that

$$\begin{aligned}
\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,r))} &\le \sum_{j\in N_0, 2^{-j+1}d_G>r} \eta(x_0, 2^{-j+1}d_G)\||f_j/F_j|^{q(\cdot)-2}f_j/F_j\|_{L^{p(\cdot)}(G)} \\
&\le \sum_{j\ge 1, 2^{-j+1}d_G>r} \eta(x_0, 2^{-j+1}d_G) \\
&\le C\omega(x_0, r)^{-1}
\end{aligned}$$

for all $0 < r < d_G$ by (ω6.3) and hence

$$\|g\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} \le C.$$

Consequently it follows that

$$\begin{aligned}\int_G f(x)g(x)dx &= \sum_{j\in N_0} \eta(x_0, 2^{-j+1}d_G) \int_G f(x)|f_j(x)/F_j|^{q(x)-2} f_j(x)/F_j dx \\ &= \sum_{j\in N_0} \eta(x_0, 2^{-j+1}d_G) F_j \\ &\geq C\|f\chi_K\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)}.\end{aligned}$$

Hence, by the monotone convergence theorem, we have

$$\sup_g \int_G f(x)g(x)dx \geq C\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)},$$

which gives the required inequality. □

Let X be a family of measurable functions on G with a norm $\|\cdot\|_X$. Then the associate space X' of X is defined as the family of all measurable functions f on G such that

$$\|f\|_{X'} = \sup_{g\in X:\|g\|_X\leq 1} \int_G |f(x)g(x)|dx < \infty.$$

Theorems 6.2, 6.3 and 6.4 give the following result.

Corollary 6.5. *For $x_0 \in G$, suppose $(\omega 6.1)$ and $(\omega 6.3)$ hold. Then*

$$\left(\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)\right)' = \underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G),$$

where $\eta(x_0, r) = \left(\log \frac{2d_G}{r}\right)^{-1} \omega(x_0, r)^{-1}$. If $(\omega 6.2)$ and $(\omega 6.3)$ hold, then the same conclusion is fulfilled with $\eta(x_0, r) = \omega(x_0, r)^{-1}$.

For $0 < q \leq \infty$, set

$$\widetilde{\mathcal{H}}^{p(\cdot),q,\omega}(G) = \sum_{x_0\in G} \overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G),$$

whose quasi-norm is defined by

$$\|f\|_{\widetilde{\mathcal{H}}^{p(\cdot),q,\omega}(G)} = \inf_{|f|=\sum_j |f_j|,\{x_j\}\subset G} \sum_j \|f_j\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_j\}}(G)}.$$

The Hölder type inequality in Theorem 6.2 or 6.3, under the same assumptions, implies

$$\begin{aligned}\int_G |f(x)g(x)|dx &= \sum_j \int_G |f(x)g_j(x)|dx \\ &\leq C\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}(G)} \sum_j \|g_j\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_j\}}(G)},\end{aligned}$$

so that

$$\int_G |f(x)g(x)|dx \leq C\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}(G)}\|g\|_{\widetilde{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)}.$$

Theorem 6.4 gives the converse inequality.

Theorems 6.2, 6.3 and 6.4 give the following result.

Corollary 6.6. *If (ω6.1) and (ω6.3) hold for all $x_0 \in G$ with the same constant Q, then*

$$\left(\widetilde{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)\right)' = \underline{\mathcal{H}}^{q(\cdot),1,\eta}(G),$$

where $\eta(x_0,r) = \left(\log \frac{2d_G}{r}\right)^{-1} \omega(x_0,r)^{-1}$. If ($\omega$6.2) and ($\omega$6.3) hold for all $x_0 \in G$ with the same constant Q, then the same conclusion is fulfilled with $\eta(x_0,r) = \omega(x_0,r)^{-1}$.

Remark 6.7. For $0 < q \le \infty$, set

$$\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0 \in G} \overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$$

and define the norm

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G)} = \sup_{x_0 \in G} \|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)},$$

as usual. Then note that

$$\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G) = \begin{cases} L^{p(\cdot)}(G) & \omega(x,0) = 0 \text{ for all } x \in G; \\ \{0\} & \omega(x,0) = \infty \text{ for all } x \in G. \end{cases}$$

For related results, we refer the reader to the paper by Di Fratta and Fiorenza [17] with logarithmic weights, and the paper by Gagatishvili and Mustafayev [19] with general weights.

Remark 6.8. If $\omega(t) = (\log(2d_G/t))^{-a}$ with $a > 0$, then (ω6.1) and (ω6.3) hold for $\eta(t) = (\log(2d_G/t))^{a-1}$; and if $\omega(t) = r^a$ with $a > 0$, then (ω6.2) and (ω6.3) hold for $\eta(t) = t^{-a}$.

7 Associate spaces of $\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$

Recall that for $x_0 \in G$ and measurable functions f on G,

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} = \sup_{0<t<d_G} \omega(x_0,t)\|f\|_{L^{p(\cdot)}(B(x_0,t))}$$

and

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)} = \int_0^{2d_G} \omega(x_0,t)\|f\|_{L^{p(\cdot)}(G\setminus B(x_0,t))}\frac{dt}{t}.$$

We have the Hölder type inequality for log type weights ω.

Theorem 7.1. *For $x_0 \in G$, suppose*

(ω7.1) *there exist constants $r_0, b, Q > 0$ such that*

$$\int_t^{2r_0} \left(\left(\log\frac{2d_G}{t}\right)^b \omega(x_0,t)^{-1}\right)^{c_p/\log(2d_G/r)} \left(\left(\log\frac{2d_G}{r}\right)^b \omega(x_0,r)^{-1}\right)^{p(x_0)}$$
$$\times \left(\log\frac{2d_G}{r}\right)^{-1}\frac{dr}{r} \le Q\left(\left(\log\frac{2d_G}{t}\right)^b \omega(x_0,t)^{-1}\right)^{p(x_0)}$$

for all $0 < t < r_0$.

Then there exists a constant $C > 0$ such that

$$\int_G |f(x)g(x)|dx \le C\|f\|_{\overline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)}\|g\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all measurable functions f, g on G, where $\eta(x_0,r) = \left(\log \frac{2d_G}{r}\right)^{-1}\omega(x_0,r)^{-1}$.

Proof. Let $x_0 \in G$. Let f and g be nonnegative measurable functions on G such that $\|f\|_{\overline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)} \le 1$ and $\|g\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} \le 1$. For $b > 0$ we have by Fubini's theorem and Hölder's inequality

$$\begin{aligned}\int_G f(x)g(x)dx &\le C\int_0^{d_G} \|f\|_{L^{q(\cdot)}(G\setminus B(x_0,t))}\left\|g\left(\log\frac{2d_G}{|\cdot - x_0|}\right)^b\right\|_{L^{p(\cdot)}(G\setminus B(x_0,t))} \\ &\qquad \times\left(\log\frac{2d_G}{t}\right)^{-b-1}\frac{dt}{t},\end{aligned}$$

as in the proof of Theorem 6.2. It suffices to show

$$\left\|g\left(\log\frac{2d_G}{|\cdot - x_0|}\right)^b\right\|_{L^{p(\cdot)}(G\setminus B(x_0,t))} \le C\left(\log\frac{2d_G}{t}\right)^b\omega(x_0,t)^{-1} = C\left(\log\frac{2d_G}{t}\right)^{b+1}\eta(x_0,t)$$

for all $0 < t < d_G$. In fact, we obtain for $0 < r_0 < d_G$

$$\begin{aligned}&\int_{B(x_0,r_0)\setminus B(x_0,t)}\left(\frac{g(x)}{(\log(2d_G/t))^b\,\omega(x_0,t)^{-1}}\right)^{p(x)}\left(\log\frac{2d_G}{|x-x_0|}\right)^{bp(x)}dx \\ \le\ & C\int_{B(x_0,r_0)\setminus B(x_0,t)}\left(\frac{g(x)}{(\log(2d_G/t))^b\,\omega(x_0,t)^{-1}}\right)^{p(x)}\left(\log\frac{2d_G}{|x-x_0|}\right)^{bp(x_0)}dx \\ \le\ & C\int_{B(x_0,r_0)\setminus B(x_0,t)}\left(\frac{g(x)}{(\log(2d_G/t))^b\,\omega(x_0,t)^{-1}}\right)^{p(x)}\left(\int_{|x-x_0|}^{2r_0}\left(\log\frac{2d_G}{r}\right)^{bp(x_0)-1}\frac{dr}{r}\right)dx \\ \le\ & C\int_t^{2r_0}\left(\int_{B(x_0,r)\setminus B(x_0,t)} g(x)^{p(x)}\left(\left(\log\frac{2d_G}{t}\right)^{-b}\omega(x_0,t)\right)^{p(x)}\left(\log\frac{2d_G}{r}\right)^{bp(x_0)-1}dx\right)\frac{dr}{r} \\ \le\ & C\left(\log\frac{2d_G}{t}\right)^{-bp(x_0)}\omega(x_0,t)^{p(x_0)}\int_t^{2r_0}\left(\left(\log\frac{2d_G}{t}\right)^b\omega(x_0,t)^{-1}\right)^{c_p/\log(2d_G/r)} \\ &\times\left(\log\frac{2d_G}{r}\right)^{bp(x_0)-1}\left(\int_{B(x_0,r)} g(x)^{p(x)}dx\right)\frac{dr}{r} \\ \le\ & C\left(\log\frac{2d_G}{t}\right)^{-bp(x_0)}\omega(x_0,t)^{p(x_0)}\int_t^{2r_0}\left(\left(\log\frac{2d_G}{t}\right)^b\omega(x_0,t)^{-1}\right)^{c_p/\log(2d_G/r)} \\ &\times\left(\left(\log\frac{2d_G}{r}\right)^b\omega(x_0,r)^{-1}\right)^{p(x_0)}\left(\log\frac{2d_G}{r}\right)^{-1}\frac{dr}{r} \\ \le\ & C\left(\log\frac{2d_G}{t}\right)^{-bp(x_0)}\omega(x_0,t)^{p(x_0)}\left(\left(\log\frac{2d_G}{t}\right)^b\omega(x_0,t)^{-1}\right)^{p(x_0)} \\ \le\ & C\end{aligned}$$

by (P2), condition (ω7.1) and Lemmas 2.1 and 2.4, which gives

$$\left\| g\left(\log\frac{2d_G}{|\cdot - x_0|}\right)^b \right\|_{L^{p(\cdot)}(B(x_0,r_0)\setminus B(x_0,t))} \le C\left(\log\frac{2d_G}{t}\right)^b \omega(x_0,t)^{-1}$$

for all $0 < t < r_0$. Moreover,

$$\left\| g\left(\log\frac{2d_G}{|\cdot - x_0|}\right)^b \right\|_{L^{p(\cdot)}(G\setminus B(x_0,r_0))} \le C\,\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,r_0))} \le C,$$

which completes the proof. □

REMARK 7.2. We show that $\omega(t) = (\log(2d_G/t))^a$ with $a > 0$ satisfies (ω7.1). To show this, for $b, c > 0$ one can find constants $r_0, Q > 0$ such that

$$\int_t^{2r_0} \left(\log\frac{2d_G}{t}\right)^{c/\log(2d_G/r)} \left(\log\frac{2d_G}{r}\right)^{b-1}\frac{dr}{r} \le Q\left(\log\frac{2d_G}{t}\right)^b$$

for all $0 < t < r_0$ and $x_0 \in G$. In fact, first find $0 < r_0 < d_G/e$ such that $\varepsilon = 1/\log(d_G/r_0) < b/2c$, and note for $\tilde{t} = 2d_G e^{-(\log(2d_G/t))^{1/2}}$

$$\begin{aligned}\int_t^{\tilde{t}} \left(\log\frac{2d_G}{t}\right)^{c/\log(2d_G/r)} \left(\log\frac{2d_G}{r}\right)^{b-1}\frac{dr}{r} &\le C\int_t^{2d_G}\left(\log\frac{2d_G}{r}\right)^{b-1}\frac{dr}{r}\\ &\le Q\left(\log\frac{2d_G}{t}\right)^b\end{aligned}$$

since $(\log(2d_G/t))^{c/\log(2d_G/r)} \le C$ for all $t < r < \tilde{t}$ and

$$\begin{aligned}\int_{\tilde{t}}^{2r_0} \left(\log\frac{2d_G}{t}\right)^{c/\log(2d_G/r)} \left(\log\frac{2d_G}{r}\right)^{b-1}\frac{dr}{r} &\le C\left(\log\frac{2d_G}{t}\right)^{c\varepsilon}\int_{\tilde{t}}^{2r_0}\left(\log\frac{2d_G}{r}\right)^{b-1}\frac{dr}{r}\\ &\le Q\left(\log\frac{2d_G}{t}\right)^{c\varepsilon+b/2} \le Q\left(\log\frac{2d_G}{t}\right)^b,\end{aligned}$$

as required.

For power weights ω, we obtain the following result.

THEOREM 7.3. *For $x_0 \in G$, suppose*

(ω7.2) *there exist constants $b, Q > 0$ such that*

$$\int_t^{2d_G} r^{-b}\omega(x_0,r)^{-1}\frac{dr}{r} \le Qr^{-b}\omega(x_0,t)^{-1}$$

for all $0 < t < d_G$.

Then there exists a constant $C > 0$ such that

$$\int_G |f(x)g(x)|dx \le C\|f\|_{\overline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)}\|g\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all measurable functions f, g on G, where $\eta(x_0,r) = \omega(x_0,r)^{-1}$.

As in the proof of Theorem 6.4, we have the following result.

THEOREM 7.4. *Let $\eta(\cdot,\cdot) \in \Omega(G)$. For $x_0 \in G$, suppose*

(ω7.3) *there exists a constant $Q > 0$ such that*

$$\int_0^t \eta(x_0, r)\frac{dr}{r} \le Q\omega(x_0, t)^{-1}$$

for all $0 < t < d_G$.

Then there exists a constant $C > 0$ such that

$$\|f\|_{\overline{\mathcal{H}}^{\{x_0\}}_{q(\cdot),1,\eta}(G)} \le C \sup_g \int_G |f(x)g(x)|dx$$

for all measurable functions f on G, where the supremum is taken over all measurable functions g on G such that $\|g\|_X \le 1$ with $X = \underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$.

Theorems 7.1, 7.3 and 7.4 give the following result.

COROLLARY 7.5. *If (ω7.1) and (ω7.3) hold for $x_0 \in G$, then*

$$\left(\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)\right)' = \overline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G),$$

where $\eta(x_0, r) = \left(\log \frac{2d_G}{r}\right)^{-1} \omega(x_0, r)^{-1}$. If ($\omega$7.2) and ($\omega$7.3) hold for $x_0 \in G$, then the same conclusion is fulfilled with $\eta(x_0, r) = \omega(x_0, r)^{-1}$.

REMARK 7.6. If $\omega(t) = (\log(2d_G/t))^a$ with $a > 0$, then (ω7.1) and (ω7.3) hold for $\eta(t) = (\log(2d_G/t))^{-a-1}$; and if $\omega(t) = t^{-a}$ with $a > 0$, then (ω7.2) and (ω7.3) hold for $\eta(t) = t^a$.

For $0 < q \le \infty$, we may consider

$$\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G) = \sum_{x_0 \in G} \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G),$$

whose quasi-norm is defined by

$$\|f\|_{\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G)} = \inf_{|f|=\sum_j |f_j|, \{x_j\} \subset G} \sum_j \|f_j\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_j\}}(G)}.$$

One can show that

$$\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G) = L^{p(\cdot)}(G).$$

For this, we only show the inclusion $L^{p(\cdot)}(G) \subset \underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G)$. Take $f \in L^{p(\cdot)}(G)$ and $x_1, x_2 \in G$ ($x_1 \ne x_2$). Write

$$f = f\chi_{B(x_2,|x_1-x_2|/2)} + f\chi_{G\setminus B(x_2,|x_1-x_2|/2)} = f_1 + f_2.$$

Then

$$\begin{aligned}\|f_1\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_1\}}(G)} &\leq \left(\int_{|x_1-x_2|/2}^{2d_G} \left(\omega(x_1,r)\|f_1\|_{L^{p(\cdot)}(B(x_1,r))}\right)^q dr/r\right)^{1/q} \\ &\leq \|f_1\|_{L^{p(\cdot)}(G)} \left(\int_{|x_1-x_2|/2}^{2d_G} \omega(x_1,r)^q dr/r\right)^{1/q} = A\|f_1\|_{L^{p(\cdot)}(G)}\end{aligned}$$

and

$$\begin{aligned}\|f_2\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_2\}}(G)} &\leq \left(\int_{|x_1-x_2|/2}^{2d_G} \left(\omega(x_2,r)\|f_2\|_{L^{p(\cdot)}(B(x_2,r))}\right)^q dr/r\right)^{1/q} \\ &\leq \|f_2\|_{L^{p(\cdot)}(G)} \left(\int_{|x_1-x_2|/2}^{2d_G} \omega(x_2,r)^q dr/r\right)^{1/q} = B\|f_2\|_{L^{p(\cdot)}(G)}.\end{aligned}$$

Hence

$$\begin{aligned}\|f\|_{\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G)} &\leq \|f_1\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_1\}}(G)} + \|f_2\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_2\}}(G)} \\ &\leq A\|f_1\|_{L^{p(\cdot)}(G)} + B\|f_2\|_{L^{p(\cdot)}(G)} \\ &\leq (A+B)\|f\|_{L^{p(\cdot)}(G)} < \infty,\end{aligned}$$

as required.

8 Associate spaces of $\underline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)$

THEOREM 8.1. *Let $\eta(\cdot,\cdot) \in \Omega(G)$, $x_0 \in G$ and $X = \underline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)$. Suppose*

(ω8.1) *there exists a constant $Q > 0$ such that*

$$\int_t^{2d_G} \omega(x_0,r)\frac{dr}{r} \leq Q\eta(x_0,t)^{-1}$$

for all $0 < t < d_G$.

Then there exists a constant $C > 0$ such that

$$\|f\|_{\overline{\mathcal{H}}^{q(\cdot),\infty,\eta}_{\{x_0\}}(G)} \leq C\|f\|_{X'}$$

for all measurable functions f on G.

Proof. Let $x_0 \in G$. First we show

$$\int_{G\setminus B(x_0,R)} f(x)g(x)dx \leq C\eta(x_0,R)^{-1}\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,R))}\|f\|_{X'} \tag{8.1}$$

for $0 < R < d_G$ and nonnegative measurable functions f, g on G. To show this, we consider

$$h = \eta(x_0,R)g\chi_{G\setminus B(x_0,R)}/\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,R))}$$

when $0 < \|g\|_{L^{p(\cdot)}(G \setminus B(x_0,R))} < \infty$. Then we have by ($\omega$8.1)

$$\int_0^{2d_G} \omega(x_0,t)\|h\|_{L^{p(\cdot)}(B(x_0,t))}\frac{dt}{t} \le \eta(x_0,R)\int_R^{2d_G} \omega(x_0,t)\frac{dt}{t} \le C,$$

and hence

$$\int_{G\setminus B(x_0,R)} f(x)h(x)dx \le C\|f\|_{X'}.$$

Now we obtain

$$\int_{G\setminus B(x_0,R)} f(x)g(x)dx \le C\eta(x_0,R)^{-1}\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,R))}\|f\|_{X'}.$$

If we take $g(x) = |f(x)/\|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))}|^{q(x)-1}\chi_{G\setminus B(x_0,R)}$ when $0 < \|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))} < \infty$, then we have by (8.1)

$$\begin{aligned} 1 &= \int_{G\setminus B(x_0,R)} \{f(x)/\|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))}\}^{q(x)}dx \\ &\le C\eta(x_0,R)^{-1}\|\{f/\|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))}\}^{q(\cdot)-1}\|_{L^{p(\cdot)}(G\setminus B(x_0,R))}\|f/\|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))}\|_{X'} \\ &\le C\eta(x_0,R)^{-1}\left\{\|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))}\right\}^{-1}\|f\|_{X'}, \end{aligned}$$

which shows

$$\eta(x_0,R)\|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))} \le C\|f\|_{X'}.$$

Thus it follows that

$$\|f\|_{\overline{\mathcal{H}}^{q(\cdot),\infty,\eta}_{\{x_0\}}(G)} \le C\|f\|_{X'},$$

as required. □

Corollary 8.2. *If (ω8.1) holds for $x_0 \in G$ and (ω6.1) holds for $x_0 \in G$, η and $q(\cdot)$, then*

$$\left(\underline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)\right)' = \overline{\mathcal{H}}^{q(\cdot),\infty,\eta}_{\{x_0\}}(G),$$

where $\eta(x_0,r) = \left(\log\frac{2d_G}{r}\right)^{-1}\omega(x_0,r)^{-1}$. If ($\omega$8.1) holds for $x_0 \in G$ and (ω6.2) holds for $x_0 \in G$, η and $q(\cdot)$, then the same conclusion is fulfilled with $\eta(x_0,r) = \omega(x_0,r)^{-1}$.

As in Fiorenza-Rakotoson [18, Corollary 1], we see that the associate and dual spaces of $\underline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)$ coincides with each other.

Remark 8.3. If $\omega(t) = (\log(2d_G/t))^{-1/a}$ with $a > 1$, then (ω8.1) holds for $\eta(t) = (\log(2d_G/t))^{-1/a'}$; and if $\omega(t) = t^{-a}$ with $a > 0$, then (ω8.1) holds for $\eta(t) = t^a$.

9 Associate space of $\overline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)$

As in the proof of Theorem 8.1, we have the following result.

Theorem 9.1. *Let $\eta(\cdot,\cdot) \in \Omega(G)$, $x_0 \in G$ and $X = \overline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)$. Suppose*

(ω9.1) *there exists a constant $Q > 0$ such that*

$$\int_0^t \omega(x_0, r)\frac{dr}{r} \le Q\eta(x_0, t)^{-1}$$

for all $0 < t < d_G$.

Then there exists a constant $C > 0$ such that

$$\|f\|_{\underline{\mathcal{H}}^{q(\cdot),\infty,\eta}_{\{x_0\}}(G)} \le C\|f\|_{X'}$$

for all measurable functions f on G.

Corollary 9.2. *If (ω9.1) holds for $x_0 \in G$ and (ω7.1) holds for $x_0 \in G$, η and $q(\cdot)$, then*

$$\left(\overline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)\right)' = \underline{\mathcal{H}}^{q(\cdot),\infty,\eta}_{\{x_0\}}(G),$$

where $\eta(x_0, r) = \left(\log \frac{2d_G}{r}\right)^{-1} \omega(x_0, r)^{-1}$. If ($\omega$9.1) holds for $x_0 \in G$ and (ω7.2) holds for $x_0 \in G$, η and $q(\cdot)$, then the same conclusion is fulfilled with $\eta(x_0, r) = \omega(x_0, r)^{-1}$.

Corollary 9.3. *If (ω9.1) holds for all $x_0 \in G$ with the same constant Q and (ω7.1) holds for η, $q(\cdot)$ and all $x_0 \in G$ with the same constant Q, then*

$$\left(\widetilde{\mathcal{H}}^{p(\cdot),1,\omega}(G)\right)' = \underline{\mathcal{H}}^{q(\cdot),\infty,\eta}(G),$$

where $\eta(x_0, r) = \left(\log \frac{2d_G}{r}\right)^{-1} \omega(x_0, r)^{-1}$. If ($\omega$9.1) holds for all $x_0 \in G$ with the same constant Q and (ω7.2) holds for η, $q(\cdot)$ and all $x_0 \in G$ with the same constant Q, then the same conclusion is fulfilled with $\eta(x_0, r) = \omega(x_0, r)^{-1}$.

This corollary gives a characterization of Morrey spaces of variable exponents; see also the paper by Gogatishvili and Mustafayev [19] for constant exponents.

Remark 9.4. If $\omega(t) = (\log(2d_G/t))^{-a-1}$ with $a > 0$, then (ω9.1) holds for $\eta(t) = (\log(2d_G/t))^a$; and if $\omega(t) = t^a$ with $a > 0$, then (ω9.1) holds for $\eta(t) = t^{-a}$.

10 Grand and small Lebesgue spaces

Following Capone-Fiorenza [11], for $0 < \theta < 1$ and measurable functions f on the unit ball $\mathbf{B} = B(0, 1)$, we define the norm

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\theta}_{\{0\}}(\mathbf{B})} = \sup_{0<t<1} \left(\log \frac{2}{t}\right)^{-\theta/p(0)} \|f\|_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}$$

and

$$\|f\|_{L^{p(\cdot)-0,\theta}(\mathbf{B})} = \sup_{0<\varepsilon<p^- -1} \varepsilon^{\theta/p(0)}\|f\|_{L^{p(\cdot)-\varepsilon}(\mathbf{B})}.$$

THEOREM 10.1. *There exists a constant $C > 0$ such that*

$$\|f\|_{L^{p(\cdot)-0,\theta}(\mathbf{B})} \le C\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\theta}_{\{0\}}(\mathbf{B})}$$

for all measurable functions f on $\mathbf{B}$.

Proof. Let f be a nonnegative measurable function on $\mathbf{B}$ such that $\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\theta}_{\{0\}}(\mathbf{B})} \le 1$ or

$$\int_{\mathbf{B}\setminus B(0,t)} \left(\left(\log \frac{2}{t}\right)^{-\theta/p(0)} f(x) \right)^{p(x)} dx \le 1 \tag{10.1}$$

for all $0 < t < 1$. For $0 < \varepsilon < p^- - 1$, we take $0 < s < 1$ such that $\varepsilon = (p^- - 1)(\log 2)/\log(2/s)$. We have

$$\begin{aligned} \int_{\mathbf{B}\setminus B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)-\varepsilon} dx &\le \int_{\mathbf{B}\setminus B(0,s)} 1\, dx + \int_{\mathbf{B}\setminus B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)} dx \\ &\le C. \end{aligned}$$

By multiplying (10.1) by $(\log(2/t))^{-b-1}$ for (large) $b > 1$, integration gives

$$\begin{aligned} \int_0^r \left(\log \frac{2}{t}\right)^{-b-1} \frac{dt}{t} &\ge \int_0^r \left(\log \frac{2}{t}\right)^{-b-1} \left(\int_{B(0,r)\setminus B(0,t)} \left(\left(\log \frac{2}{t}\right)^{-\theta/p(0)} f(x) \right)^{p(x)} dx \right) \frac{dt}{t} \\ &\ge \int_0^r \left(\log \frac{2}{t}\right)^{-b-1} \left(\int_{B(0,r)\setminus B(0,t)} \left(\log \frac{2}{t}\right)^{-\theta - c_p/\log(2/|x|)} f(x)^{p(x)} dx \right) \frac{dt}{t} \\ &= \int_{B(0,r)} f(x)^{p(x)} \left(\int_0^{|x|} \left(\log \frac{2}{t}\right)^{-b-1-\theta-c_p/\log(2/|x|)} \frac{dt}{t} \right) dx \\ &\ge C \int_{B(0,r)} f(x)^{p(x)} \left(\log \frac{2}{|x|}\right)^{-b-\theta} dx, \end{aligned}$$

or

$$\int_{B(0,r)} f(x)^{p(x)} \left(\log \frac{2}{|x|}\right)^{-b-\theta} dx \le C \left(\log \frac{2}{r}\right)^{-b}$$

for $0 < r < 1$.

First consider the case when

$$A = \int_{B(0,s)} \left(\log \frac{2}{|x|}\right)^{(p(0)-\varepsilon)(\theta+b)/\varepsilon} dx \ge 1.$$

For $k > 1$, we obtain

$$\begin{aligned}
&\int_{B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)-\varepsilon} dx \\
\le\ & \int_{B(0,s)} \left(\varepsilon^k A^{-1/p(0)} \left(\log \frac{2}{|x|}\right)^{(\theta+b)/\varepsilon}\right)^{p(x)-\varepsilon} dx \\
&+ \int_{B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)-\varepsilon} \left(\frac{\varepsilon^{\theta/p(0)} f(x)}{\varepsilon^k A^{-1/p(0)} (\log(2/|x|))^{(\theta+b)/\varepsilon}}\right)^{\varepsilon} dx \\
\le\ & C\Bigg\{\varepsilon^{kp(0)} \int_{B(0,s)} A^{-(p(x)-\varepsilon)/p(0)} \left(\log \frac{2}{|x|}\right)^{(p(0)-\varepsilon)(\theta+b)/\varepsilon} dx \\
&+ \varepsilon^{\theta} A^{\varepsilon/p(0)} \int_{B(0,s)} f(x)^{p(x)} \left(\log \frac{2}{|x|}\right)^{-(\theta+b)} dx\Bigg\}
\end{aligned}$$

since $\varepsilon^{p(x)-\varepsilon} \le C\varepsilon^{p(0)}$ by (P2) for all $x \in B(0,s)$. Since $\log(2/t) \le (2^a/a)t^{-a}$ for $0 < t < 1$ and $a = \varepsilon/\{2(p(0)-\varepsilon)(\theta+b)\}$, we find

$$A \le \int_{B(0,s)} \left(\frac{2^a}{a}|x|^{-a}\right)^{1/(2a)} dx \le \left(\frac{2^a}{a}\right)^{1/(2a)} \int_{\mathbf{B}} |x|^{-1/2} dx \le Ca^{-1/(2a)},$$

so that we have by (P2)

$$A^{-p(x)/p(0)} \le CA^{-1+c\varepsilon/p(0)} \qquad \text{for } x \in B(0,s) \text{ and some constant } c > 0$$

and

$$A^{\varepsilon/p(0)} \le C\varepsilon^{-(b+\theta)}.$$

Hence we have

$$\begin{aligned}
&\int_{B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)-\varepsilon} dx \\
\le\ & C\Bigg\{\varepsilon^{kp(0)} A^{-(p(0)-(1+c)\varepsilon)/p(0)} \int_{B(0,s)} \left(\log \frac{2}{|x|}\right)^{(p(0)-\varepsilon)(\theta+b)/\varepsilon} dx \\
&+ \varepsilon^{\theta} A^{\varepsilon/p(0)} \int_{B(0,s)} f(x)^{p(x)} \left(\log \frac{2}{|x|}\right)^{-(\theta+b)} dx\Bigg\} \\
\le\ & C\Bigg\{\varepsilon^{kp(0)} A^{\varepsilon(1+c)/p(0)} + \varepsilon^{\theta} A^{\varepsilon/p(0)} \int_{B(0,s)} f(x)^{p(x)} \left(\log \frac{2}{|x|}\right)^{-(\theta+b)} dx\Bigg\} \\
\le\ & C\left\{\varepsilon^{kp(0)} A^{\varepsilon(1+c)/p(0)} + \varepsilon^{\theta+b} A^{\varepsilon/p(0)}\right\} \\
\le\ & C\left\{\varepsilon^{kp(0)-(b+\theta)(1+c)} + 1\right\}.
\end{aligned}$$

If we take b and k such that $kp(0) - (b+\theta)(1+c) \ge 0$, then the present case is obtained.

If $A \le 1$, then we obtain by (P2)

$$\begin{aligned}
\int_{B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)-\varepsilon} dx &\le \int_{B(0,s)} \left(\log \frac{2}{|x|}\right)^{(\theta+b)(p(x)-\varepsilon)/\varepsilon} dx \\
&\quad + \int_{B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)-\varepsilon} \left(\frac{\varepsilon^{\theta/p(0)} f(x)}{(\log(2/|x|))^{(\theta+b)/\varepsilon}}\right)^{\varepsilon} dx \\
&\le C + \int_{B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)} \left(\log \frac{2}{|x|}\right)^{-(\theta+b)} dx \\
&\le C\left\{1 + \varepsilon^{\theta} \int_{B(0,s)} f(x)^{p(x)} \left(\log \frac{2}{|x|}\right)^{-(\theta+b)} dx\right\} \\
&\le C\left\{1 + \varepsilon^{\theta} \left(\log \frac{2}{s}\right)^{-b}\right\} \\
&\le C,
\end{aligned}$$

which completes the proof. □

Given f on $\mathbf{R}^n$, recall the definition of the symmetric decreasing rearrangement of f by

$$f^{\star}(x) = \int_0^{\infty} \chi_{E_f(t)^{\star}}(x)dt,$$

where $E^{\star} = \{x : |B(0,|x|)| < |E|\}$ and $E_f(t) = \{y : |f(y)| > t\}$; see Burchard [6].

Theorem 10.2. *There exists a constant $C > 0$ such that*

$$\|f^{\star}\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\theta}_{\{0\}}(\mathbf{B})} \le C\|f^{\star}\|_{L^{p(\cdot)-0,\theta}(\mathbf{B})}$$

for all measurable functions f on $\mathbf{B}$.

Proof. Let f be a nonnegative measurable function on $\mathbf{B}$ such that $\|f^{\star}\|_{L^{p(\cdot)-0,\theta}(\mathbf{B})} \le 1$. Note that

$$\int_{\mathbf{B}\setminus B(0,t/2)} \left(\varepsilon^{-\theta/p(0)} f^{\star}(x)\right)^{p(x)-\varepsilon} dx \le 1 \qquad (10.2)$$

for all $0 < t < 1$ and $\varepsilon = (p^- - 1)(\log 2)/\log(2/t)$. We have

$$\begin{aligned}
\int_{\mathbf{B}\setminus B(0,t)} \left(\varepsilon^{-\theta/p(0)} f^{\star}(x)\right)^{p(x)} dx &\le C\left(\frac{1}{|B(0,t)\setminus B(0,t/2)|}\int_{B(0,t)\setminus B(0,t/2)} f^{\star}(x)dx\right)^{\varepsilon} \\
&\quad \times \int_{\mathbf{B}\setminus B(0,t)} \varepsilon^{-\theta p(x)/p(0)} f^{\star}(x)^{p(x)-\varepsilon} dx
\end{aligned}$$

since $f^{\star}$ is radially decreasing. Set

$$I = \frac{1}{|B(0,t)\setminus B(0,t/2)|}\int_{B(0,t)\setminus B(0,t/2)} f^{\star}(x)dx$$

and

$$J = \left(\frac{1}{|B(0,t)\setminus B(0,t/2)|}\int_{B(0,t)\setminus B(0,t/2)} f^{\star}(x)^{p(x)-\varepsilon}dx\right)^{1/(p(0)-\varepsilon)}.$$

If $J \geq 1$, then we have by (10.2)

$$\begin{aligned} I &\leq J + C\frac{1}{|B(0,t) \setminus B(0,t/2)|}\int_{B(0,t)\setminus B(0,t/2)} f^{\star}(x)\left(\frac{f^{\star}(x)}{J}\right)^{p(x)-\varepsilon-1} dx \\ &\leq J + CJ^{-p(0)+\varepsilon+1}\frac{1}{|B(0,t) \setminus B(0,t/2)|}\int_{B(0,t)\setminus B(0,t/2)} f^{\star}(x)^{p(x)-\varepsilon}dx \\ &\leq CJ \end{aligned}$$

by (P2) since $J \leq Ct^{-n/p(0)}$ for all $0 < t < 1$ and if $J \leq 1$, then

$$I \leq 1 + \frac{1}{|B(0,t) \setminus B(0,t/2)|}\int_{B(0,t)\setminus B(0,t/2)} f^{\star}(x)^{p(x)-\varepsilon}dx \leq C.$$

Hence

$$I^{\varepsilon} \leq C\left(t^{-n\varepsilon/p(0)} + 1\right) \leq C,$$

so that

$$\int_{\mathbf{B}\setminus B(0,t)} \left(\varepsilon^{-\theta/p(0)} f^{\star}(x)\right)^{p(x)} dx \leq C\int_{\mathbf{B}\setminus B(0,t/2)} \left(\varepsilon^{-\theta/p(0)} f^{\star}(x)\right)^{p(x)-\varepsilon} dx \leq C,$$

which completes the proof. □

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