Hardy's inequality in Musielak-Orlicz-Sobolev spaces

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Abstract

Our aim in this paper is to treat Hardy's inequalities for Musielak-Orlicz-Sobolev functions on proper open subset of $\mathbf{R}^N$.

1 Introduction

The higher dimensional Hardy's inequality of the form

$$\int_\Omega |u(x)|^p \delta(x)^{-p+\beta} dx \le C \int_\Omega |\nabla u(x)|^p \delta(x)^\beta dx,\ u \in C_0^\infty(\Omega)$$

appeared in [12] for bounded Lipschitz domains $\Omega \subset \mathbf{R}^N$, $1 < p < \infty$ and $\beta < p-1$, where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. For related results, we refer to [1], [2], [6], [7], [8] and [13].

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions. Harjulehto-Hästö-Koskenoja [4] proved Hardy's inequality for Sobolev functions $u \in W_0^{1,p(\cdot)}(\Omega)$ when Ω is bounded and $p(\cdot)$ is a variable exponent satisfying the log-Hölder conditions on Ω, as an extension of [2]. In fact they proved the following:

Theorem A. *Let Ω be an open and bounded subset of $\mathbf{R}^N$. Suppose $1 < p^- \le p^+ < \infty$, where $p^- := \inf_{x\in\mathbf{R}^N} p(x)$ and $p^+ := \sup_{x\in\mathbf{R}^N} p(x)$. Assume that Ω satisfies the measure density condition, that is, there exists a constant $k > 0$ such that*

$$|B(z,r) \cap \Omega^c| \ge k|B(z,r)| \tag{1.1}$$

for every $z \in \partial\Omega$ and $r > 0$ (see [3]). *Then there exist positive constants C and b_0 such that the inequality*

$$\|\delta^{b-1} u\|_{L^{p(\cdot)}(\Omega)} \le C\|\delta^b |\nabla u|\|_{L^{p(\cdot)}(\Omega)} \tag{1.2}$$

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holds for all $u \in W_0^{1,p(\cdot)}(\Omega)$ *and all* $0 \le b < b_0$, *where* $\delta(x) = dist(x, \partial\Omega)$.

In the case when $b = 0$, Hästö [5, Theorem 3.2] proved Theorem A without the assumption that Ω is bounded. It is also shown in [4] that if $p^- > N$ then (1.2) holds without the measure density condition (1.1).

Recently, these results have been extended to the two variable exponents Sobolev spaces $W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$ in [10], where $\Phi_{p(\cdot),q(\cdot)}(x,t) = \left(t(\log(c_0+t))^{q(x)}\right)^{p(x)}$ with $p(\cdot)$ as above and a measurable bounded function $q(\cdot)$. In fact, the following results are shown in [10]:

Theorem B ([10, Theorem 1.1]). *Let* $\Omega \neq \mathbf{R}^N$ *be an open set. Suppose* $1 < p^- \le p^+ < \infty$ *and* Ω *satisfies the measure density condition* (1.1). *Then, for* $0 < A < N/p^+$, $A \le 1$, *there exist positive constants* C *and* b_0 *such that the inequality*

$$\|\delta^{\alpha+b-1}u\|_{\Phi_{p_\alpha(\cdot),q(\cdot)}(\Omega)} \le C\|\delta^b|\nabla u|\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}$$

holds for all $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$, $0 \le \alpha \le A$ *and* $0 \le b < b_0$, *where* $1/p_\alpha(x) = 1/p(x) - \alpha/N$.

Theorem B′ ([10, Theorem 1.2]). *If* $N < p^- \le p^+ < \infty$, *then the same conclusion as in Theorem B holds without the measure density condition* (1.1).

Our aim in this paper is to extend these results to functions in general Musielak-Orlicz-Sobolev spaces $W_0^{1,\Phi}(\Omega)$ defined by a general function $\Phi(x,t)$ satisfying certain conditions (see Section 2 for the definitions of Φ and $W_0^{1,\Phi}(\Omega)$). Corresponding to the functions $\Phi_{p_\alpha(\cdot),q(\cdot)}(x,t)$ in [10], we shall introduce functions $\Psi_\alpha(x,t)$ to state our main results Theorem 4.4 and Theorem 5.2, which are extensions of Theorem B and Theorem B′, respectively.

2 Preliminaries

Throughout this paper, let C denote various constants independent of the variables in question and $C(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$.

We consider a function

$$\Phi(x,t) = t\phi(x,t) : \mathbf{R}^N \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions (Φ1) – (Φ4):

(Φ1) $\phi(\,\cdot\,, t)$ is measurable on $\mathbf{R}^N$ for each $t \ge 0$ and $\phi(x, \cdot\,)$ is continuous on $[0,\infty)$ for each $x \in \mathbf{R}^N$;

(Φ2) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \phi(x,1) \le A_1 \quad \text{for all } x \in \mathbf{R}^N;$$

(Φ3) $\phi(x,\cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \ge 1$ such that

$$\phi(x,t) \le A_2\phi(x,s) \quad \text{for all } x \in \mathbf{R}^N \quad \text{whenever } 0 \le t < s;$$

(Φ4) there exists a constant $A_3 \geq 1$ such that

$$\phi(x, 2t) \leq A_3 \phi(x, t) \quad \text{for all } x \in \mathbf{R}^N \text{ and } t > 0.$$

Note that (Φ2), (Φ3) and (Φ4) imply

$$0 < \inf_{x \in \mathbf{R}^N} \phi(x, t) \leq \sup_{x \in \mathbf{R}^N} \phi(x, t) < \infty$$

for each $t > 0$.

If $\Phi(x, \cdot)$ is convex for each $x \in \mathbf{R}^N$, then (Φ3) holds with $A_2 = 1$; namely $\phi(x, \cdot)$ is non-decreasing for each $x \in \mathbf{R}^N$.

Let $\bar{\phi}(x, t) = \sup_{0 \leq s \leq t} \phi(x, s)$ and

$$\overline{\Phi}(x, t) = \int_0^t \bar{\phi}(x, r)\, dr$$

for $x \in \mathbf{R}^N$ and $t \geq 0$. Then $\overline{\Phi}(x, \cdot)$ is convex and

$$\frac{1}{2A_3} \Phi(x, t) \leq \overline{\Phi}(x, t) \leq A_2 \Phi(x, t)$$

for all $x \in \mathbf{R}^N$ and $t \geq 0$.

By (Φ3), we see that

$$\Phi(x, at) \begin{cases} \leq A_2 a \Phi(x, t) & \text{if } 0 \leq a \leq 1 \\ \geq A_2^{-1} a \Phi(x, t) & \text{if } a \geq 1. \end{cases} \tag{2.1}$$

We shall also consider the following conditions:

(Φ5) for every $\gamma > 0$, there exists a constant $B_\gamma \geq 1$ such that

$$\phi(x, t) \leq B_\gamma \phi(y, t)$$

whenever $|x - y| \leq \gamma t^{-1/N}$ and $t \geq 1$;

(Φ6) there exist a function $g \in L^1(\mathbf{R}^N)$ and a constant $B_\infty \geq 1$ such that $0 \leq g(x) < 1$ for all $x \in \mathbf{R}^N$ and

$$B_\infty^{-1} \phi(x, t) \leq \phi(x', t) \leq B_\infty \phi(x, t)$$

whenever $|x'| \geq |x|$ and $g(x) \leq t \leq 1$.

Example 2.1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1, \ldots, k$, be measurable functions on $\mathbf{R}^N$ such that

(P1) $1 \leq p^- := \inf_{x \in \mathbf{R}^N} p(x) \leq \sup_{x \in \mathbf{R}^N} p(x) =: p^+ < \infty$

and

(Q1) $-\infty < q_j^- := \inf_{x \in \mathbf{R}^N} q_j(x) \leq \sup_{x \in \mathbf{R}^N} q_j(x) =: q_j^+ < \infty$

for all $j=1,\dots,k$.

Set $L_c(t)=\log(c+t)$ for $c\ge e$ and $t\ge 0$, $L_c^{(1)}(t)=L_c(t)$, $L_c^{(j+1)}(t)=L_c(L_c^{(j)}(t))$ and

$$\Phi(x,t)=t^{p(x)}\prod_{j=1}^{k}(L_c^{(j)}(t))^{q_j(x)}.$$

Then, $\Phi(x,t)$ satisfies (Φ1), (Φ2) and (Φ4). It satisfies (Φ3) if there is a constant $K\ge 0$ such that $K(p(x)-1)+q_j(x)\ge 0$ for all $x\in G$ and $j=1,\dots,k$; in particular if $p^->1$ or $q_j^-\ge 0$ for all $j=1,\dots,k$.

Moreover, we see that $\Phi(x,t)$ satisfies (Φ5) if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x)-p(y)|\le\frac{C_p}{L_e(1/|x-y|)}$$

with a constant $C_p\ge 0$ and

(Q2) $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely

$$|q_j(x)-q_j(y)|\le\frac{C_{q_j}}{L_e^{(j+1)}(1/|x-y|)}$$

with constants $C_{q_j}\ge 0$, $j=1,\dots k$.

Finally, we see that $\Phi(x,t)$ satisfies (Φ6) with $g(x)=1/(1+|x|)^{N+1}$ if $p(\cdot)$ is log-Hölder continuous at ∞, namely if it satisfies

(P3) $|p(x)-p(x')|\le\dfrac{C_{p,\infty}}{L_e(|x|)}$ whenever $|x'|\ge|x|$ with a constant $C_{p,\infty}\ge 0$.

In fact, if $1/(1+|x|)^{N+1}<t\le 1$, then $t^{-|p(x)-p(x')|}\le e^{(N+1)C_\infty}$ for $|x'|\ge|x|$ and $L_c^{(j)}(t)^{|q_j(x)-q_j(x')|}\le L_c^{(j)}(1)^{q_j^+-q_j^-}$.

Example 2.2. Let $p_1(\cdot)$, $p_2(\cdot)$, $q_1(\cdot)$ and $q_2(\cdot)$ be measurable functions on $\mathbf{R}^N$ satisfying (P1) and (Q1).

Then,

$$\Phi(x,t)=(1+t)^{p_1(x)}(1+1/t)^{-p_2(x)}L_c(t)^{q_1(x)}L_c(1/t)^{-q_2(x)}$$

satisfies (Φ1), (Φ2) and (Φ4). It satisfies (Φ3) if $p_j^->1$, $j=1,2$ or $q_j^-\ge 0$, $j=1,2$. As a matter of fact, it satisfies (Φ3) if and only if $p_j(\cdot)$, $q_j(\cdot)$ satisfies the following conditions:

(1) $q_j(x)\ge 0$ at points x where $p_j(x)=1$, $j=1,2$;

(2) $\sup_{x:p_j(x)>1}\{\min(q_j(x),0)\log(p_j(x)-1)\}<\infty$, $j=1,2$.

Moreover, we see that $\Phi(x,t)$ satisfies (Φ5) if $p_1(\cdot)$ is log-Hölder continuous and $q_1(\cdot)$ is 2-log-Hölder continuous.

Finally, we see that $\Phi(x,t)$ satisfies (Φ6) with $g(x)=1/(1+|x|)^{N+1}$ if $p_2(\cdot)$ is log-Hölder continuous at ∞ and

(Q3) $q_2(\cdot)$ is 2-log-Hölder continuous at ∞, namely

$$|q_2(x) - q_2(x')| \le \frac{C_{q_2,\infty}}{L_c^{(2)}(|x|)} \quad \text{whenever } |x'| \ge |x|$$

with a constant $C_{q_2,\infty} \ge 0$.

In fact, if $1/(1+|x|)^{N+1} < t \le 1$, then $(1+t)^{|p_1(x)-p_1(x')|} \le 2^{p_1^+ - 1}$, $(1+1/t)^{|p_2(x)-p_2(x')|} \le e^{(N+1)C_{p,\infty}}$, $(\log(e+t))^{|q_1(x)-q_1(x')|} \le (\log(e+1))^{q_1^+ - q_1^-}$ and $(\log(e+1/t))^{|q_2(x)-q_2(x')|} \le C(N, C_{q,\infty})$ for $|x'| \ge |x|$.

Let Ω be an open set in $\mathbf{R}^N$. Given $\Phi(x,t)$ as above, the associated Musielak-Orlicz space

$$L^{\Phi}(\Omega) = \left\{ f \in L^1_{loc}(\Omega)\,;\ \int_{\Omega} \Phi\big(y, |f(y)|\big)\, dy < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\Omega)} = \inf \left\{ \lambda > 0\,;\ \int_{\Omega} \overline{\Phi}\big(y, |f(y)|/\lambda\big)\, dy \le 1 \right\}$$

(cf. [11]). Further, we define the Musielak-Orlicz-Sobolev space by

$$W^{1,\Phi}(\Omega) = \{u \in L^{\Phi}(\Omega) : |\nabla u| \in L^{\Phi}(\Omega)\}.$$

The norm

$$\|u\|_{W^{1,\Phi}(\Omega)} = \|u\|_{L^{\Phi}(\Omega)} + \||\nabla u|\|_{L^{\Phi}(\Omega)}$$

makes $W^{1,\Phi}(\Omega)$ a Banach space. We denote the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\Phi}(\Omega)$ by $W_0^{1,\Phi}(\Omega)$. As usual, let $W^{1,\Phi}_{loc}(\mathbf{R}^N)$ denote the set of functions u on $\mathbf{R}^N$ such that $u|_{\Omega} \in W^{1,\Phi}(\Omega)$ for every bounded open set Ω. By (Φ2) and (Φ3), $W^{1,\Phi}_{loc}(\mathbf{R}^N) \subset W^{1,1}_{loc}(\mathbf{R}^N)$.

3 Lemmas

We denote by $B(x,r)$ the open ball centered at x of radius r. For a measurable set E, we denote by $|E|$ the Lebesgue measure of E.

For a locally integrable function f on Ω, the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap\Omega} |f(y)|\, dy.$$

We know the following boundedness of maximal operator on $L^{\Phi}(\Omega)$.

Lemma 3.1 ([9, Corollary 4.4]). *Suppose that $\Phi(x,t)$ satisfies (Φ5), (Φ6) and further assume:*

($\Phi 3^*$) $t \mapsto t^{-\varepsilon_0}\phi(x,t)$ *is uniformly almost increasing on* $(0,\infty)$ *for some* $\varepsilon_0 > 0$, *namely there is a constant* $A_{2,\varepsilon_0} \ge 1$ *such that*

$$t^{-\varepsilon_0}\phi(x,t) \le A_{2,\varepsilon_0} s^{-\varepsilon_0}\phi(x,s) \quad \text{for all } x \in \mathbf{R}^N \text{ whenever } 0 < t < s.$$

Then the maximal operator M *is bounded from* $L^{\Phi}(\Omega)$ *into itself, namely, there is a constant* $C > 0$ *such that*

$$\|Mf\|_{L^{\Phi}(\Omega)} \le C\|f\|_{L^{\Phi}(\Omega)}$$

for all $f \in L^{\Phi}(\Omega)$.

For $\lambda \ge 1$, $x \in \mathbf{R}^N$ and $t \ge 0$, set

$$\Phi_\lambda(x,t) = \Phi(x,t^{1/\lambda}) = t\phi_\lambda(x,t),$$

where $\phi_\lambda(x,t) = t^{1/\lambda-1}\phi(x,t^{1/\lambda})$.

Lemma 3.2. (1) *$\Phi_\lambda(x,t)$ satisfies the conditions* ($\Phi 2$) *and* ($\Phi 4$).

(2) *Suppose $\Phi(x,t)$ satisfies* ($\Phi 3^*$). *Then $\Phi_\lambda(x,t)$ satisfies* ($\Phi 1$) *and* ($\Phi 3$) *when* $\lambda \le 1+\varepsilon_0$, *and it satisfies* ($\Phi 3^*$) *when* $\lambda < 1+\varepsilon_0$ (*with* ε_0 *replaced by* $(1+\varepsilon_0-\lambda)/\lambda$).

(3) *If $\Phi(x,t)$ satisfies* ($\Phi 5$), *then so does* $\Phi_\lambda(x,t)$.

(4) *If $\Phi(x,t)$ satisfies* ($\Phi 6$), *then so does* $\Phi_\lambda(x,t)$.

Proof. (1) ($\Phi 2$) for Φ immediately implies that for Φ_λ. For ($\Phi 4$), note that $\phi_\lambda(x,2t) \le 2^{1/\lambda-1}A_2A_3\phi_\lambda(x,t)$.

(2) The assertions of (2) follow from ($\Phi 3^*$) and the equality

$$\phi_\lambda(x,t) = t^{(1+\varepsilon_0)/\lambda-1}(t^{1/\lambda})^{-\varepsilon_0}\phi(x,t^{1/\lambda}).$$

(3) It is enough to note that $t^{-\lambda/N} \le t^{-1/N}$ for $t \ge 1$.

(4) It is enough to note that $g(x) \le g(x)^{1/\lambda}$ when $0 \le g(x) < 1$. □

From Lemma 3.1 and the above lemma, we obtain

Corollary 3.3. *Suppose that $\Phi(x,t)$ satisfies* ($\Phi 5$), ($\Phi 6$) *and* ($\Phi 3^*$). *Then the maximal operator M is bounded from $L^{\Phi_\lambda}(\Omega)$ into itself for* $1 \le \lambda < 1+\varepsilon_0$.

Set

$$\Phi^{-1}(x,s) = \sup\{t > 0\,;\, \Phi(x,t) < s\}$$

for $x \in \mathbf{R}^N$ and $s > 0$.

Lemma 3.4 (cf. [9, Lemma 5.1]). *$\Phi^{-1}(x,\cdot)$ is non-decreasing,*

$$\Phi(x,\Phi^{-1}(x,t)) = t$$

and

$$A_2^{-1}t \le \Phi^{-1}(x,\Phi(x,t)) \le A_2^2 t \tag{3.1}$$

for all $x \in \mathbf{R}^N$ and $t > 0$.

We shall consider the following condition:

($\Phi 6^*$) $\Phi(x,t)$ satisfies ($\Phi 6$) with $g(x) \le (1+|x|)^{-\beta}$ for some $\beta > N$.

LEMMA 3.5. *If $\Phi(x,t)$ satisfies ($\Phi 6^*$), then there exists $0<\lambda<1$ such that*

$$\Phi(x, \lambda g^*(x)) \le (2|x|)^{-N} \quad \textit{for all } x \in \mathbf{R}^N,$$

where $g^*(x) = \max\big(g(x),\, Mg(x)\big)$.

Proof. Since $g(x) \le (1+|x|)^{-\beta}$ with $\beta > N$, $Mg(x) \le C(1+|x|)^{-N}$, so that $g^*(x) \le C(1+|x|)^{-N}$. Hence

$$\Phi(x, \lambda g^*(x)) \le \lambda C(1+|x|)^{-N} A_2 \phi(x, \lambda C) \le 2^N \lambda C A_2 (2|x|)^{-N} \phi(x, \lambda C).$$

Thus, the required inequality holds if $\lambda \le (2^N C A_1 A_2^2)^{-1}$. □

LEMMA 3.6. *$r \mapsto r^{\sigma_0}\Phi^{-1}(x, r^{-N})$ is uniformly almost decreasing on $(0,\infty)$, where $\sigma_0 = N\Big/\big(1 + (\log A_3)/(\log 2)\big)$.*

Proof. By ($\Phi 4$), we see that

$$\Phi^{-1}\left(x, \frac{1}{2A_3} s\right) \le \frac{1}{2}\Phi^{-1}(x,s) \tag{3.2}$$

for all $x \in \mathbf{R}^N$ and $s > 0$. If $0 < \lambda < 1$, then choosing $k \in \mathbf{N}$ such that $(2A_3)^{-k} \le \lambda < (2A_3)^{-k+1}$ and applying (3.2), we have

$$\Phi^{-1}(x, \lambda s) \le 2^{-k+1}\Phi^{-1}(x,s) \le 2\lambda^{1/(1+\sigma)}\Phi^{-1}(x,s),$$

where $\sigma = (\log A_3)/(\log 2)$. Note that $\sigma_0 = N/(1+\sigma)$. Thus, for $a > 1$, we have

$$\begin{aligned}(ar)^{\sigma_0}\Phi^{-1}\big(x, (ar)^{-N}\big) &\le (ar)^{\sigma_0} 2 (a^{-N})^{1/(1+\sigma)}\Phi^{-1}\big(x, r^{-N}\big) \\ &= 2r^{\sigma_0}\Phi^{-1}(x, r^{-N}),\end{aligned}$$

which shows the assertion of the lemma. □

LEMMA 3.7. *Suppose that $\Phi(x,t)$ satisfies ($\Phi 5$) and ($\Phi 6^*$). Let $0 < \alpha < \sigma_0$ for σ_0 given in Lemma 3.6. Then there exists a constant $C > 0$ such that*

$$\int_{B(x,2|x|)\setminus B(x,r)} |x-y|^{\alpha - N} f(y)\, dy \le C r^{\alpha}\Phi^{-1}(x, r^{-N}) \tag{3.3}$$

and

$$\int_{B(x,r)} f(y)\, dy \le C r^N \Phi^{-1}(x, r^{-N}) \tag{3.4}$$

for all $x \in \mathbf{R}^N$, $0 < r \le 2|x|$, and $f \ge 0$ satisfying $\|f\|_{L^{\Phi}(\mathbf{R}^N)} \le 1$.

Proof. Condition ($\Phi\kappa J$) in [9] with $\kappa(x,r) = r^N$ and $J(x,r) = r^{\alpha - N}$ is satisfied by Lemma 3.6, if $0 < \alpha < \sigma_0$. Hence, (3.3) follows from [9, Lemma 6.3] in view of Lemma 3.5. (3.4) follows from [9, Lemma 5.3] and Lemma 3.5. □

Hereafter, let Ω is an open set in $\mathbf{R}^N$ such that $\Omega \neq \mathbf{R}^N$, and let $\delta(x) = \mathrm{dist}(x, \partial\Omega)$.

The following is a key lemma:

LEMMA 3.8. (1) *If Ω satisfies*

$$|B(z,r) \cap \Omega^c| \geq k|B(z,r)| \tag{3.5}$$

for every $z \in \partial\Omega$ and $r > 0$ with a constant $k > 0$ ($k \leq 1$), then there exists a constant $C = C(N,k) > 0$ such that

$$|u(x)| \leq C \int_{B(x,2\delta(x))} |x-y|^{1-N} |\nabla u(y)| \, dy$$

for almost every $x \in \Omega$, whenever $u \in W^{1,1}_{loc}(\mathbf{R}^N)$ and $u = 0$ outside Ω.

(2) *Let $\lambda > N$. Then there exists a constant $C > 0$ such that*

$$|v(x)| \leq C \left(\delta(x)^{\lambda-N} \int_{B(x,2\delta(x))} |\nabla v(y)|^{\lambda} \, dy \right)^{1/\lambda}$$

for every $x \in \Omega$, whenever $v \in W^{1,\lambda}_{loc}(\mathbf{R}^N)$ and $v = 0$ outside Ω.

For (1) see [10, Lemma 2.1]; for (2) see e.g. [6, (3.1)] (also cf. [2, Proposition 1]). Here note that (2) holds without the assumption (3.5).

We consider

$$H(f; x, \alpha) = \delta(x)^{\alpha-1} \int_{B(x,2\delta(x))} |x-y|^{1-N} f(y) \, dy$$

for $x \in \Omega$, $0 \leq \alpha \leq 1$ and $f \in L^1_{loc}(\mathbf{R}^N)$ such that $f \geq 0$, $f = 0$ outside Ω.

We know (by integration by parts)

$$H(f; x, 0) \leq CMf(x). \tag{3.6}$$

for all $x \in \Omega$.

LEMMA 3.9. *Let $\Omega \neq \mathbf{R}^N$ be an open set and suppose that $\Phi(x,t)$ satisfies ($\Phi5$) and ($\Phi6^*$).*

(1) *Let $\alpha \in [0, \sigma_0) \cap [0,1]$. Then there exists a constant $C > 0$ such that*

$$H(f; x, \alpha) \leq CMf(x)\Phi(x, Mf(x))^{-\alpha/N} \tag{3.7}$$

for all $x \in \Omega$ and $f \geq 0$ such that $f = 0$ outside Ω and $\|f\|_{L^\Phi(\Omega)} \leq 1$.

(2) *Let $\alpha \in [0, \sigma_0]$. Then there exists a constant $C > 0$ such that*

$$\delta(x)^{\alpha-N} \int_{B(x,2\delta(x))} f(y) \, dy \leq CMf(x)\Phi(x, Mf(x))^{-\alpha/N} \tag{3.8}$$

for all $x \in \Omega$ and $f \geq 0$ such that $f = 0$ outside Ω and $\|f\|_{L^\Phi(\Omega)} \leq 1$.

Proof. We have only to consider the case $\alpha > 0$. Without loss of generality, we may assume that $0 \in \partial\Omega$, so that $\delta(x) \le |x|$. Let $f \ge 0$ with $f = 0$ outside Ω and $\|f\|_{L^\Phi(\Omega)} \le 1$.

(1) For $0 < r \le \delta(x)$, we have by (3.3) in Lemma 3.7

$$\begin{aligned} H(f; x, \alpha) &\le C\left\{\delta(x)^{\alpha-1} r Mf(x) + \int_{B(x,2\delta(x))\setminus B(x,r)} |x-y|^{\alpha-N} f(y)\,dy\right\} \\ &\le C\left\{r^\alpha Mf(x) + r^\alpha \Phi^{-1}(x, r^{-N})\right\}. \end{aligned}$$

Suppose $\Phi(x, Mf(x))^{-1/N} > \delta(x)$. Then we have by (3.6)

$$H(f; x, \alpha) = \delta(x)^\alpha H(f; x, 0) \le C\delta(x)^\alpha Mf(x) \le CMf(x)\Phi(x, Mf(x))^{-\alpha/N},$$

which is (3.7).

Next, if $\Phi(x, Mf(x))^{-1/N} \le \delta(x)$, then take $r = \Phi(x, Mf(x))^{-1/N}$. Then, in view of (3.1) in Lemma 3.4, we obtain (3.7).

(2) By (3.4),

$$\delta(x)^{\alpha-N} \int_{B(x,2\delta(x))} f(y)\,dy \le C\delta(x)^\alpha \Phi^{-1}(x, \delta(x)^{-N}).$$

If $\alpha \le \sigma_0$, then $r \mapsto r^\alpha \Phi^{-1}(x, r^{-N})$ is uniformly almost decreasing in view of Lemma 3.6. Hence

$$\delta(x)^{\alpha-N} \int_{B(x,2\delta(x))} f(y)\,dy \le Cr^\alpha \Phi^{-1}(x, r^{-N})$$

for $0 < r \le \delta(x)$. Thus, by the same arguments as above we obtain (3.8). □

4 Hardy's inequality I

LEMMA 4.1. *Let $\Omega \ne \mathbf{R}^N$ be an open set satisfying* (3.5). *Suppose $\Phi(x,t)$ satisfies* (Φ5), (Φ6) *and* ($\Phi3^*$). *Then there exist constants $C > 0$ and $0 < b_0 < 1$ such that*

$$\|\delta^{b-1} u\|_{L^\Phi(\Omega)} \le C\|\delta^b |\nabla u|\|_{L^\Phi(\Omega)} \tag{4.1}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ and $0 \le b \le b_0$. If $u \in W_0^{1,\Phi}(\Omega)$ and $\delta^b|\nabla u| \in L^\Phi(\Omega)$ for $0 \le b \le b_0$, then $\delta^b u$ extended by 0 outside Ω belongs to $W^{1,\Phi}(\mathbf{R}^N)$.

Proof. Without loss of generality, we may assume that $0 \in \partial\Omega$. For $u \in W_0^{1,\Phi}(\Omega)$ and $b \ge 0$, let

$$u_b(x) = \begin{cases} \delta(x)^b u(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \in \Omega^c. \end{cases}$$

We first treat $u \in C_0^\infty(\Omega)$. Note that δ and $1/\delta$ are bounded on support of u and $\delta \in W^{1,\infty}(\Omega)$. Hence $u_b \in W^{1,\Phi}(\mathbf{R}^N) \subset W_{loc}^{1,1}(\mathbf{R}^N)$ for every $b \ge 0$. Applying Lemma 3.8 (1) to this function, we have

$$\delta(x)^b |u(x)| \le C \int_{B(x,2\delta(x))\cap\Omega} |x-y|^{1-N} \{b\delta(y)^{b-1}|u(y)| + \delta(y)^b |\nabla u(y)|\}\,dy, \tag{4.2}$$

so that

$$\delta(x)^{b-1}|u(x)| \le C\left\{bM(\delta^{b-1}u)(x) + M(\delta^b|\nabla u|)(x)\right\}$$

for a.e. $x \in \Omega$ with a constant C independent of b. In view of Lemma 3.1, we find

$$\|\delta^{b-1}u\|_{L^\Phi(\Omega)} \le C_0\left\{b\|\delta^{b-1}u\|_{L^\Phi(\Omega)} + \|\delta^b|\nabla u|\|_{L^\Phi(\Omega)}\right\},$$

which gives

$$(1 - C_0 b)\|\delta^{b-1}u\|_{L^\Phi(\Omega)} \le C_0\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)}.$$

Hence, taking b_0 such that $1 - C_0 b_0 > 0$, we have (4.1) for $0 \le b \le b_0$.

We next treat $u \in W_0^{1,\Phi}(\Omega)$ such that $u = 0$ outside $B(0,R)$ for some $R > 0$. Then we can find a sequence $\varphi_j \in C_0^\infty(\Omega)$ such that $\varphi_j \to u$ in $W_0^{1,\Phi}(\Omega)$ and $\varphi_j = 0$ outside $B(0,2R)$ for each j. By the above discussions, for $0 < b \le b_0$, we have

$$\|\delta^{b-1}\varphi_j\|_{L^\Phi(\Omega)} \le C\|\delta^b|\nabla\varphi_j|\|_{L^\Phi(\Omega)} \tag{4.3}$$

for all j and

$$\|\delta^{b-1}(\varphi_j - \varphi_{j'})\|_{L^\Phi(\Omega)} \le C\|\delta^b|\nabla\varphi_j - \nabla\varphi_{j'}|\|_{L^\Phi(\Omega)} \tag{4.4}$$

for all $j,\ j'$. Since δ is bounded on $B(0,2R)$, we see that

$$\|\delta^b|\nabla\varphi_j|\|_{L^\Phi(\Omega)} \to \|\delta^b|\nabla u|\|_{L^\Phi(\Omega)}$$

as $j \to \infty$. Similarly

$$\|\delta^b|\nabla\varphi_j - \nabla\varphi_{j'}|\|_{L^\Phi(\Omega)} \to 0$$

as $j,\ j' \to \infty$. Hence by (4.4), $\{\delta^{b-1}\varphi_j\}$ is a Cauchy sequence in $L^\Phi(\Omega)$, which implies that $\delta^{b-1}\varphi_j \to \delta^{b-1}u$ in $L^\Phi(\Omega)$. Thus, letting $j \to \infty$ in (4.3), we obtain (4.1). Further, $(\varphi_j)_b \to u_b$ in $L^\Phi(\mathbf{R}^N)$ and

$$\begin{aligned}\nabla(\varphi_j)_b &= \begin{cases} b\delta^{b-1}\varphi_j\nabla\delta + \delta^b\nabla\varphi_j & \text{on } \Omega \\ 0 & \text{on } \Omega^c\end{cases}\\ &\to \begin{cases} b\delta^{b-1}u\nabla\delta + \delta^b\nabla u & \text{on } \Omega \\ 0 & \text{on } \Omega^c\end{cases}\end{aligned}$$

in $L^\Phi(\mathbf{R}^N)$ as $j \to \infty$. It then follows that

$$\nabla u_b = \begin{cases} b\delta^{b-1}u\nabla\delta + \delta^b\nabla u & \text{on } \Omega \\ 0 & \text{on } \Omega^c,\end{cases}$$

which belongs to $L^\Phi(\mathbf{R}^N)$, and hence $u_b \in W^{1,\Phi}(\mathbf{R}^N)$.

Finally we treat a general $u \in W_0^{1,\Phi}(\Omega)$. For each $n \in \mathbf{N}$, we consider a C^1-function H_n on $[0,\infty)$ such that $0 \le H_n \le 1$ on $[0,\infty)$, $H_n = 1$ on $[0,n]$, $H_n = 0$ on $[3n,\infty)$, $0 \le -H_n'(t) \le t^{-1}$ for $t \in (n,3n)$. The existence of such H_n is assured since $\int_n^{3n} t^{-1}\,dt = \log 3 > 1$. Set $u_n(x) = H_n(|x|)u(x)$, $n = 1,2,\ldots$. Then we know by the above that

$$\|\delta^{b-1}u_n\|_{L^\Phi(\Omega)} \le C\|\delta^b|\nabla(u_n)|\|_{L^\Phi(\Omega)}. \tag{4.5}$$

Since $\delta^{b-1}|u_n| \uparrow \delta^{b-1}|u|$ $(n \to \infty)$,

$$\|\delta^{b-1}u_n\|_{L^\Phi(\Omega)} \to \|\delta^{b-1}u\|_{L^\Phi(\Omega)} \quad (n \to \infty).$$

On the other hand,

$$\begin{aligned}|\nabla u_n(x)| &\le |H_n{}'(|x|)||u(x)| + H_n(|x|)|\nabla u(x)| \\ &\le \frac{1}{|x|}|u(x)|\chi_{B(0,3n)\setminus B(0,n)}(x) + |\nabla u(x)|.\end{aligned}$$

Since $\delta(x)^b/|x| \le |x|^{b-1} \le n^{b-1}$ for $|x| \ge n$ and $b < 1$,

$$\delta(x)^b|\nabla u_n(x)| \le n^{b-1}|u(x)| + \delta(x)^b|\nabla u(x)|,$$

so that

$$\begin{aligned}\|\delta^b|\nabla u_n|\|_{L^\Phi(\Omega)} &\le n^{b-1}\|u\|_{L^\Phi(\Omega)} + \|\delta^b|\nabla u|\|_{L^\Phi(\Omega)} \\ &\to \|\delta^b|\nabla u|\|_{L^\Phi(\Omega)} \quad (n \to \infty).\end{aligned}$$

Therefore, by letting $n \to \infty$ in (4.5), we obtain (4.1), which also implies that $u_b \in W^{1,\Phi}(\mathbf{R}^N)$. □

For $\alpha \ge 0$, we consider a function $\Psi_\alpha(x,t) : \mathbf{R}^N \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Ψ1) $\Psi_\alpha(\cdot,t)$ is measurable on $\mathbf{R}^N$ for each $t \ge 0$ and $\Psi_\alpha(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in \mathbf{R}^N$;

(Ψ2) $\Psi_\alpha(x,\cdot)$ is uniformly almost increasing on $[0,\infty)$, namely there is a constant $A_4 \ge 1$ such that $\Psi_\alpha(x,t) \le A_4\Psi_\alpha(x,s)$ for all $x \in \mathbf{R}^N$, whenever $0 \le t < s$;

(Ψ3) there exists a constant $A_5 \ge 1$ such that

$$\Psi_\alpha\left(x, t\Phi(x,t)^{-\alpha/N}\right) \le A_5\Phi(x,t)$$

for all $x \in \mathbf{R}^N$ and $t > 0$.

Note that we may take $\Psi_0(x,t) = \Phi(x,t)$.

Example 4.2. Let $\Phi(x,t)$ be as in Example 2.1. Set

$$\Psi_\alpha(x,t) = \left(t\prod_{j=1}^{k}(L_e^{(j)}(t))^{q_j(x)/p(x)}\right)^{p^\sharp(x)},$$

where $1/p^\sharp(x) = 1/p(x) - \alpha/N$. If $0 \le \alpha < N/p^+$, then Ψ_α satisfies (Ψ1), (Ψ2) and (Ψ3).

Example 4.3. Let $\Phi(x,t)$ be as in Example 2.2. Set

$$\Psi_\alpha(x,t) = \left((1+t)L_c(t)^{q_1(x)/p_1(x)}\right)^{p_1^\sharp(x)}\left((1+1/t)L_c(1/t)^{-q_2(x)/p_2(x)}\right)^{p_2^\sharp(x)}.$$

If $0 \le \alpha < \min\{N/p_1^+, N/p_2^+\}$, then Ψ_α satisfies (Ψ1), (Ψ2) and (Ψ3).

THEOREM 4.4. *Let $\Omega \neq \mathbf{R}^N$ be an open set satisfying* (3.5). *Suppose $\Phi(x,t)$ satisfies* ($\Phi5$), ($\Phi3^*$) *and* ($\Phi6^*$) *and let $\alpha \in [0,\sigma_0) \cap [0,1]$ for σ_0 given in Lemma 3.6. Then there exist constants $C^* > 0$ and $0 < b_0 < 1$ such that*

$$\int_\Omega \Psi_\alpha\big(x, \delta(x)^{\alpha+b-1}|u(x)|/C^*\big)\, dx \le 1$$

for all $u \in W_0^{1,\Phi}(\Omega)$ with $\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)} \le 1$ and $0 \le b \le b_0$.

Proof. Let b_0 be the number given in Lemma 4.1 and let $0 \le b \le b_0$. Let $u \in W_0^{1,\Phi}(\Omega)$ with $\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)} \le 1$. By Lemma 4.1, $\delta^b u$ extended by 0 outside Ω belongs to $W^{1,1}_{loc}(\mathbf{R}^N)$, so that by Lemma 3.8 (1), (4.2) holds a.e. $x \in \Omega$. Hence

$$\delta(x)^{\alpha+b-1}|u(x)| \le C\delta(x)^{\alpha-1} \int_{B(x,2\delta(x))} |x-y|^{1-N} f_u(y)\, dy$$

for a.e $x \in \Omega$, where $f_u(y) = b\delta(y)^{b-1}|u(y)| + \delta(y)^b|\nabla u(y)|$ for $y \in \Omega$ and $f_u(y) = 0$ for $y \in \Omega^c$. By Lemma 4.1, there is a constant $C_1 \ge 1$ such that $\|f_u\|_{L^\Phi(\Omega)} \le C_1$. Applying Lemma 3.9 (1) to f_u/C_1 and using ($\Phi4$), we have

$$\delta(x)^{\alpha+b-1}|u(x)| \le C_2 Mf_u(x)\Phi(x, Mf_u(x))^{-\alpha/N}$$

a.e. $x \in \Omega$. Hence by ($\Psi2$) and ($\Psi3$) we have

$$\int_\Omega \Psi_\alpha(x, \delta(x)^{\alpha+b-1}|u(x)|/C_2)\, dx \le A_4A_5 \int_\Omega \Phi(x, Mf_u(x))\, dx \tag{4.6}$$

whenever $\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)} \le 1$. By Lemma 3.1, $\|Mf_u\|_{L^\Phi(\Omega)} \le C_3$, which implies $\int_\Omega \Phi(x, Mf_u(x))\, dx \le C_4$ ($C_4 \ge 1$).

Now let $0 < \varepsilon \le 1$. Since

$$\Phi(x, Mf_{\varepsilon u}(x)) = \Phi(x, \varepsilon Mf_u(x)) \le A_2\varepsilon\Phi(x, Mf_u(x))$$

by (2.1), applying (4.6) to εu, we have

$$\begin{aligned}\int_\Omega \Psi_\alpha(x,\delta(x)^{\alpha+b-1}|\varepsilon u(x)|/C_2)\, dx &\le A_4A_5 \int_\Omega \Phi(x, Mf_{\varepsilon u}(x))\, dx \\ &\le A_2A_4A_5\varepsilon \int_\Omega \Phi(x, Mf_u(x))\, dx \le A_2A_4A_5C_4\varepsilon.\end{aligned}$$

Thus, taking $\varepsilon = (A_2A_4A_5C_4)^{-1}$ and $C^* = C_2/\varepsilon$, we obtain the required result. □

Applying Theorem 4.4 to special Φ and Ψ_α given in Examples 2.1 and 4.2, we obtain the following corollary, which is an extension of Theorem B.

COROLLARY 4.5. *Let Φ and Ψ_α be as in Examples 2.1 and 4.2 and let $\Omega \neq \mathbf{R}^N$ be an open set satisfying* (3.5). *Suppose $p^- > 1$ and let $\alpha \in [0, N/p^+) \cap [0,1]$. Then there exist constants $C > 0$ and $0 < b_0 < 1$ such that*

$$\|\delta^{\alpha+b-1}u\|_{L^{\Psi_\alpha}(\Omega)} \le C\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ and $0 \le b \le b_0$.

Similarly, applying Theorem 4.4 to special Φ and Ψ_α given in Examples 2.2 and 4.3, we obtain another extension of Theorem B:

Corollary 4.6. *Let Φ and Ψ_α be as in Examples 2.2 and 4.3 and let $\Omega \neq \mathbf{R}^N$ be an open set satisfying* (3.5). *Suppose* $\min(p_1^-, p_2^-) > 1$ *and let* $\alpha \in [0, \min(N/p_1^+, N/p_2^+)) \cap [0,1]$. *Then there exist constants $C > 0$ and $0 < b_0 < 1$ such that*

$$\|\delta^{\alpha+b-1} u\|_{L^{\Psi_\alpha}(\Omega)} \le C \|\delta^b |\nabla u|\|_{L^\Phi(\Omega)}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ and $0 \le b \le b_0$.

5 Hardy's inequality II

For a proof of next theorem, we prepare the following lemma instead of Lemma 4.1.

Lemma 5.1. *Let $\Omega \neq \mathbf{R}^N$ be an open set. Suppose that $\Phi(x,t)$ satisfies* ($\Phi5$), ($\Phi6$) *and* ($\Phi3^*$) *for $\varepsilon_0 > N-1$. Then there exist constants $C > 0$ and $0 < b_1 < 1$ such that*

$$\|\delta^{b-1} u\|_{L^\Phi(\Omega)} \le C \|\delta^b |\nabla u|\|_{L^\Phi(\Omega)}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ and $0 \le b \le b_1$. If $u \in W_0^{1,\Phi}(\Omega)$ and $\delta^b|\nabla u| \in L^\Phi(\Omega)$ for $0 \le b \le b_1$, then $\delta^b u$ extended by 0 outside Ω belongs to $W^{1,\Phi}(\mathbf{R}^N)$.

Proof. Take λ such that $N < \lambda < \varepsilon_0 + 1$. Then $W^{1,\Phi}(\mathbf{R}^N) \subset W^{1,\lambda}_{loc}(\mathbf{R}^N)$.

First, let $u \in C_0^\infty(\Omega)$ and $b \ge 0$. Let u_b be the function $\delta^b u$ extended by 0 outside Ω. Then $u_b \in W^{1,\Phi}(\mathbf{R}^N) \subset W^{1,\lambda}_{loc}(\mathbf{R}^N)$ and applying Lemma 3.8 (2) to $v = u_b$, we have

$$[\delta(x)^{b-1}|u(x)|]^\lambda \le C\delta(x)^{-N} \int_{B(x,2\delta(x))\cap\Omega} f_u(y)\,dy \le CMf_u(x) \tag{5.1}$$

for all $x \in \Omega$, where $f_u(y) = [b\delta(y)^{b-1}|u(y)| + \delta(y)^b|\nabla u(y)|]^\lambda$. In view of Corollary 3.3, we find

$$\|[\delta^{b-1}|u|]^\lambda\|_{L^{\Phi_\lambda}(\Omega)} \le C\|f_u\|_{L^{\Phi_\lambda}(\Omega)}.$$

Since $\|f\|_{L^{\Phi_\lambda}(\Omega)} = \|f^{1/\lambda}\|^\lambda_{L^\Phi(\Omega)}$ for every $f \in L^{\Phi_\lambda}(\Omega)$, we obtain

$$\|\delta^{b-1}u\|_{L^\Phi(\Omega)} \le C^{1/\lambda}\|f_u^{1/\lambda}\|_{L^\Phi(\Omega)} \le C_1\left\{b\|\delta^{b-1}u\|_{L^\Phi(\Omega)} + \|\delta^b|\nabla u|\|_{L^\Phi(\Omega)}\right\},$$

which gives

$$(1 - C_1 b)\|\delta^{b-1}u\|_{L^\Phi(\Omega)} \le C_1\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)}.$$

Take b_1 such that $1 - C_1 b_1 > 0$. Then, in the same way as the last half of the proof of Lemma 4.1, we obtain the required results for $u \in W_0^{1,\Phi}(\Omega)$ and $0 \le b \le b_1$. □

THEOREM 5.2. *Let $\Omega \neq \mathbf{R}^N$ be an open set. Suppose $\Phi(x,t)$ satisfies (Φ5), ($\Phi6^*$) and ($\Phi3^*$) with $\varepsilon_0 > N-1$. Let $\alpha \in [0,\sigma_0]$. Then there exist $C^* > 0$ and $0 < b_1 < 1$ such that*

$$\int_\Omega \Psi_\alpha(x, \delta(x)^{\alpha+b-1}|u(x)|/C^*)\,dx \le 1$$

for all $u \in W_0^{1,\Phi}(\Omega)$ with $\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)} \le 1$ and $0 \le b \le b_1$.

Proof. Let b_1 be as in the above lemma and let $0 \le b \le b_1$. Let $u \in W_0^{1,\Phi}(\Omega)$ with $\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)} \le 1$. Take λ such that $N < \lambda < \varepsilon_0 + 1$. By the above lemma, $\delta^b u$ extended by 0 outside Ω belongs to $W^{1,\lambda}_{loc}(\mathbf{R}^N)$, so that by (5.1) we have

$$[\delta(x)^{\alpha+b-1}|u(x)|]^\lambda \le C\delta(x)^{\alpha\lambda-N}\int_{B(x,2\delta(x))} f_u(y)\,dy$$

for all $x \in \Omega$, where $f_u(y) = [b\delta(y)^{b-1}|u(y)| + \delta(y)^b|\nabla u(y)|]^\lambda$ for $y \in \Omega$ and $f_u(y) = 0$ for $y \in \Omega^c$. By Lemma 5.1, there is a constant $C_1 \ge 1$ such that $\|f_u^{1/\lambda}\|_{L^\Phi(\Omega)} \le C_1$, so that $\|f_u\|_{L^{\Phi_\lambda}(\Omega)} \le {C_1}^\lambda$.

Here we note that $\Phi_\lambda(x,t)$ satisfies ($\Phi6^*$) with g^λ in place of g and that $r \mapsto r^{\lambda\sigma_0}\Phi_\lambda^{-1}(x, r^{-N})$ is uniformly almost decreasing on $(0,\infty)$. Since $\lambda\alpha \in [0,\lambda\sigma_0]$, we can apply Lemma 3.9 (2) to $f_u/{C_1}^\lambda$, $\lambda\alpha$ and Φ_λ in place of f, α and Φ respectively, and using (Φ4), we obtain

$$\begin{aligned}\delta(x)^{\alpha+b-1}|u(x)| &\le C[Mf_u(x)]^{1/\lambda}\Phi_\lambda(x, Mf_u(x)/{C_1}^\lambda)^{-\alpha/N}\\ &\le C_2[Mf_u(x)]^{1/\lambda}\Phi(x,[Mf_u(x)]^{1/\lambda})^{-\alpha/N}\end{aligned}$$

for all $x \in \Omega$. Hence by (Ψ2) and (Ψ3)

$$\begin{aligned}\int_\Omega \Psi_\alpha(x,\delta(x)^{\alpha+b-1}|u(x)|/C_2)\,dx &\le A_4A_5\int_\Omega \Phi(x,[Mf_u(x)]^{1/\lambda})\,dx\\ &= A_4A_5\int_\Omega \Phi_\lambda(x, Mf_u(x))\,dx. \qquad (5.2)\end{aligned}$$

By Corollary 3.3, $\|Mf_u\|_{L^{\Phi_\lambda}(\Omega)} \le C_3$, which implies $\int_\Omega \Phi_\lambda(x, Mf_u(x))\,dx \le C_4$.

Let $0 < \varepsilon \le 1$. Since

$$\begin{aligned}\Phi_\lambda(x, Mf_{\varepsilon u}(x)) &= \Phi_\lambda(x, \varepsilon^\lambda Mf_u(x)) = \Phi(x, \varepsilon[Mf_u(x)]^{1/\lambda})\\ &\le A_2\varepsilon\Phi(x,[Mf_u(x)]^{1/\lambda}) = A_2\varepsilon\Phi_\lambda(x, Mf_u(x))\end{aligned}$$

by (2.1), applying (5.2) to εu, we have

$$\begin{aligned}\int_\Omega \Psi_\alpha(x,\delta(x)^{\alpha+b-1}|\varepsilon u(x)|/C_2)\,dx &\le A_4A_5\int_\Omega \Phi_\lambda(x, Mf_{\varepsilon u}(x))\,dx\\ &\le A_2A_4A_5\varepsilon\int_\Omega \Phi_\lambda(x, Mf_u(x))\,dx \le A_2A_4A_5C_4\varepsilon.\end{aligned}$$

Thus, taking $\varepsilon = (A_2A_4A_5C_4)^{-1}$ and $C^* = C_2/\varepsilon$, we obtain the required result. □

Applying Theorem 5.2 to special Φ and Ψ_α given in Examples 2.1 and 4.2, we obtain the following corollary, which is an extension of Theorem B′.

Corollary 5.3. *Let Φ and Ψ_α be as in Examples 2.1 and 4.2. Suppose $p^- > N$ and let $0 \le \alpha < N/p^+$. Then there exist constants $C > 0$ and $0 < b_1 < 1$ such that*

$$\|\delta^{\alpha+b-1}u\|_{L^{\Psi_\alpha}(\Omega)} \le C\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ and $0 \le b \le b_1$.

Similarly, applying Theorem 5.2 to special Φ and Ψ_α given in Examples 2.2 and 4.3, we obtain another extension of Theorem B′:

Corollary 5.4. *Let Φ and Ψ_α be as in Examples 2.2 and 4.3. Suppose $\min(p_1^-, p_2^-) > N$ and let $0 \le \alpha < \min(N/p_1^+, N/p_2^+)$. Then there exist constants $C > 0$ and $0 < b_1 < 1$ such that*

$$\|\delta^{\alpha+b-1}u\|_{L^{\Psi_\alpha}(\Omega)} \le C\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ and $0 \le b \le b_1$.

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