Hardy’s inequality in Musielak-Orlicz-Sobolev spaces

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Abstract

Our aim in this paper is to treat Hardy’s inequalities for Musielak-Orlicz-Sobolev functions on proper open subset of $\mathbb{R}^N$.

1 Introduction

The higher dimensional Hardy’s inequality of the form

$$\int_{\Omega} |u(x)|^p \delta(x)^{-p+\beta} dx \leq C \int_{\Omega} |\nabla u(x)|^p \delta(x)^{\beta} dx, \ u \in C_0^\infty(\Omega)$$

appeared in [12] for bounded Lipschitz domains $\Omega \subset \mathbb{R}^N$, $1 < p < \infty$ and $\beta < p-1$, where $\delta(x) = \text{dist}(x, \partial \Omega)$. For related results, we refer to [1], [2], [6], [7], [8] and [13].

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions. Harjulehto-Hästö-Koskenoja [4] proved Hardy’s inequality for Sobolev functions $u \in W^{1,p}(\cdot)(\Omega)$ when $\Omega$ is bounded and $p(\cdot)$ is a variable exponent satisfying the log-Hölder conditions on $\Omega$, as an extension of [2]. In fact they proved the following:

**Theorem A.** Let $\Omega$ be an open and bounded subset of $\mathbb{R}^N$. Suppose $1 < p^- \leq p^+ < \infty$, where $p^- := \inf_{x \in \mathbb{R}^N} p(x)$ and $p^+ := \sup_{x \in \mathbb{R}^N} p(x)$. Assume that $\Omega$ satisfies the measure density condition, that is, there exists a constant $k > 0$ such that

$$|B(z, r) \cap \Omega^c| \geq k|B(z, r)| \quad (1.1)$$

for every $z \in \partial \Omega$ and $r > 0$ (see [3]). Then there exist positive constants $C$ and $b_0$ such that the inequality

$$\|\delta^{b-1}u\|_{L^p(\cdot)(\Omega)} \leq C\|\delta^b|\nabla u||_{L^p(\cdot)(\Omega)} \quad (1.2)$$

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holds for all \( u \in W^{1,p(x)}_0(\Omega) \) and all \( 0 \leq b < b_0 \), where \( \delta(x) = \text{dist}(x, \partial \Omega) \).

In the case when \( b = 0 \), H"ast"o [5, Theorem 3.2] proved Theorem A without the assumption that \( \Omega \) is bounded. It is also shown in [4] that if \( p^- > N \) then (1.2) holds without the measure density condition (1.1).

Recently, these results have been extended to the two variable exponents Sobolev spaces \( W^{1,\Phi(p(\cdot),q(\cdot))}_0(\Omega) \) in [10], where \( \Phi(p(\cdot),q(\cdot))(x,t) = (t(\log(c_0 + t))^q(x))^{p(x)} \) with \( p(\cdot) \) as above and a measurable bounded function \( q(\cdot) \). In fact, the following results are shown in [10]:

**Theorem B** ([10, Theorem 1.1]). Let \( \Omega \neq \mathbb{R}^N \) be an open set. Suppose \( 1 < p^- \leq p^+ < \infty \) and \( \Omega \) satisfies the measure density condition (1.1). Then, for \( 0 < A < N/p^+ \), \( A \leq 1 \), there exist positive constants \( C \) and \( b_0 \) such that the inequality

\[
\| \delta^{\alpha+b-1} u \|_{\Phi(p(\cdot),q(\cdot))(\Omega)} \leq C \| \delta^b |\nabla u| \|_{\Phi(p(\cdot),q(\cdot))(\Omega)}
\]

holds for all \( u \in W^{1,\Phi(p(\cdot),q(\cdot))}_0(\Omega) \), \( 0 \leq \alpha \leq A \) and \( 0 \leq b < b_0 \), where \( 1/p_0(x) = 1/p(x) - \alpha/N \).

**Theorem B'** ([10, Theorem 1.2]). If \( N < p^- \leq p^+ < \infty \), then the same conclusion as in Theorem B holds without the measure density condition (1.1).

Our aim in this paper is to extend these results to functions in general Musielak-Orlicz-Sobolev spaces \( W^{1,\Phi}_0(\Omega) \) defined by a general function \( \Phi(x,t) \) satisfying certain conditions (see Section 2 for the definitions of \( \Phi \) and \( W^{1,\Phi}_0(\Omega) \)). Corresponding to the functions \( \Phi^{p(\cdot),q(\cdot)}(x,t) \) in [10], we shall introduce functions \( \Psi_{\alpha}(x,t) \) to state our main results Theorem 4.4 and Theorem 5.2, which are extensions of Theorem B and Theorem B', respectively.

## 2 Preliminaries

Throughout this paper, let \( C \) denote various constants independent of the variables in question and \( C(a,b,\cdots) \) be a constant that depends on \( a, b, \cdots \).

We consider a function

\[
\Phi(x,t) = t \phi(x,t) : \mathbb{R}^N \times [0,\infty) \to [0,\infty)
\]

satisfying the following conditions (\( \Phi_1 \) – (\( \Phi_4 \)):

(\( \Phi_1 \)) \( \phi(\cdot,t) \) is measurable on \( \mathbb{R}^N \) for each \( t \geq 0 \) and \( \phi(x,\cdot) \) is continuous on \( [0,\infty) \) for each \( x \in \mathbb{R}^N \);

(\( \Phi_2 \)) there exists a constant \( A_1 \geq 1 \) such that

\[
A_1^{-1} \leq \phi(x,1) \leq A_1 \quad \text{for all} \ x \in \mathbb{R}^N;
\]

(\( \Phi_3 \)) \( \phi(x,\cdot) \) is uniformly almost increasing, namely there exists a constant \( A_2 \geq 1 \) such that

\[
\phi(x,t) \leq A_2 \phi(x,s) \quad \text{for all} \ x \in \mathbb{R}^N \quad \text{whenever} \ 0 \leq t < s;
\]
(Φ4) there exists a constant $A_3 \geq 1$ such that

$$\phi(x, 2t) \leq A_3 \phi(x, t) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t > 0.$$  

Note that (Φ2), (Φ3) and (Φ4) imply

$$0 < \inf_{x \in \mathbb{R}^N} \phi(x, t) \leq \sup_{x \in \mathbb{R}^N} \phi(x, t) < \infty$$  

for each $t > 0$.

If $\Phi(x, \cdot)$ is convex for each $x \in \mathbb{R}^N$, then (Φ3) holds with $A_2 = 1$; namely $\phi(x, \cdot)$ is non-decreasing for each $x \in \mathbb{R}^N$.

Let $\bar{\Phi}(x, t) = \sup_{0 \leq s \leq t} \phi(x, s)$ and

$$\Phi(x, t) = \int_0^t \bar{\phi}(x, r) \, dr$$

for $x \in \mathbb{R}^N$ and $t \geq 0$. Then $\bar{\Phi}(x, \cdot)$ is convex and

$$\frac{1}{2 A_3} \Phi(x, t) \leq \bar{\Phi}(x, t) \leq A_2 \Phi(x, t)$$

for all $x \in \mathbb{R}^N$ and $t \geq 0$.

By (Φ3), we see that

$$\Phi(x, at) \left\{ \begin{array}{ll}
& \leq A_2 a \Phi(x, t) & \text{if } 0 \leq a \leq 1 \\
& \geq A_2^{-1} a \Phi(x, t) & \text{if } a \geq 1.
\end{array} \right. \quad (2.1)$$

We shall also consider the following conditions:

(Φ5) for every $\gamma > 0$, there exists a constant $B_\gamma \geq 1$ such that

$$\phi(x, t) \leq B_{\gamma} \phi(y, t)$$  

whenever $|x - y| \leq \gamma t^{-1/N}$ and $t \geq 1$;

(Φ6) there exist a function $g \in L^1(\mathbb{R}^N)$ and a constant $B_\infty \geq 1$ such that $0 \leq g(x) < 1$ for all $x \in \mathbb{R}^N$ and

$$B_\infty^{-1} \phi(x, t) \leq \phi(x', t) \leq B_\infty \phi(x, t)$$

whenever $|x'| \geq |x|$ and $g(x) \leq t \leq 1$.

Example 2.1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1, \ldots, k$, be measurable functions on $\mathbb{R}^N$ such that

(P1) $1 \leq p^+ := \inf_{x \in \mathbb{R}^N} p(x) \leq \sup_{x \in \mathbb{R}^N} p(x) =: p^+ < \infty$

and

(Q1) $-\infty < q_j^+ := \inf_{x \in \mathbb{R}^N} q_j(x) \leq \sup_{x \in \mathbb{R}^N} q_j(x) =: q_j^+ < \infty$
for all \(j = 1, \ldots, k\).

Set \(L_c(t) = \log(c+t)\) for \(c \geq e\) and \(t \geq 0\), \(L_c^{(1)}(t) = L_c(t)\), \(L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))\) and

\[
\Phi(x, t) = t^{p(x)} \prod_{j=1}^{k} (L_c^{(j)}(t))^{q_j(x)}.
\]

Then, \(\Phi(x, t)\) satisfies (\(\Phi_1\)), (\(\Phi_2\)) and (\(\Phi_4\)). It satisfies (\(\Phi_3\)) if there is a constant \(K \geq 0\) such that \(K(p(x)-1) + q_j(x) \geq 0\) for all \(x \in G\) and \(j = 1, \ldots, k\); in particular if \(p^- > 1\) or \(q_j^- > 0\) for all \(j = 1, \ldots, k\).

Moreover, we see that \(\Phi(x, t)\) satisfies (\(\Phi_5\)) if

\[\text{(P2)} \quad p(\cdot) \text{ is log-Hölder continuous, namely}\]

\[
|p(x) - p(y)| \leq \frac{C_p}{L_c(1/|x-y|)}
\]

with a constant \(C_p \geq 0\) and

\[\text{(Q2)} \quad q_j(\cdot) \text{ is (j + 1)-log-Hölder continuous, namely}\]

\[
|q_j(x) - q_j(y)| \leq \frac{C_{q_j}}{L_c^{(j+1)}(1/|x-y|)}
\]

with constants \(C_{q_j} \geq 0\), \(j = 1, \ldots, k\).

Finally, we see that \(\Phi(x, t)\) satisfies (\(\Phi_6\)) with \(g(x) = 1/(1 + |x|)^{N+1}\) if \(p(\cdot)\) is log-Hölder continuous at \(\infty\), namely if it satisfies

\[\text{(P3)} \quad |p(x) - p(x')| \leq \frac{C_{p, \infty}}{L_c(|x|)} \text{ whenever } |x'| \geq |x| \text{ with a constant } C_{p, \infty} \geq 0.
\]

In fact, if \(1/(1 + |x|)^{N+1} < t \leq 1\), then \(t^{-|p(x) - p(x')|} \leq e^{(N+1)C_{\infty}}\) for \(|x'| \geq |x|\) and \(L_c^{(j)}(t)|q_j(x) - q_j(x')| \leq L_c^{(j)}(1)|q_j^+ - q_j^-|\).

**Example 2.2.** Let \(p_1(\cdot)\), \(p_2(\cdot)\), \(q_1(\cdot)\) and \(q_2(\cdot)\) be measurable functions on \(\mathbb{R}^N\) satisfying (P1) and (Q1).

Then,

\[
\Phi(x, t) = (1 + t)^{p_1(x)}(1 + 1/t)^{-p_2(x)}L_c(t)^{q_1(x)}L_c(1/t)^{-q_2(x)}
\]

satisfies (\(\Phi_1\)), (\(\Phi_2\)) and (\(\Phi_4\)). It satisfies (\(\Phi_3\)) if \(p_j^- > 1\), \(j = 1, 2\) or \(q_j^- \geq 0\), \(j = 1, 2\). As a matter of fact, it satisfies (\(\Phi_3\)) if and only if \(p_j(\cdot)\), \(q_j(\cdot)\) satisfies the following conditions:

1. \(q_j(x) \geq 0\) at points \(x\) where \(p_j(x) = 1\), \(j = 1, 2\);
2. \(\sup_{x:p_j(x)>1}\{\min(q_j(x), 0) \log(p_j(x) - 1)\} < \infty\), \(j = 1, 2\).

Moreover, we see that \(\Phi(x, t)\) satisfies (\(\Phi_5\)) if \(p_1(\cdot)\) is log-Hölder continuous and \(q_1(\cdot)\) is 2-log-Hölder continuous.

Finally, we see that \(\Phi(x, t)\) satisfies (\(\Phi_6\)) with \(g(x) = 1/(1 + |x|)^{N+1}\) if \(p_2(\cdot)\) is log-Hölder continuous at \(\infty\) and
(Q3) $q_2(\cdot)$ is 2-log-Hölder continuous at $\infty$, namely

$$|q_2(x) - q_2(x')| \leq \frac{C_{q_2,\infty}}{L^{(2)}(|x|)}$$

whenever $|x'| \geq |x|$ with a constant $C_{q_2,\infty} \geq 0$.

In fact, if $1/(1 + |x|)^{N+1} < t \leq 1$, then $(1 + t)^{|p_1(x) - p_1(x')|} \leq 2^{p_t} - 1$, $(1 + 1/t)^{|p_2(x) - p_2(x')|} \leq e^{(N+1)c_{p,\infty}}$, $(\log(e + t))^{-|q_1(x) - q_1(x')|} \leq (\log(e + 1))^{-q_t}$ and $(\log(e + 1/t))^{-|q_2(x) - q_2(x')|} \leq C(N, C_{q,\infty})$ for $|x'| \geq |x|$.

Let $\Omega$ be an open set in $\mathbb{R}^N$. Given $\Phi(x, t)$ as above, the associated Musielak-Orlicz space

$$L^\Phi(\Omega) = \left\{ f \in L^1_{\text{loc}}(\Omega) : \int_\Omega \Phi(y, |f(y)|) \, dy < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \overline{\Phi}(y, |f(y)|/\lambda) \, dy \leq 1 \right\}$$

(cf. [11]). Further, we define the Musielak-Orlicz-Sobolev space by

$$W^{1,\Phi}(\Omega) = \{ u \in L^\Phi(\Omega) : |\nabla u| \in L^\Phi(\Omega) \}.$$  

The norm

$$\|u\|_{W^{1,\Phi}(\Omega)} = \|u\|_{L^\Phi(\Omega)} + \|\nabla u\|_{L^\Phi(\Omega)}$$

makes $W^{1,\Phi}(\Omega)$ a Banach space. We denote the closure of $C^\infty_0(\Omega)$ in $W^{1,\Phi}(\Omega)$ by $W^{1,\Phi}_0(\Omega)$. As usual, let $W^{1,\Phi}_{\text{loc}}(\mathbb{R}^N)$ denote the set of functions $u$ on $\mathbb{R}^N$ such that $u|_{\Omega} \in W^{1,\Phi}(\Omega)$ for every bounded open set $\Omega$. By (\Phi2) and (\Phi3), $W^{1,\Phi}_{\text{loc}}(\mathbb{R}^N) \subset W^{1,1}_{\text{loc}}(\mathbb{R}^N)$.

3 Lemmas

We denote by $B(x, r)$ the open ball centered at $x$ of radius $r$. For a measurable set $E$, we denote by $|E|$ the Lebesgue measure of $E$.

For a locally integrable function $f$ on $\Omega$, the Hardy-Littlewood maximal function $Mf$ is defined by

$$Mf(x) = \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r) \cap \Omega} |f(y)| \, dy.$$

We know the following boundedness of maximal operator on $L^\Phi(\Omega)$.

**Lemma 3.1** ([9, Corollary 4.4]). Suppose that $\Phi(x, t)$ satisfies (\Phi5), (\Phi6) and further assume:
\((\Phi_3^*)\)  \(t \mapsto t^{-\varepsilon_0}\phi(x,t)\) is uniformly almost increasing on \((0,\infty)\) for some \(\varepsilon_0 > 0\), namely there is a constant \(A_{2,\varepsilon_0} \geq 1\) such that
\[ t^{-\varepsilon_0}\phi(x,t) \leq A_{2,\varepsilon_0}s^{-\varepsilon_0}\phi(x,s) \quad \text{for all } x \in \mathbb{R}^N \text{ whenever } 0 < t < s. \]

Then the maximal operator \(M\) is bounded from \(L^\Phi(\Omega)\) into itself, namely, there is a constant \(C > 0\) such that 
\[
\|Mf\|_{L^\Phi(\Omega)} \leq C\|f\|_{L^\Phi(\Omega)}
\]
for all \(f \in L^\Phi(\Omega)\).

For \(\lambda \geq 1, x \in \mathbb{R}^N\) and \(t \geq 0\), set 
\[
\Phi_\lambda(x,t) = \Phi(x,t^{1/\lambda}) = t\phi_\lambda(x,t),
\]
where \(\phi_\lambda(x,t) = t^{1/\lambda - 1}\phi(x,t^{1/\lambda}).\)

**Lemma 3.2.**
1. \(\Phi_\lambda(x,t)\) satisfies the conditions \((\Phi_2)\) and \((\Phi_4)\).
2. Suppose \(\Phi(x,t)\) satisfies \((\Phi_3^*)\). Then \(\Phi_\lambda(x,t)\) satisfies \((\Phi_1)\) and \((\Phi_3)\) when \(\lambda \leq 1 + \varepsilon_0\), and it satisfies \((\Phi_3^*)\) when \(\lambda < 1 + \varepsilon_0\) (with \(\varepsilon_0\) replaced by \((1 + \varepsilon_0 - \lambda)/\lambda\)).
3. If \(\Phi(x,t)\) satisfies \((\Phi_5)\), then so does \(\Phi_\lambda(x,t)\).
4. If \(\Phi(x,t)\) satisfies \((\Phi_6)\), then so does \(\Phi_\lambda(x,t)\).

**Proof.**
1. \((\Phi_2)\) for \(\Phi\) immediately implies that for \(\Phi_\lambda\). For \((\Phi_4)\), note that \(\phi_\lambda(x,2t) \leq 2^{1/\lambda - 1}A_2A_3\phi_\lambda(x,t)\).
2. The assertions of (2) follow from \((\Phi_3^*)\) and the equality
\[
\phi_\lambda(x,t) = t^{(1+\varepsilon_0)/\lambda - 1}\phi(x,t^{1/\lambda}).
\]
3. It is enough to note that \(t^{-\lambda/N} \leq t^{-1/N}\) for \(t \geq 1\).
4. It is enough to note that \(g(x) \leq g(x)^{1/\lambda}\) when \(0 \leq g(x) < 1\).

From Lemma 3.1 and the above lemma, we obtain

**Corollary 3.3.** Suppose that \(\Phi(x,t)\) satisfies \((\Phi_5)\), \((\Phi_6)\) and \((\Phi_3^*)\). Then the maximal operator \(M\) is bounded from \(L^{\Phi_\lambda}(\Omega)\) into itself for \(1 \leq \lambda < 1 + \varepsilon_0\).

Set
\[
\Phi^{-1}(x,s) = \sup\{t > 0; \Phi(x,t) < s\}
\]
for \(x \in \mathbb{R}^N\) and \(s > 0\).

**Lemma 3.4 (cf. [9, Lemma 5.1]).** \(\Phi^{-1}(x,\cdot)\) is non-decreasing,
\[
\Phi(x,\Phi^{-1}(x,t)) = t
\]
and
\[
A_2^{-1}t \leq \Phi^{-1}(x,\Phi(x,t)) \leq A_2^2t
\]
for all \(x \in \mathbb{R}^N\) and \(t > 0\).

We shall consider the following condition:
(Φ6*) Φ(x, t) satisfies (Φ6) with \( g(x) \leq (1 + |x|)^{-\beta} \) for some \( \beta > N \).

**Lemma 3.5.** If \( \Phi(x, t) \) satisfies (Φ6*), then there exists \( 0 < \lambda < 1 \) such that
\[
\Phi(x, \lambda g^*(x)) \leq (2|x|)^{-N} \quad \text{for all} \quad x \in \mathbb{R}^N,
\]
where \( g^*(x) = \max\{g(x), Mg(x)\} \).

**Proof.** Since \( g(x) \leq (1 + |x|)^{-\beta} \) with \( \beta > N \), \( Mg(x) \leq C(1 + |x|)^{-N} \), so that \( g^*(x) \leq C(1 + |x|)^{-N} \). Hence
\[
\Phi(x, \lambda g^*(x)) \leq \lambda C(1 + |x|)^{-N} A_2 \Phi(x, \lambda C) \leq 2^N \lambda C A_2 (2|x|)^{-N} \Phi(x, \lambda C).
\]
Thus, the required inequality holds if \( \lambda \leq \left(2^N C A_1 A_2^2\right)^{-1} \).

**Lemma 3.6.** \( r \mapsto r^{\sigma_0} \Phi^{-1}(x, r^{-N}) \) is uniformly almost decreasing on \((0, \infty)\), where \( \sigma_0 = N/(1 + (\log A_3)/(\log 2)) \).

**Proof.** By (Φ4), we see that
\[
\Phi^{-1} \left( x, \frac{1}{2A_3} s \right) \leq \frac{1}{2} \Phi^{-1}(x, s) \tag{3.2}
\]
for all \( x \in \mathbb{R}^N \) and \( s > 0 \). If \( 0 < \lambda < 1 \), then choosing \( k \in \mathbb{N} \) such that \((2A_3)^{-k} \leq \lambda < (2A_3)^{-k+1}\) and applying (3.2), we have
\[
\Phi^{-1}(x, \lambda s) \leq 2^{-k+1} \Phi^{-1}(x, s) \leq 2\lambda^{1/(1+\sigma)} \Phi^{-1}(x, s),
\]
where \( \sigma = (\log A_3)/(\log 2) \). Note that \( \sigma_0 = N/(1 + \sigma) \). Thus, for \( a > 1 \), we have
\[
(ar)^{\sigma_0} \Phi^{-1}(x, (ar)^{-N}) \leq (ar)^{\sigma_0} 2(a^{-N})^{1/(1+\sigma)} \Phi^{-1}(x, r^{-N})
\]
\[
= 2r^{\sigma_0} \Phi^{-1}(x, r^{-N}),
\]
which shows the assertion of the lemma.

**Lemma 3.7.** Suppose that \( \Phi(x, t) \) satisfies (Φ5) and (Φ6*). Let \( 0 < \alpha < \sigma_0 \) for \( \sigma_0 \) given in Lemma 3.6. Then there exists a constant \( C > 0 \) such that
\[
\int_{B(x,2|x|) \setminus B(x,r)} |x - y|^{\alpha - N} f(y) \, dy \leq Cr^\alpha \Phi^{-1}(x, r^{-N}) \tag{3.3}
\]
and
\[
\int_{B(x,r)} f(y) \, dy \leq Cr^N \Phi^{-1}(x, r^{-N}) \tag{3.4}
\]
for all \( x \in \mathbb{R}^N \), \( 0 < r \leq 2|x| \), and \( f \geq 0 \) satisfying \( \|f\|_{L^1(\mathbb{R}^N)} \leq 1 \).

**Proof.** Condition (ΦκJ) in [9] with \( \kappa(x, r) = r^N \) and \( J(x, r) = r^{\alpha-N} \) is satisfied by Lemma 3.6, if \( 0 < \alpha < \sigma_0 \). Hence, (3.3) follows from [9, Lemma 6.3] in view of Lemma 3.5. (3.4) follows from [9, Lemma 5.3] and Lemma 3.5.
Hereafter, let $\Omega$ is an open set in $\mathbb{R}^N$ such that $\Omega \neq \mathbb{R}^N$, and let $\delta(x) = \text{dist}(x, \partial \Omega)$.

The following is a key lemma:

**Lemma 3.8.** (1) If $\Omega$ satisfies

$$|B(z, r) \cap \Omega^c| \geq k|B(z, r)|$$  \hspace{1cm} (3.5)

for every $z \in \partial \Omega$ and $r > 0$ with a constant $k > 0$ ($k \leq 1$), then there exists a constant $C = C(N, k) > 0$ such that

$$|u(x)| \leq C \int_{B(x, 2\delta(x))} |x - y|^{1-N} |\nabla u(y)| \, dy$$

for almost every $x \in \Omega$, whenever $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^N)$ and $u = 0$ outside $\Omega$.

(2) Let $\lambda > N$. Then there exists a constant $C > 0$ such that

$$|v(x)| \leq C \left( \delta(x)^{\lambda-N} \int_{B(x, 2\delta(x))} |\nabla v(y)|^\lambda \, dy \right)^{1/\lambda}$$

for every $x \in \Omega$, whenever $v \in W^{1,\lambda}_{\text{loc}}(\mathbb{R}^N)$ and $v = 0$ outside $\Omega$.

For (1) see [10, Lemma 2.1]; for (2) see e.g. [6, (3.1)] (also cf. [2, Proposition 1]). Here note that (2) holds without the assumption (3.5).

We consider

$$H(f; x, \alpha) = \delta(x)^{\alpha-1} \int_{B(x, 2\delta(x))} |x - y|^{1-N} f(y) \, dy$$

for $x \in \Omega$, $0 \leq \alpha \leq 1$ and $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $f \geq 0$, $f = 0$ outside $\Omega$.

We know (by integration by parts)

$$H(f; x, 0) \leq CMf(x).$$ \hspace{1cm} (3.6)

for all $x \in \Omega$.

**Lemma 3.9.** Let $\Omega \neq \mathbb{R}^N$ be an open set and suppose that $\Phi(x, t)$ satisfies $(\Phi5)$ and $(\Phi6^*)$.

(1) Let $\alpha \in [0, \sigma_0) \cap [0, 1]$. Then there exists a constant $C > 0$ such that

$$H(f; x, \alpha) \leq CMf(x)\Phi(x, Mf(x))^{-\alpha/N}$$ \hspace{1cm} (3.7)

for all $x \in \Omega$ and $f \geq 0$ such that $f = 0$ outside $\Omega$ and $\|f\|_{L^\Phi(\Omega)} \leq 1$.

(2) Let $\alpha \in [0, \sigma_0]$. Then there exists a constant $C > 0$ such that

$$\delta(x)^{\alpha-N} \int_{B(x, 2\delta(x))} f(y) \, dy \leq CMf(x)\Phi(x, Mf(x))^{-\alpha/N}$$ \hspace{1cm} (3.8)

for all $x \in \Omega$ and $f \geq 0$ such that $f = 0$ outside $\Omega$ and $\|f\|_{L^\Phi(\Omega)} \leq 1$. 

Proof. We have only to consider the case $\alpha > 0$. Without loss of generality, we may assume that $0 \in \partial \Omega$, so that $\delta(x) \leq |x|$. Let $f \geq 0$ with $f = 0$ outside $\Omega$ and $\|f\|_{L^\Phi(\Omega)} \leq 1$.

(1) For $0 < r \leq \delta(x)$, we have by (3.3) in Lemma 3.7

$$H(f; x, \alpha) \leq C \left\{ \delta(x)^{\alpha-1} Mf(x) + \int_{B(x,2\delta(x)) \setminus B(x,r)} |x-y|^{\alpha-N} f(y) \, dy \right\} \leq C \left\{ r^\alpha Mf(x) + r^\alpha \Phi^{-1}(x, r^{-N}) \right\}.$$

Suppose $\Phi(x, Mf(x))^{-1/N} > \delta(x)$. Then we have by (3.6)

$$H(f; x, \alpha) = \delta(x)^\alpha H(f; x, 0) \leq C \delta(x)^\alpha Mf(x) \leq CMf(x) \Phi(x, Mf(x))^{-\alpha/N},$$

which is (3.7).

Next, if $\Phi(x, Mf(x))^{-1/N} \leq \delta(x)$, then take $r = \Phi(x, Mf(x))^{-1/N}$. Then, in view of (3.1) in Lemma 3.4, we obtain (3.7).

(2) By (3.4),

$$\delta(x)^{\alpha-N} \int_{B(x,2\delta(x))} f(y) \, dy \leq C \delta(x)^\alpha \Phi^{-1}(x, \delta(x)^{-N}).$$

If $\alpha \leq \sigma_0$, then $r \mapsto r^\alpha \Phi^{-1}(x, r^{-N})$ is uniformly almost decreasing in view of Lemma 3.6. Hence

$$\delta(x)^{\alpha-N} \int_{B(x,2\delta(x))} f(y) \, dy \leq Cr^\alpha \Phi^{-1}(x, r^{-N})$$

for $0 < r \leq \delta(x)$. Thus, by the same arguments as above we obtain (3.8). \qed

4 Hardy’s inequality I

**Lemma 4.1.** Let $\Omega \neq \mathbb{R}^N$ be an open set satisfying (3.5). Suppose $\Phi(x, t)$ satisfies (\Phi5), (\Phi6) and (\Phi3*). Then there exist constants $C > 0$ and $0 < b_0 < 1$ such that

$$\|b^{\sigma-1} u\|_{L^\Phi(\Omega)} \leq C \|b^{\sigma} |\nabla u|\|_{L^\Phi(\Omega)} \tag{4.1}$$

for all $u \in W^{1,\Phi}_0(\Omega)$ and $0 \leq b \leq b_0$. If $u \in W^{1,\Phi}_0(\Omega)$ and $b^\sigma |\nabla u| \in L^\Phi(\Omega)$ for $0 \leq b \leq b_0$, then $b^\sigma u$ extended by $0$ outside $\Omega$ belongs to $W^{1,\Phi}(\mathbb{R}^N)$.

**Proof.** Without loss of generality, we may assume that $0 \in \partial \Omega$. For $u \in W^{1,\Phi}_0(\Omega)$ and $b \geq 0$, let

$$u_b(x) = \begin{cases} \delta(x)^b u(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

We first treat $u \in C^{\infty}_0(\Omega)$. Note that $\delta$ and $1/\delta$ are bounded on support of $u$ and $\delta \in W^{1,\infty}(\Omega)$. Hence $u_b \in W^{1,\Phi}(\mathbb{R}^N) \subset W^{1,1}_{loc}(\mathbb{R}^N)$ for every $b \geq 0$. Applying Lemma 3.8 (1) to this function, we have

$$\delta(x)^b |u(x)| \leq C \int_{B(x,2\delta(x)) \cap \Omega} |x-y|^{1-N} \left\{ b \delta(y)^{b-1} |u(y)| + \delta(y)^b |\nabla u(y)| \right\} \, dy, \tag{4.2}$$
so that
\[ \delta(x)^{b-1}|u(x)| \leq C \left\{ bM(\delta^{b-1}u)(x) + M(\delta^{b}|\nabla u|)(x) \right\} \]
for a.e. \( x \in \Omega \) with a constant \( C \) independent of \( b \). In view of Lemma 3.1, we find
\[ \|\delta^{b-1}u\|_{L^s(\Omega)} \leq C_0 \left\{ b\|\delta^{b-1}u\|_{L^s(\Omega)} + \|\delta^b|\nabla u|\|_{L^s(\Omega)} \right\}, \]
which gives
\[ (1 - C_0b)\|\delta^{b-1}u\|_{L^s(\Omega)} \leq C_0\|\delta^b|\nabla u|\|_{L^s(\Omega)} \]
Hence, taking \( b_0 \) such that \( 1 - C_0b_0 > 0 \), we have (4.1) for \( 0 \leq b \leq b_0 \).

We next treat \( u \in W_{0,1}^1(\Omega) \) such that \( u = 0 \) outside \( B(0,R) \) for some \( R > 0 \). Then we can find a sequence \( \varphi_j \in C_0^\infty(\Omega) \) such that \( \varphi_j \to u \) in \( W_{0,1}^1(\Omega) \) and \( \varphi_j = 0 \) outside \( B(0,2R) \) for each \( j \). By the above discussions, for \( 0 < b \leq b_0 \), we have
\[ \|\delta^{b-1}\varphi_j\|_{L^s(\Omega)} \leq C\|\delta^b|\nabla \varphi_j|\|_{L^s(\Omega)} \]  \hspace{1cm} (4.3)
for all \( j \) and
\[ \|\delta^{b-1}(\varphi_j - \varphi_j')\|_{L^s(\Omega)} \leq C\|\delta^b|\nabla \varphi_j - \nabla \varphi_j'|\|_{L^s(\Omega)} \]  \hspace{1cm} (4.4)
for all \( j, j' \). Since \( \delta \) is bounded on \( B(0,2R) \), we see that
\[ \|\delta^b|\nabla \varphi_j|\|_{L^s(\Omega)} \to \|\delta^b|\nabla u|\|_{L^s(\Omega)} \]
as \( j \to \infty \). Similarly
\[ \|\delta^b|\nabla \varphi_j - \nabla \varphi_j'|\|_{L^s(\Omega)} \to 0 \]
as \( j, j' \to \infty \). Hence by (4.4), \( \{\delta^{b-1}\varphi_j\} \) is a Cauchy sequence in \( L^s(\Omega) \), which implies that \( \delta^{b-1}\varphi_j \to \delta^{b-1}u \) in \( L^s(\Omega) \). Thus, letting \( j \to \infty \) in (4.3), we obtain (4.1). Further, \( (\varphi_j)_b \to u_b \) in \( L^s(\mathbb{R}^N) \) and
\[ \nabla(\varphi_j)_b = \begin{cases} b\delta^{b-1}\varphi_j \nabla \delta + \delta^b \nabla \varphi_j & \text{on } \Omega \\ 0 & \text{on } \Omega^c \end{cases} \]
\[ \to \begin{cases} b\delta^{b-1}u \nabla \delta + \delta^b \nabla u & \text{on } \Omega \\ 0 & \text{on } \Omega^c \end{cases} \]
in \( L^s(\mathbb{R}^N) \) as \( j \to \infty \). It then follows that
\[ \nabla u_b = \begin{cases} b\delta^{b-1}u \nabla \delta + \delta^b \nabla u & \text{on } \Omega \\ 0 & \text{on } \Omega^c \end{cases}, \]
which belongs to \( L^s(\mathbb{R}^N) \), and hence \( u_b \in W_{0,1}^1(\mathbb{R}^N) \).

Finally we treat a general \( u \in W_{0,1}^1(\Omega) \). For each \( n \in \mathbb{N} \), we consider a \( C^1 \)-function \( H_n \) on \([0,\infty)\) such that \( 0 \leq H_n \leq 1 \) on \([0,\infty)\), \( H_n = 1 \) on \([0,n]\), \( H_n = 0 \) on \([3n,\infty)\), \( 0 \leq -H_n'(t) \leq t^{-1} \) for \( t \in (n,3n) \). The existence of such \( H_n \) is assured since \( \int_n^{3n} t^{-1} dt = \log 3 > 1 \). Set \( u_n(x) = H_n(|x|)u(x) \), \( n = 1,2,\ldots \). Then we know by the above that
\[ \|\delta^{b-1}u_n\|_{L^s(\Omega)} \leq C\|\delta^b|\nabla (u_n)|\|_{L^s(\Omega)} \]  \hspace{1cm} (4.5)
Since $\delta^{b-1}|u_n| \uparrow \delta^{b-1}|u|$ $(n \to \infty)$,

$$\|\delta^{b-1}u_n\|_{L^p(\Omega)} \to \|\delta^{b-1}u\|_{L^p(\Omega)} \quad (n \to \infty).$$

On the other hand,

$$|\nabla u_n(x)| \leq |H_n'(|x|)||u(x)| + H_n(|x|)|\nabla u(x)|$$

$$\leq \frac{1}{|x|}|u(x)|\chi_{B(0,3n)\setminus B(0,n)}(x) + |\nabla u(x)|.$$

Since $\frac{\delta(x)^b}{|x|} \leq \frac{|x|^{b-1}}{n}$ for $|x| \geq n$ and $b < 1$,

$$\delta(x)^b|\nabla u_n(x)| \leq n^{b-1}|u(x)| + \delta(x)^b|\nabla u(x)|,$$

so that

$$\|\delta^b\nabla u_n\|_{L^p(\Omega)} \leq n^{b-1}\|u\|_{L^p(\Omega)} + \|\delta^b\nabla u\|_{L^p(\Omega)}$$

$$\to \|\delta^b\nabla u\|_{L^p(\Omega)} \quad (n \to \infty).$$

Therefore, by letting $n \to \infty$ in (4.5), we obtain (4.1), which also implies that $u_b \in W^{1,\Phi}(\mathbb{R}^N)$.

For $\alpha \geq 0$, we consider a function $\Psi_\alpha(x,t) : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Ψ1) $\Psi_\alpha(\cdot,t)$ is measurable on $\mathbb{R}^N$ for each $t \geq 0$ and $\Psi_\alpha(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in \mathbb{R}^N$;

(Ψ2) $\Psi_\alpha(x,\cdot)$ is uniformly almost increasing on $[0,\infty)$, namely there is a constant $A_4 \geq 1$ such that $\Psi_\alpha(x,t) \leq A_4 \Psi_\alpha(x,s)$ for all $x \in \mathbb{R}^N$, whenever $0 \leq t < s$;

(Ψ3) there exists a constant $A_5 \geq 1$ such that

$$\Psi_\alpha(x,t\Phi(x,t)^{-\alpha/N}) \leq A_5 \Phi(x,t)$$

for all $x \in \mathbb{R}^N$ and $t > 0$.

Note that we may take $\Psi_0(x,t) = \Phi(x,t)$.

**Example 4.2.** Let $\Phi(x,t)$ be as in Example 2.1. Set

$$\Psi_\alpha(x,t) = \left( t \prod_{j=1}^{k} (L^j_{\alpha}(t))^{\eta_j(x)/p(x)} \right)^{p(x)^{\sharp}(x)},$$

where $1/p^\sharp(x) = 1/p(x) - \alpha/N$. If $0 \leq \alpha < N/p^+$, then $\Psi_\alpha$ satisfies (Ψ1), (Ψ2) and (Ψ3).

**Example 4.3.** Let $\Phi(x,t)$ be as in Example 2.2. Set

$$\Psi_\alpha(x,t) = \left( (1 + t)L_{\alpha}(t)^{q_1(x)/p_1(x)} \right)^{p_1^\sharp(x)} \left( (1 + 1/t)L_{\alpha}(1/t)^{-q_2(x)/p_2(x)} \right)^{p_2^\sharp(x)}.$$

If $0 \leq \alpha < \min \{N/p_1^+, N/p_2^+\}$, then $\Psi_\alpha$ satisfies (Ψ1), (Ψ2) and (Ψ3).
Theorem 4.4. Let \( \Omega \neq \mathbb{R}^N \) be an open set satisfying (3.5). Suppose \( \Phi(x,t) \) satisfies (\( \Phi_5 \)), (\( \Phi_3^* \)) and (\( \Phi^* \)) and let \( \alpha \in [0,\sigma_0] \cap [0,1] \) for \( \sigma_0 \) given in Lemma 3.6. Then there exist constants \( C^* > 0 \) and \( 0 < b_0 < 1 \) such that

\[
\int_\Omega \Psi_\alpha(x,\delta(x)^{\alpha+b-1}|u(x)|/C^*) \, dx \leq 1
\]

for all \( u \in W_0^{1,\Phi}(\Omega) \) with \( \|\delta^b|\nabla u|\|_{L^\Phi(\Omega)} \leq 1 \) and \( 0 \leq b \leq b_0 \).

Proof. Let \( b_0 \) be the number given in Lemma 4.1 and let \( 0 \leq b \leq b_0 \). Let \( u \in W_0^{1,\Phi}(\Omega) \) with \( \|\delta^b|\nabla u|\|_{L^\Phi(\Omega)} \leq 1 \). By Lemma 4.1, \( \delta^b u \) extended by 0 outside \( \Omega \) belongs to \( W_0^{1,1}(\mathbb{R}^N) \), so that by Lemma 3.8 (1), (4.2) holds a.e. \( x \in \Omega \). Hence

\[
\delta(x)^{\alpha+b-1}|u(x)| \leq C\delta(x)^{\alpha-1} \int_{B(x,\delta(x))} |x-y|^{1-N} f_u(y) \, dy
\]

for a.e \( x \in \Omega \), where \( f_u(y) = b\delta(y)^{b-1}|u(y)| + \delta(y)^b|\nabla u(y)| \) for \( y \in \Omega \) and \( f_u(y) = 0 \) for \( y \in \Omega^c \). By Lemma 4.1, there is a constant \( C_1 \geq 1 \) such that \( \|f_u\|_{L^\Phi(\Omega)} \leq C_1 \). Applying Lemma 3.9 (1) to \( f_u/C_1 \) and using (\( \Phi_4 \)), we have

\[
\delta(x)^{\alpha+b-1}|u(x)| \leq C_2 M f_u(x) \Phi(x, Mf_u(x))^{-a/N}
\]
a.e. \( x \in \Omega \). Hence by (\( \Phi_2 \)) and (\( \Phi_3 \)) we have

\[
\int_\Omega \Psi_\alpha(x,\delta(x)^{\alpha+b-1}|u(x)|/C_2) \, dx \leq A_4 A_5 \int_\Omega \Phi(x, Mf_u(x)) \, dx \tag{4.6}
\]

whenever \( \|\delta^b|\nabla u|\|_{L^\Phi(\Omega)} \leq 1 \). By Lemma 3.1, \( \|M f_u\|_{L^\Phi(\Omega)} \leq C_3 \), which implies \( \int_\Omega \Phi(x, Mf_u(x)) \, dx \leq C_4 (C_4 \geq 1) \).

Now let \( 0 < \varepsilon \leq 1 \). Since

\[
\Phi(x, Mf_{\varepsilon u}(x)) = \Phi(x, \varepsilon Mf_u(x)) \leq A_2 \varepsilon \Phi(x, Mf_u(x))
\]

by (2.1), applying (4.6) to \( \varepsilon u \), we have

\[
\int_\Omega \Psi_\alpha(x,\delta(x)^{\alpha+b-1}|u(x)|/C_2) \, dx \leq A_4 A_5 \int_\Omega \Phi(x, Mf_{\varepsilon u}(x)) \, dx
\]

\[
\leq A_2 A_4 A_5 \varepsilon \int_\Omega \Phi(x, Mf_u(x)) \, dx \leq A_2 A_4 A_5 C_4 \varepsilon.
\]

Thus, taking \( \varepsilon = (A_2 A_4 A_5 C_4)^{-1} \) and \( C^* = C_2/\varepsilon \), we obtain the required result. \( \square \)

Applying Theorem 4.4 to special \( \Phi \) and \( \Psi_\alpha \) given in Examples 2.1 and 4.2, we obtain the following corollary, which is an extension of Theorem B.

Corollary 4.5. Let \( \Phi \) and \( \Psi_\alpha \) be as in Examples 2.1 and 4.2 and let \( \Omega \neq \mathbb{R}^N \) be an open set satisfying (3.5). Suppose \( p^- > 1 \) and let \( \alpha \in [0,N/p^+] \cap [0,1] \). Then there exist constants \( C > 0 \) and \( 0 < b_0 < 1 \) such that

\[
\|\delta^{\alpha+b-1} u\|_{L^\Psi_\alpha(\Omega)} \leq C\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)}
\]

for all \( u \in W_0^{1,\Phi}(\Omega) \) and \( 0 \leq b \leq b_0 \).
Similarly, applying Theorem 4.4 to special \( \Phi \) and \( \Psi_\alpha \) given in Examples 2.2 and 4.3, we obtain another extension of Theorem B:

**Corollary 4.6.** Let \( \Phi \) and \( \Psi_\alpha \) be as in Examples 2.2 and 4.3 and let \( \Omega \neq \mathbb{R}^N \) be an open set satisfying \((3.5)\). Suppose \( \min(p_1^-, p_2^-) > 1 \) and let \( \alpha \in [0, \min(N/p_1^-, N/p_2^-)) \cap [0, 1] \). Then there exist constants \( C > 0 \) and \( 0 < b_0 < 1 \) such that

\[
\|\delta^{\alpha+b-1}u\|_{L^{\Phi_\alpha}(\Omega)} \leq C\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)}
\]

for all \( u \in W_0^{1, \Phi}(\Omega) \) and \( 0 \leq b \leq b_0 \).

**5 Hardy’s inequality II**

For a proof of next theorem, we prepare the following lemma instead of Lemma 4.1.

**Lemma 5.1.** Let \( \Omega \neq \mathbb{R}^N \) be an open set. Suppose that \( \Phi(x, t) \) satisfies \((\Phi5), (\Phi6)\) and \((\Phi3^*)\) for \( \varepsilon > N - 1 \). Then there exist constants \( C > 0 \) and \( 0 < b_1 < 1 \) such that

\[
\|\delta^{-1}u\|_{L^{\Phi}(\Omega)} \leq C\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)}
\]

for all \( u \in W_0^{1, \Phi}(\Omega) \) and \( 0 \leq b \leq b_1 \). If \( u \in W_0^{1, \Phi}(\Omega) \) and \( \delta^b|\nabla u| \in L^\Phi(\Omega) \) for \( 0 \leq b \leq b_1 \), then \( \delta^b u \) extended by 0 outside \( \Omega \) belongs to \( W_0^{1, \Phi}(\mathbb{R}^N) \).

**Proof.** Take \( \lambda \) such that \( N < \lambda < \varepsilon_0 + 1 \). Then \( W_0^{1, \Phi}(\mathbb{R}^N) \subset W_0^{1, \lambda}(\mathbb{R}^N) \).

First, let \( u \in C_0^\infty(\Omega) \) and \( b \geq 0 \). Let \( u_b \) be the function \( \delta^b u \) extended by 0 outside \( \Omega \). Then \( u_b \in W_0^{1, \Phi}(\mathbb{R}^N) \subset W_0^{1, \lambda}(\mathbb{R}^N) \) and applying Lemma 3.8 (2) to \( v = u_b \), we have

\[
|\delta(x)^{b-1}|u(x)||^\lambda \leq C\delta(x)^{-N}\int_{B(x, 2\delta(x)) \cap \Omega} f_u(y) \, dy \leq CM f_u(x) \quad (5.1)
\]

for all \( x \in \Omega \), where \( f_u(y) = [b\delta(y)^{b-1}|u(y)| + \delta(y)^b|\nabla u(y)|]^\lambda \). In view of Corollary 3.3, we find

\[
\|\delta^{-1}u\|_{L^{\Phi_\lambda}(\Omega)} \leq C\|\delta^b|\nabla u|\|_{L^\Phi(\Omega)},
\]

Since \( \|f\|_{L^{\Phi_\lambda}(\Omega)} = \|f^{1/\lambda}\|_{L^{\Phi}(\Omega)} \) for every \( f \in L^{\Phi_\lambda}(\Omega) \), we obtain

\[
\|\delta^{-1}u\|_{L^{\Phi}(\Omega)} \leq C^{1/\lambda}\|f_u^{1/\lambda}\|_{L^{\Phi}(\Omega)} \leq C_1 \{ b\|\delta^{b-1}u\|_{L^{\Phi}(\Omega)} + \|\delta^b|\nabla u|\|_{L^\Phi(\Omega)} \},
\]

which gives

\[
(1 - C_1 b)\|\delta^{-1}u\|_{L^{\Phi}(\Omega)} \leq C_1 \|\delta^b|\nabla u|\|_{L^\Phi(\Omega)}.
\]

Take \( b_1 \) such that \( 1 - C_1 b_1 > 0 \). Then, in the same way as the last half of the proof of Lemma 4.1, we obtain the required results for \( u \in W_0^{1, \Phi}(\Omega) \) and \( 0 \leq b \leq b_1 \).

\[\square\]
THEOREM 5.2. Let Ω ≠ ℝ^N be an open set. Suppose Φ(x, t) satisfies (Φ5), (Φ6^*) and (Φ3^*) with ε_0 > N - 1. Let α ∈ [0, σ_0]. Then there exist C^* > 0 and 0 < b_1 < 1 such that

\[ \int_{Ω} Ψ_α(x, δ(x)^{α+b-1} |u(x)| / C^*) dx \leq 1 \]

for all u ∈ W^{1,Φ}_0(Ω) with \( \|δ^b|\nabla u|\|_{L^p(Ω)} \leq 1 \) and 0 ≤ b ≤ b_1.

Proof. Let b_1 be as in the above lemma and let 0 ≤ b ≤ b_1. Let u ∈ W^{1,Φ}_0(Ω) with \( \|δ^b|\nabla u|\|_{L^p(Ω)} \leq 1 \). Take λ such that N < λ < ε_0 + 1. By the above lemma, δ^b u extended by 0 outside Ω belongs to W^{1,Λ}_{loc}(ℝ^N), so that by (5.1) we have

\[ [δ(x)^{α+b-1}|u(x)|]^λ \leq Cδ(x)^{α-λN} \int_{B(x,2δ(x))} f_u(y) dy \]

for all x ∈ Ω, where f_u(y) = |bδ(y)^{b-1}|u(y)| + δ(y)^b|\nabla u(y)| for y ∈ Ω and f_u(y) = 0 for y ∈ Ω^c. By Lemma 5.1, there is a constant C_1 ≥ 1 such that \( \|f_u^{1/λ}\|_{L^p(Ω)} \leq C_1 \), so that \( \|f_u\|_{L^p(Ω)} \leq C_1^λ \).

Here we note that Φ_λ(x, t) satisfies (Φ6^*) with g^λ in place of g and that r → r^{λσ_0}Φ_1^{-}(x, r^{-N}) is uniformly almost decreasing on (0, ∞). Since λα ∈ [0, λσ_0], we can apply Lemma 3.9 (2) to \( f_u / C_1^λ \), λα and Φ_λ in place of f, α and Φ respectively, and using (Φ4), we obtain

\[ δ(x)^{α+b-1}|u(x)| \leq C[Mf_u(x)]^{1/λ}Φ_λ(x, Mf_u(x) / C_1^λ)^{-α/N} \leq C_2[Mf_u(x)]^{1/λ}Φ(x, [Mf_u(x)]^{1/λ})^{-α/N} \]

for all x ∈ Ω. Hence by (Ψ2) and (Ψ3)

\[ \int_{Ω} Ψ_α(x, δ(x)^{α+b-1} |u(x)| / C_2) dx \leq A_4A_5 \int_{Ω} Φ(x, [Mf_u(x)]^{1/λ}) dx = A_4A_5 \int_{Ω} Φ_λ(x, Mf_u(x)) dx. \] (5.2)

By Corollary 3.3, \( \|Mf_u\|_{L^p(Ω)} \leq C_3 \), which implies \( \int_{Ω} Φ_λ(x, Mf_u(x)) dx \leq C_4 \).

Let 0 < ε ≤ 1. Since

\[ Φ_λ(x, Mf_{εu}(x)) = Φ_λ(x, ε^λ Mf_u(x)) = Φ(x, ε[Mf_u(x)]^{1/λ}) \leq A_2εΦ(x, [Mf_u(x)]^{1/λ}) = A_2εΦ_λ(x, Mf_u(x)) \]

by (2.1), applying (5.2) to εu, we have

\[ \int_{Ω} Ψ_α(x, δ(x)^{α+b-1} |εu(x)| / C_2) dx \leq A_4A_5 \int_{Ω} Φ_λ(x, Mf_{εu}(x)) dx \leq A_2A_4A_5ε \int_{Ω} Φ_λ(x, Mf_u(x)) dx \leq A_2A_4A_5C_4ε. \]

Thus, taking ε = (A_2A_4A_5C_4)^{-1} and C^* = C_2 / ε, we obtain the required result.

□
Applying Theorem 5.2 to special $\Phi$ and $\Psi_\alpha$ given in Examples 2.1 and 4.2, we obtain the following corollary, which is an extension of Theorem B’.

**Corollary 5.3.** Let $\Phi$ and $\Psi_\alpha$ be as in Examples 2.1 and 4.2. Suppose $p^- > N$ and let $0 \leq \alpha < N/p^+$. Then there exist constants $C > 0$ and $0 < b_1 < 1$ such that

$$\|\delta^{\alpha+b^1-1}u\|_{L^{\Psi_\alpha}(\Omega)} \leq C\|\delta^b|\nabla u|\|_{L^{\Psi}(\Omega)}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ and $0 \leq b \leq b_1$.

Similarly, applying Theorem 5.2 to special $\Phi$ and $\Psi_\alpha$ given in Examples 2.2 and 4.3, we obtain another extension of Theorem B’:

**Corollary 5.4.** Let $\Phi$ and $\Psi_\alpha$ be as in Examples 2.2 and 4.3. Suppose $\min(p^-_1, p^-_2) > N$ and let $0 \leq \alpha < \min(N/p^+_1, N/p^+_2)$. Then there exist constants $C > 0$ and $0 < b_1 < 1$ such that

$$\|\delta^{\alpha+b^1-1}u\|_{L^{\Psi_\alpha}(\Omega)} \leq C\|\delta^b|\nabla u|\|_{L^{\Psi}(\Omega)}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ and $0 \leq b \leq b_1$.

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