# COMPACT EMBEDDINGS FOR SOBOLEV SPACES OF VARIABLE EXPONENTS AND EXISTENCE OF SOLUTIONS FOR NONLINEAR ELLIPTIC PROBLEMS INVOLVING THE p(x)-LAPLACIAN AND ITS CRITICAL EXPONENT

YOSHIHIRO MIZUTA, TAKAO OHNO, TETSU SHIMOMURA AND NAOKI SHIOJI

Abstract. We give a sufficient condition for the compact embedding from $W_0^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ in case $\operatorname{ess\,inf}_{x\in\Omega}(Np(x)/(N-kp(x))-q(x))=0$, where $\Omega$ is a bounded open set in $\mathbb{R}^N$. As an application, we find a nontrivial nonnegative weak solution of the nonlinear elliptic equation

$$-\mathrm{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right)=|u(x)|^{q(x)-2}u(x) \quad \text{in } \Omega, \qquad u(x)=0 \quad \text{on } \partial\Omega.$$

We also consider the existence of a weak solution to the problem above even if the embedding is not compact.

## 1. Introduction

In recent years, many authors have studied the generalized Lebesgue spaces; see [2, 5, 8–23, 26–29, 32]. First, let us recall some definitions. Following Orlicz [29] and Kovácik and Rákosník [22], for an open set $\Omega$ in $\mathbb{R}^N$ with $N \geq 1$ and a measurable function $p(\cdot):\Omega\to[1,\infty)$, we define the $L^{p(\cdot)}(\Omega)$-norm of a measurable function $f$ on $\Omega$ by

$$\|f\|_{L^{p(\cdot)}(\Omega)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{f(x)}{\lambda}\right|^{p(x)}dx\leq 1\right\}$$

and denote by $L^{p(\cdot)}(\Omega)$ the family of all measurable functions whose $L^{p(\cdot)}(\Omega)$-norms are finite. Further we denote by $W^{k,p(\cdot)}(\Omega)$ with $k\in\mathbb{N}$ the family of all measurable functions $u$ on $\Omega$ such that

$$\|u\|_{W^{k,p(\cdot)}(\Omega)}=\sum_{0\leq|\alpha|\leq k}\|D^\alpha u\|_{L^{p(\cdot)}(\Omega)}<\infty$$

and by $W_0^{k,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$.

Recently, Kurata and the fourth author [23] posed the following problem: if a variable exponent $q(\cdot)$ satisfies $2<\operatorname{ess\,inf}_{x\in\Omega}q(x)\leq\operatorname{ess\,sup}_{x\in\Omega}q(x)\leq 2N/(N-2)$ $(N\geq 3)$ and $q(\cdot)$ is equal to $2N/(N-2)$ at a point, then does the problem

$$(1.1)\qquad -\Delta u(x)=|u(x)|^{q(x)-2}u(x) \quad \text{in } \Omega \quad \text{and} \quad u(x)=0 \quad \text{on } \partial\Omega$$

2000 *Mathematics Subject Classification.* 35J20, 46B50, 46E30.

*Key words and phrases.* Sobolev spaces of variable exponents, compact embeddings, nonlinear elliptic problems.

The fourth author was supported in part by Grant-in-Aid for Scientific Research (C)(No.17540149) Japan Society for Promotion of Science.

have a positive solution? When $q(\cdot)$ is a constant, problem (1.1) has been studied by many researchers. If $q(\cdot)$ is a constant smaller than $2N/(N-2)$, then the embedding from $W_0^{1,2}(\Omega)$ to $L^q(\cdot)(\Omega)$ is compact, and hence the existence of a positive solution to (1.1) is easily obtained by the standard mountain pass theorem. When $q(\cdot) \equiv 2N/(N-2)$, problem (1.1) is quite interesting. If $\Omega$ is star-shaped, then Pohozaev [31] showed that there is no solution. If $\Omega$ has a nontrivial topology in the sense of $\mathbb{Z}_2$-homology, then Bahri and Coron [3] showed that the problem has a positive solution; see also [7]. Even if $\Omega$ is contractible, then, under some condition on the shape of $\Omega$, Passaseo [30] obtained a positive solution. In the case when $q(\cdot)$ is a variable exponent and $q(\cdot)$ coincides with $2N/(N-2)$ at a point in $\Omega$, since the embedding of $W_0^{1,2}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ may not be compact, the existence of positive solution to (1.1) is not trivial. Kurata and the fourth author showed that if there exist $x_0 \in \Omega$, $C_0 > 0$, $\eta > 0$ and $0 < l < 1$ such that $\operatorname{ess\,sup}_{x\in\Omega\setminus B_\eta(x_0)} q(x) < 2N/(N-2)$ and

$$q(x) \le \frac{2N}{N-2} - \frac{C_0}{(\log(1/|x-x_0|))^l} \quad \text{for almost every } x \in \Omega \cap B_\eta(x_0), \tag{1.2}$$

then the embedding from $W_0^{1,2}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact; see [23, Theorem 2]. As an application of the compact embedding, they obtained a positive solution to (1.1).

Our first aim in this paper is to establish the compact embedding from $W_0^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ when $q(\cdot)$ is an exponent satisfying a condition weaker than (1.2). As an application, we show the existence of a nontrivial nonnegative weak solution to the nonlinear elliptic equation

$$\begin{cases} -\operatorname{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right) = |u(x)|^{q(x)-2}u(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here $u$ is called a weak solution of (1.3) if $u \in W_0^{1,p(\cdot)}(\Omega)$ and

$$\int_\Omega \left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x) - |u(x)|^{q(x)-2}u(x)v(x)\right) dx = 0$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$. Our final goal is to find nontrivial nonnegative weak solutions to (1.3), even if the embedding might not be compact.

## 2. Preliminaries

Throughout this paper, we use the symbol $C$ to denote various positive constants independent of the variables in question. We only use $N$ as the dimension of the Euclidean space $\mathbb{R}^N$ and we set $B_r(x) = \{y \in \mathbb{R}^N : |y-x| < r\}$ for $x \in \mathbb{R}^N$ and $r > 0$. For a measurable subset $E$ of $\mathbb{R}^N$, we denote by $|E|$ the Lebesgue measure of $E$. For a measurable function $u$, we set $u^+ = \max\{u, 0\}$. Unless otherwise stated, we assume that $N \ge 2$ and $\Omega$ is a bounded open set in $\mathbb{R}^N$.

A measurable function $p(\cdot) : \Omega \to [1,\infty)$ is called a variable exponent on $\Omega$. We set

$$p_* = \operatorname*{ess\,inf}_{x\in\Omega} p(x) \quad \text{and} \quad p^* = \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$

It is worth noting the next result, which follows readily from the definition of $L^{p(\cdot)}$-norm (see [17, Theorem 1.3]).

**Lemma 2.1.** *If $p(\cdot)$ is a variable exponent on $\Omega$ satisfying $1 \le p_* \le p^* < \infty$, then*

$$\min\left\{\|u\|^{p_*}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^*}_{L^{p(\cdot)}(\Omega)}\right\} \le \int_\Omega |u(x)|^{p(x)}dx \le \max\left\{\|u\|^{p_*}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^*}_{L^{p(\cdot)}(\Omega)}\right\}.$$

A variable exponent $p(\cdot)$ is said to satisfy the log-Hölder condition on $\Omega$ if

$$|p(x)-p(y)| \le \frac{C}{\log(1/|x-y|)} \quad \text{for each } x, y \in \Omega \text{ with } |x-y| < \frac{1}{2},$$

where $C$ is a positive constant. We set

$$p^\sharp_k(x) = \begin{cases} Np(x)/(N-kp(x)) & \text{if } 1 \le p(x) < N/k, \\ \infty & \text{if } p(x) \ge N/k \end{cases}$$

for each $k \in \mathbb{N}$.

We know the following Sobolev inequality for functions in $W^{1,p(\cdot)}_0(\Omega)$; see [20, Proposition 4.2 (1)].

**Lemma 2.2.** *Let $p(\cdot)$ be a variable exponent on $\Omega$ satisfying the log-Hölder condition and $1 \le p_* \le p^* < \infty$. If $p^* < N$, then there exists a constant $C > 0$ such that*

$$\|u\|_{L^{p^\sharp_1(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

*for $u \in W^{1,p(\cdot)}_0(\Omega)$.*

**Corollary 2.3.** *Let $p(\cdot)$ be as in the previous lemma. If $p^* < N/k$ with $k \in \mathbb{N}$, then there exists a constant $C > 0$ such that*

$$\|u\|_{L^{p^\sharp_k(\cdot)}(\Omega)} \le C\sum_{|\alpha|=k}\|D^\alpha u\|_{L^{p(\cdot)}(\Omega)}$$

*for $u \in W^{k,p(\cdot)}_0(\Omega)$.*

*Proof.* Assume $p^* < N/k$ with $k \in \mathbb{N}$. Let $u \in W^{k,p(\cdot)}_0(\Omega)$ and let $\ell$ be a positive integer with $\ell \le k$. Then we see from Lemma 2.2 that $u \in W^{k-\ell,p^\sharp_\ell(\cdot)}_0(\Omega)$, so that

$$\|D^\alpha u\|_{L^{p^\sharp_\ell(\cdot)}(\Omega)} \le C\sum_{|\beta|=k-\ell+1}\|D^\beta u\|_{L^{p^\sharp_{\ell-1}(\cdot)}(\Omega)}$$

for $|\alpha| = k-\ell$, where $p^\sharp_0(x) = p(x)$. This proves the required result. □

## 3. Compact embeddings

In this section, we assume that $p(\cdot)$ is a variable exponent on $\Omega$ satisfying the log-Hölder condition and $1 \le p_* \le p^* < \infty$. For a set $K$ in $\mathbb{R}^N$, we define

$$K(r) = \{x \in \mathbb{R}^N : \delta_K(x) \le r\} \quad \text{for } r > 0,$$

where $\delta_K(x)$ denotes the distance of $x$ to $K$.

First, as in [23], we show the following noncompact embedding from $W^{k,p(\cdot)}_0(\Omega)$ to $L^{q(\cdot)}(\Omega)$.

**Proposition 3.1.** *Let $x_0 \in \Omega$ and $k \in \mathbb{N}$, and let $q(\cdot) : \Omega \to [1,\infty)$ be a variable exponent on $\Omega$ such that there exist $C > 0$ and $\eta > 0$ satisfying*

$$q(x) \ge p^\sharp_k(x) - \frac{C}{\log(1/|x-x_0|)} \quad \textit{for almost every } x \in \Omega \cap B_\eta(x_0). \tag{3.1}$$

*If $p(x_0) < N/k$, then the embedding from $W^{k,p(\cdot)}_0(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is not compact.*

*Proof.* Assume $p(x_0) < N/k$. We may assume that $x_0 = 0$ and $B_1(0) \subset \Omega$. Let $\psi \in C_0^\infty(\mathbb{R})$ be a function such that $0 \le \psi(r) \le 1, \psi(r) = 0$ for $r > 1$ and $\psi(r) = 1$ for $0 \le r < 1/2$. Set

$$\psi_n(x) = n^{N/p_k^\sharp(0)}\psi(n|x|)$$

for each $n \in \mathbb{N}$. Then, for $n \ge 2$ and $0 \le |\alpha| \le k$, we note

$$\begin{aligned}\int_\Omega |D^\alpha \psi_n(x)|^{p(x)} dx &\le C \int_{B_{1/n}(0)} n^{(N/p_k^\sharp(0)+|\alpha|)p(x)} dx \\ &\le C n^{(N/p_k^\sharp(0)+|\alpha|)(p(0)+C/\log n)} \int_{B_{1/n}(0)} dx \le C\end{aligned}$$

by the log-Hölder condition on $p(\cdot)$. Using (3.1), we have

$$\begin{aligned}\int_\Omega |\psi_n(x)|^{q(x)} dx &\ge \int_{B_{1/(2n)}(0)} n^{Nq(x)/p_k^\sharp(0)} |\psi(n|x|)|^{q(x)} dx \\ &\ge C n^N \int_{B_{1/(2n)}(0)} dx = C > 0,\end{aligned}$$

which implies that the embedding from $W_0^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is not compact since $\int_\Omega |\psi_n(x)|^{p(x)} dx \to 0$ as $n \to \infty$. □

As a direct consequence, we have the following result:

**Corollary 3.2.** *Let $K$ be a set in $\mathbb{R}^N$, and let $x_0 \in K \cap \Omega$ and $k \in \mathbb{N}$. Let $q(\cdot) : \Omega \to [1, \infty)$ be a variable exponent on $\Omega$ such that there exist $C > 0$ and $r > 0$ satisfying*

$$q(x) \ge p_k^\sharp(x) - \frac{C}{\log(1/\delta_K(x))} \quad \text{for almost every } x \in K(r) \cap \Omega.$$

*If $p(x_0) < N/k$, then the embedding from $W_0^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is not compact.*

*Proof.* Assume $p(x_0) < N/k$. Since $\delta_K(x) \le |x - x_0|$ for each $x \in \mathbb{R}^N$, we obtain the conclusion by the previous proposition. □

For the compact embeddings, we first give the following result.

**Proposition 3.3.** *Assume that $p^* < N/k$ with some $k \in \mathbb{N}$. Let $q(\cdot)$ be a variable exponent on $\Omega$ such that $1 \le q_*$ and*

$$\operatorname*{ess\,inf}_{x \in \Omega} \left(p_k^\sharp(x) - q(x)\right) > 0. \tag{3.2}$$

*Then the following hold.*

(i) *The embedding of $W_0^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact.*
(ii) *If $\Omega$ satisfies the cone condition, then the embedding of $W^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact.*

The case (i) in the proposition is essentially a special case of [22, Theorem 3.8]; the case (ii) is a slight generalization of [14, Theorem 1.3] to the case $1 \le p_*$.

*Proof of Proposition 3.3.* We only give a proof of (ii), since (i) can be proved similarly. Assume that $\Omega$ satisfies the cone condition. By (3.2), take $\varepsilon > 0$ such that $p_k^\sharp(x) - q(x) > 2\varepsilon > 0$ for almost every $x \in \Omega$. Since $p(\cdot)$ is uniformly continuous on

$\Omega$, one can find open balls $\{B_j\}_{j=1}^l$ and $\{\tilde{B}_j\}_{j=1}^l$ with $l \in \mathbb{N}$ such that $\overline{\Omega} \subset \bigcup_{i=1}^l B_i$, $\overline{B}_j \subset \tilde{B}_j$ and

$$\inf_{x \in \tilde{B}_j \cap \Omega} p_k^\sharp(x) - \varepsilon \geq \sup_{x \in \tilde{B}_j \cap \Omega} p_k^\sharp(x) - 2\varepsilon \geq \operatorname*{ess\,sup}_{x \in \tilde{B}_j \cap \Omega} q(x) \quad \text{for each } j = 1, \dots, l.$$

Setting $p_j = \inf_{x \in \tilde{B}_j \cap \Omega} p(x)$ and $q_j = \operatorname{ess\,sup}_{x \in \tilde{B}_j \cap \Omega} q(x)$, we see that $q_j < Np_j/(N - kp_j)$ and the embedding from $\{u \in W^{k,p(\cdot)}(\Omega) : u = 0 \text{ on } \Omega \setminus \tilde{B}_j\}$ to $W^{k,p_j}(\Omega)$ and the embedding from $\{u \in L^{q_j}(\Omega) : u = 0 \text{ on } \Omega \setminus \tilde{B}_j\}$ to $L^{q(\cdot)}(\Omega)$ are continuous. By the Rellich-Kondrachov theorem (see [1, Theorem 6.3]), $W^{k,p_j}(\Omega)$ is compactly embedded into $L^{q_j}(\Omega)$. Now, take $\varphi_j \in C^1(\Omega; [0,1])$ such that $|\nabla \varphi_j| \leq C$ on $\Omega$, $\varphi_j = 1$ on $\Omega \cap B_j$ and $\varphi_j = 0$ on $\Omega \setminus \tilde{B}_j$. It is easy to see that the linear operator $u \mapsto \varphi_j u$ is continuous on $W^{k,p(\cdot)}(\Omega)$. Noting $\varphi_j u = 0$ on $\Omega \setminus \tilde{B}_j$ for each $u \in W^{k,p(\cdot)}(\Omega)$, we can infer that $\{\varphi_j u : u \in W^{k,p(\cdot)}(\Omega)\}$ is compactly embedded into $L^{q(\cdot)}(\Omega)$. Passing to subsequences repeatedly, we obtain the conclusion. □

For a compact set $K$ in $\mathbb{R}^N$ and $s \in [0, N]$, following Mattila [25], we say that the $(N-s)$-dimensional upper Minkowski content of $K$ is finite if

$$|K(r)| \leq Cr^s \quad \text{for small } r > 0.$$

Now we are concerned with the compact embedding from $W_0^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ when $q(\cdot)$ and $p_k^\sharp(\cdot)$ coincides on some part of $\Omega$.

**Theorem 3.4.** *Let $\varphi(\cdot) \colon [1/r_0, \infty) \to (0, \infty)$ be a continuous function such that*

(i) *$\varphi(r)/\log r$ is nonincreasing on $[1/r_0, \infty)$,*
(ii) *$\varphi(r) \to \infty$ as $r \to \infty$*

*for some $r_0 \in (0, 1/e)$. Let $K$ be a compact set in $\mathbb{R}^N$ whose $(N-s)$-dimensional upper Minkowski content is finite for some $s$ with $0 < s \leq N$. Let $k \in \mathbb{N}$ and let $q(\cdot)$ be a variable exponent on $\Omega$ such that*

(iii) $1 \leq q_* \leq q^* < \infty$,
(iv) $\operatorname{ess\,inf}_{\Omega \setminus K(r_0)} \left( p_k^\sharp(x) - q(x) \right) > 0$,
(v) $q(x) \leq p_k^\sharp(x) - \dfrac{\varphi(1/\delta_K(x))}{\log(1/\delta_K(x))}$ *for almost every $x \in K(r_0) \cap \Omega$.*

*Then the embedding from $W_0^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact.*

*Proof.* Without loss of generality, we may assume $\varphi(r)/\log r \to 0$ as $r \to \infty$; otherwise, we have $\operatorname{ess\,inf}_{x \in \Omega}(p_k^\sharp(x) - q(x)) > 0$, so that the conclusion follows from Proposition 3.3 (i).

First, consider the case $p^* < N/k$. Let us prove that

$$\lim_{\varepsilon \to +0} \sup \left\{ \int_{K(\varepsilon) \cap \Omega} |v(x)|^{q(x)} dx : v \in W_0^{k,p(\cdot)}(\Omega), \|v\|_{W^{k,p(\cdot)}(\Omega)} \leq 1 \right\} = 0. \tag{3.3}$$

For this purpose, take $\beta$ with $0 < \beta < s/(p^*)_k^\sharp$. Let $\varepsilon > 0$ such that $\varepsilon^{-1} > 1/r_0$ and $\varphi(1/\varepsilon) \geq 1$. We set $\eta_n = \varepsilon^{-\beta n}$ for each $n \in \mathbb{N}$. Then, by the assumptions on $\varphi$, we have for each $n \in \mathbb{N}$ and $x \in (K(\varepsilon^n) \setminus K(\varepsilon^{n+1})) \cap \Omega$,

$$\begin{aligned} \eta_n^{q(x) - p_k^\sharp(x)} &\leq \eta_n^{-\frac{\varphi(1/\delta_K(x))}{\log(1/\delta_K(x))}} \leq \eta_n^{-\frac{\varphi(1/\varepsilon^{n+1})}{\log(1/\varepsilon^{n+1})}} \\ &= \exp(-(\beta n/(n+1))\varphi(1/\varepsilon^{n+1})) \equiv A_n. \end{aligned}$$

Since

$$|K(r) \cap \Omega| \le Cr^s \quad \text{for all } r > 0$$

by the boundedness of $\Omega$, we have

$$\int_{(K(\varepsilon^n)\setminus K(\varepsilon^{n+1}))\cap\Omega} \eta_n^{q(x)} dx \le \eta_n^{(p^*)^\sharp_k} \int_{K(\varepsilon^n)\cap\Omega} dx \le C\varepsilon^{n(s-\beta(p^*)^\sharp_k)}.$$

Hence we have

$$\begin{aligned}
&\int_{(K(\varepsilon^n)\setminus K(\varepsilon^{n+1}))\cap\Omega} |v(x)|^{q(x)} dx \\
&\le \int_{(K(\varepsilon^n)\setminus K(\varepsilon^{n+1}))\cap\Omega} |v(x)|^{q(x)} \left(\frac{|v(x)|}{\eta_n}\right)^{p^\sharp_k(x)-q(x)} dx + \int_{(K(\varepsilon^n)\setminus K(\varepsilon^{n+1}))\cap\Omega} \eta_n^{q(x)} dx \\
&\le A_n \int_{(K(\varepsilon^n)\setminus K(\varepsilon^{n+1}))\cap\Omega} |v(x)|^{p^\sharp_k(x)} dx + C\varepsilon^{n(s-\beta(p^*)^\sharp_k)},
\end{aligned}$$

so that for each $n_0 \in \mathbb{N}$, we obtain

$$\begin{aligned}
\int_{K(\varepsilon^{n_0})\cap\Omega} |v(x)|^{q(x)} dx &= \sum_{n=n_0}^{\infty} \int_{(K(\varepsilon^n)\setminus K(\varepsilon^{n+1}))\cap\Omega} |v(x)|^{q(x)} dx \\
&\le \left(\sup_{n\ge n_0} A_n\right) \int_\Omega |v(x)|^{p^\sharp_k(x)} dx + C \sum_{n=n_0}^{\infty} \varepsilon^{n(s-\beta(p^*)^\sharp_k)}.
\end{aligned}$$

Since $A_n \to 0$ as $n \to \infty$, $s - \beta(p^*)^\sharp_k > 0$ and $\|v\|_{L^{p^\sharp_k(\cdot)}(\Omega)} \le C\|v\|_{W^{k,p(\cdot)}(\Omega)}$ for all $v \in W_0^{k,p(\cdot)}(\Omega)$ by Corollary 2.3, (3.3) is obtained by letting $n_0 \to \infty$.

Let $\{v_j\}$ be a bounded sequence in $W_0^{k,p(\cdot)}(\Omega)$. We may assume that it converges weakly to some $v \in W_0^{k,p(\cdot)}(\Omega)$. By Proposition 3.3 (ii), the embedding from $W^{k,p(\cdot)}(B)$ to $L^{q(\cdot)}(B)$ is compact for each ball $B \subset \Omega$ such that $\operatorname{ess\,inf}_{x\in B}(p^\sharp_k(x) - q(x)) > 0$. Let $n \in \mathbb{N}$. Since $\Omega \setminus K(2^{-n})$ is a bounded open set in $\mathbb{R}^N$, there exists a finite family of balls contained in $\mathbb{R}^N \setminus K(2^{-n-1})$ whose union contains $\Omega \setminus K(2^{-n})$. Since $\operatorname{ess\,inf}_{x\in\Omega\setminus K(2^{-n-1})}(p^\sharp_k(x) - q(x)) > 0$, we can find a subsequence $\{v_{j_{k,n}}\}$ of $\{v_j\}$ such that $v_{j_{k,n}} \to v$ in $L^{q(\cdot)}(\Omega \setminus K(2^{-n}))$ as well as almost everywhere on $\Omega \setminus K(2^{-n})$. Using the diagonal method, we can find a subsequence $\{v_{j_n}\}$ such that $v_{j_n} \to v$ in $L^{q(\cdot)}(\Omega \setminus K(\varepsilon))$ for each small $\varepsilon > 0$ and $v_{j_n} \to v$ almost everywhere on $\Omega$. It follows that

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \int_\Omega |v_{j_n}(x) - v(x)|^{q(x)} dx \\
&= \overline{\lim_{n\to\infty}} \left( \int_{K(\varepsilon)\cap\Omega} |v_{j_n}(x) - v(x)|^{q(x)} dx + \int_{\Omega\setminus K(\varepsilon)} |v_{j_n}(x) - v(x)|^{q(x)} dx \right) \\
&= \overline{\lim_{n\to\infty}} \int_{K(\varepsilon)\cap\Omega} |v_{j_n}(x) - v(x)|^{q(x)} dx
\end{aligned}$$

for each small $\varepsilon > 0$, which together with (3.3) implies that $\|v_{j_n} - v\|_{L^{q(\cdot)}(\Omega)} \to 0$ as $n \to \infty$.

Next consider the general case. We choose $\varepsilon_0 > 0$ such that

$$q^* < N(N/k - \varepsilon_0)/(k\varepsilon_0) - \varphi(1/r_0)/\log(1/r_0).$$

We set $p_{\varepsilon_0}(x) = \min\{p(x), N/k - \varepsilon_0\}$. Since the embedding from $W_0^{k,p(\cdot)}(\Omega)$ to $W_0^{k,p_{\varepsilon_0}(\cdot)}(\Omega)$ is bounded, we can apply the first considerations to obtain the required result. □

As a special case of Theorem 3.4, we have the following corollary, which gives an extension of [23, Theorem 2]. We put $\log^1 r = \log r$ and $\log^{n+1} r = \log(\log^n r)$, inductively.

**Corollary 3.5.** *Let $k \in \mathbb{N}$ and let $q(\cdot)$ be a variable exponent on $\Omega$ such that $1 \le q_* \le q^* < \infty$. Suppose there exist $x_0 \in \Omega$, $C > 0$, $n \in \mathbb{N}$ and small $r_0 > 0$ such that*

$$\operatorname*{ess\,inf}_{x\in\Omega\setminus B_{r_0}(x_0)} \left(p_k^\sharp(x) - q(x)\right) > 0$$

*and*

$$q(x) \le p_k^\sharp(x) - \frac{C\log^n(1/|x-x_0|)}{\log(1/|x-x_0|)} \quad \text{for almost every } x \in B_{r_0}(x_0).$$

*Then the embedding from $W_0^{k,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact.*

## 4. Existence of a solution to (1.3): compact embedding case

In this section, we assume that $p(\cdot)$ is a variable exponent on $\Omega$ satisfying the log-Hölder condition and $1 < p_* \le p^* < N$. Further let $q(\cdot)$ be a variable exponent on $\Omega$ such that $p^* < q_* \le q(x) \le p_1^\sharp(x)$ for almost every $x \in \Omega$.

As an application of Theorem 3.4, we show an existence result of nontrivial nonnegative weak solutions to (1.3) as follows.

**Theorem 4.1.** *Assume that the embedding from $W_0^{1,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact. Then there exists a nontrivial nonnegative weak solution of* (1.3).

In the case of $\operatorname{ess\,inf}_{x\in\Omega}(p_1^\sharp(x) - q(x)) > 0$, Fan and Zhang obtained such a result in [15, Theorem 4.7]. Although $q(\cdot)$ can be equal to $p_1^\sharp(\cdot)$ at some points, the proof in [15] also works in our case with minor changes since we consider the case that the embedding from $W_0^{1,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact. However, for the reader's convenience, we give a proof of our theorem.

Let $X$ be a Banach space. We say that $u \in X$ is a critical point of $I \in C^1(X;\mathbb{R})$ if the Fréchet derivative $I'(u)$ of $I$ at $u$ is zero. We say that $\{u_n\} \subset X$ is a Palais-Smale sequence for $I$ if $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$ in the dual space of $X$. We also say that $I$ satisfies the Palais-Smale condition if every Palais-Smale sequence for $I$ has a convergent subsequence.

We consider a functional $I\colon W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \int_\Omega \left(\frac{1}{p(x)}|\nabla u(x)|^{p(x)} - \frac{1}{q(x)}u^+(x)^{q(x)}\right) dx \quad \text{for } u \in W_0^{1,p(\cdot)}(\Omega).$$

The Gâteaux derivative $I'(u)$ of $I$ at $u \in W_0^{1,p(\cdot)}(\Omega)$ is given by

$$\begin{aligned}\langle I'(u), v\rangle &= \lim_{t\to 0} \frac{I(u+tv) - I(u)}{t} \\ &= \int_\Omega \left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x) - u^+(x)^{q(x)-1}v(x)\right) dx\end{aligned}$$

for each $v \in W_0^{1,p(\cdot)}(\Omega)$. By the Vitali convergence theorem, we see that $I'$ is continuous from $W_0^{1,p(\cdot)}(\Omega)$ to its dual space $(W_0^{1,p(\cdot)}(\Omega))'$, and hence $I \in C^1(W_0^{1,p(\cdot)}(\Omega);\mathbb{R})$.

The following is essentially due to Boccardo and Murat [4, Theorem 2.1].

**Proposition 4.2.** *Let $\{u_n\} \subset W_0^{1,p(\cdot)}(\Omega)$ be a Palais-Smale sequence for $I$. Then $\{u_n\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Further there exist a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ and $u \in W_0^{1,p(\cdot)}(\Omega)$ such that $\{\nabla u_{n_i}(x)\}$ converges to $\nabla u(x)$ for almost every $x \in \Omega$.*

*Proof.* Setting $\beta = \sup_{n\in\mathbb{N}} I(u_n)$, we have

$$\int_\Omega \left(\frac{1}{p^*}|\nabla u_n(x)|^{p(x)} - \frac{1}{q_*}u_n^+(x)^{q(x)}\right) dx \le I(u_n) \le \beta \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

Since $I'(u_n) \to 0$ as $n \to \infty$ in $(W_0^{1,p(\cdot)}(\Omega))'$, we have

$$\int_\Omega \left(|\nabla u_n(x)|^{p(x)} - u_n^+(x)^{q(x)}\right) dx = \langle I'(u_n), u_n\rangle \ge -\|u_n\|_{W^{1,p(\cdot)}(\Omega)} \tag{4.2}$$

for each large positive integer $n$. Subtracting (4.2) divided by $q_*$ from (4.1) gives

$$\left(\frac{1}{p^*} - \frac{1}{q_*}\right)\int_\Omega |\nabla u_n(x)|^{p(x)}dx \le \beta + \frac{1}{q_*}\|u_n\|_{W^{1,p(\cdot)}(\Omega)} \le C(\|\nabla u_n\|_{L^{p(\cdot)}(\Omega)} + 1);$$

we used Lemma 2.2 in the second inequality. Thus Lemma 2.1 gives

$$\|\nabla u_n\|_{L^{p(\cdot)}(\Omega)} + 1 \ge C \min\left\{\|\nabla u_n\|^{p_*}_{L^{p(\cdot)}(\Omega)}, \|\nabla u_n\|^{p^*}_{L^{p(\cdot)}(\Omega)}\right\},$$

so that $\{u_n\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Hence, passing to a subsequence, we may assume that $\{u_n\}$ converges weakly to some $u$ in $W_0^{1,p(\cdot)}(\Omega)$ and $\{u_n(x)\}$ converges to $u(x)$ for almost every $x \in \Omega$. For $\eta > 0$, let $T_\eta : \mathbb{R} \to \mathbb{R}$ be a function such that

$$T_\eta(t) = t \quad \text{for } |t| \le \eta, \qquad T_\eta(t) = \eta t/|t| \quad \text{for } |t| > \eta.$$

Since $\{T_\eta(u_n - u)\}$ converges weakly to 0 in $W_0^{1,p(\cdot)}(\Omega)$ and $\{u_n\}$ is bounded in $L^{q(\cdot)}(\Omega)$ by Lemma 2.2, we have

$$\begin{aligned}\varlimsup_{n\to\infty} &\int_\Omega \left(|\nabla u_n(x)|^{p(x)-2}\nabla u_n(x) - |\nabla u(x)|^{p(x)-2}\nabla u(x)\right)\nabla(T_\eta(u_n(x) - u(x)))\,dx \\ &= \varlimsup_{n\to\infty} \int_\Omega u_n^+(x)^{q(x)-1}T_\eta(u_n(x) - u(x))\,dx \le C\eta,\end{aligned}$$

where $C > 0$ is a constant which is independent of $\eta > 0$. We set

$$\rho_n(x) = \left(|\nabla u_n(x)|^{p(x)-2}\nabla u_n(x) - |\nabla u(x)|^{p(x)-2}\nabla u(x)\right)(\nabla u_n(x) - \nabla u(x)).$$

We note that $\rho_n \ge 0$ almost everywhere for each $n \in \mathbb{N}$. Further we set

$$E_n = \{x \in \Omega : |u_n(x) - u(x)| \le \eta\}, \quad F_n = \{x \in \Omega : |u_n(x) - u(x)| > \eta\}$$

for each $n \in \mathbb{N}$. We fix $\theta \in (0, 1)$. Since

$$\int_\Omega \rho_n(x)^\theta\,dx \le \left(\int_{E_n} \rho_n(x)\,dx\right)^\theta |E_n|^{1-\theta} + \left(\int_{F_n} \rho_n(x)\,dx\right)^\theta |F_n|^{1-\theta} \quad \text{for each } n \in \mathbb{N},$$

$|F_n| \to 0$ and $\{\rho_n\}$ is bounded in $L^1(\Omega)$, we have

$$\varlimsup_{n\to\infty} \int_\Omega \rho_n(x)^\theta\,dx \le (C\eta)^\theta |\Omega|^{1-\theta}.$$

Letting $\eta \to 0$, we have $\int_\Omega \rho_n(x)^\theta \, dx \to 0$. Thus we may assume $\{\rho_n(x)\}$ converges to 0 for almost every $x \in \Omega$. Since $p_* > 1$, we see that a subsequence of $\{\nabla u_n(x)\}$ converges to $\nabla u(x)$ for almost every $x \in \Omega$. □

**Lemma 4.3.** *Suppose the embedding from $W_0^{1,p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ is compact. Then the functional $I$ satisfies the Palais-Smale condition.*

*Proof.* Let $\{u_n\} \subset W_0^{1,p(\cdot)}(\Omega)$ be a Palais-Smale sequence for $I$. By the previous proposition, we may assume that $\{u_n\}$ converges weakly to some $u \in W_0^{1,p(\cdot)}(\Omega)$, and $\{u_n(x)\}$ and $\{\nabla u_n(x)\}$ converge to $u(x)$ and $\nabla u(x)$ almost every $x \in \Omega$, respectively. Since $\langle I'(u_n), u\rangle \to 0$, the Vitali convergence theorem implies that

$$\int_\Omega |\nabla u(x)|^{p(x)} \, dx = \int_\Omega u^+(x)^{q(x)} \, dx.$$

This equality together with $\langle I'(u_n), u_n\rangle \to 0$ and the compact embedding assumption give

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega |\nabla u_n(x)|^{p(x)} \, dx &= \lim_{n\to\infty} \int_\Omega u_n^+(x)^{q(x)} \, dx \\ &= \int_\Omega u^+(x)^{q(x)} \, dx = \int_\Omega |\nabla u(x)|^{p(x)} \, dx. \end{aligned} \tag{4.3}$$

Now, we consider the function

$$w_n(x) = 2^{p^*-1} \left( |\nabla u_n(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \right) - |\nabla u_n(x) - \nabla u(x)|^{p(x)} .$$

Since $w_n(x) \geq 0$ for almost every $x \in \Omega$, we see from Fatou's lemma and (4.3) that

$$\begin{aligned} & 2^{p^*} \int_\Omega |\nabla u(x)|^{p(x)} dx - \overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n(x) - \nabla u(x)|^{p(x)} \, dx \\ & \geq \int_\Omega \underline{\lim_{n\to\infty}} \, w_n(x) dx = 2^{p^*} \int_\Omega |\nabla u(x)|^{p(x)} dx, \end{aligned}$$

so that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n(x) - \nabla u(x)|^{p(x)} \, dx = 0.$$

Hence we see that $\{u_n\}$ converges strongly to $u$ in $W_0^{1,p(\cdot)}(\Omega)$. □

We recall the following variant of the mountain pass theorem; see e.g., [34].

**Theorem 4.4.** *Let $X$ be a Banach space and let $I$ be a $C^1$ functional on $X$ such that $I(0) = 0$,*

(i) *there exist positive constants $\kappa, r > 0$ such that $I(u) \geq \kappa$ for all $u \in X$ with $\|u\| = r$, and*
(ii) *there exists an element $v \in X$ such that $I(v) < 0$ and $\|v\| > r$.*

*Define*

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I(\gamma(t)),$$

*where*

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0,\ I(\gamma(1)) < 0,\ \|\gamma(1)\| > r\}. \tag{4.4}$$

*Then $c > 0$ and for each $\varepsilon > 0$, there exists $u \in X$ such that $|I(u) - c| \leq \varepsilon$ and $\|I'(u)\| \leq \varepsilon$.*

Now we are ready to prove Theorem 4.1.

*Proof of Theorem 4.1.* First we find $r > 0$ such that

$$\inf\{I(u) : u \in W_0^{1,p(\cdot)}(\Omega),\ \|u\|_{W^{1,p(\cdot)}(\Omega)} = r\} > 0. \tag{4.5}$$

Taking $r > 0$ so small, by Lemma 2.2, we have $\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \le 1$ and $\|u\|_{L^{q(\cdot)}(\Omega)} \le 1$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$ with $\|u\|_{W^{1,p(\cdot)}(\Omega)} = r$. Then for each $u \in W_0^{1,p(\cdot)}(\Omega)$ with $\|u\|_{W^{1,p(\cdot)}(\Omega)} = r$, we have

$$\int_\Omega u^+(x)^{q(x)} dx \le \|u\|^{q_*}_{L^{q(\cdot)}(\Omega)} \le C\|u\|^{q_*}_{L^{p_1^\sharp(\cdot)}(\Omega)} \le C\|\nabla u\|^{q_*}_{L^{p(\cdot)}(\Omega)}$$

by Lemmas 2.1 and 2.2, so that

$$I(u) \ge \frac{1}{p^*}\|\nabla u\|^{p^*}_{L^{p(\cdot)}(\Omega)} - \frac{C}{q_*}\|\nabla u\|^{q_*}_{L^{p(\cdot)}(\Omega)}.$$

Since $p^* < q_*$, we have (4.5) if $r > 0$ is small.

Next we prove $I(tu) \to -\infty$ as $t \to \infty$ for $u \in W_0^{1,p(\cdot)}(\Omega)$ with $u^+ \ne 0$. In fact, if $u \in W_0^{1,p(\cdot)}(\Omega)$ such that $u^+ \ne 0$, then we see that

$$I(tu) \le t^{p^*} \int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)} dx - t^{q_*} \int_\Omega \frac{1}{q(x)} u^+(x)^{q(x)} dx \to -\infty$$

as $t \to \infty$, since $p^* < q_*$.

Now the required result follows from Lemma 4.3 and Theorem 4.4. □

As a direct consequence of Theorem 4.1, we have the following:

**Corollary 4.5.** *Suppose all hypotheses in Theorem 3.4 hold for $k = 1$. Then there exists a nontrivial nonnegative weak solution of* (1.3).

## 5. Existence of a solution to (1.3): noncompact embedding case

Our final aim is to deal with the existence result of a nontrivial nonnegative weak solution to (1.3) in the case that the embedding may not be compact.

For real sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n = b_n + o(1)$ or $a_n \le b_n + o(1)$ if $\lim_n(a_n - b_n) = 0$ or $\overline{\lim}_n(a_n - b_n) \le 0$, respectively.

**Proposition 5.1.** *Let $p(\cdot)$ be a log-Hölder continuous function on $\Omega$ with $1 < p_* \le p^* < N$ and let $q(\cdot)$ be a measurable function on $\Omega$ such that $p^* < q_* \le q(x) \le p_1^\sharp(x)$ for almost every $x \in \Omega$. Assume $\inf_{u\in\mathcal{N}_I} I(u) < \inf_{u\in\mathcal{N}_J} J(u)$, where*

$$I(u) = \int_\Omega \left(\frac{1}{p(x)}|\nabla u(x)|^{p(x)} - \frac{1}{q(x)}u^+(x)^{q(x)}\right) dx \quad \textit{for } u \in W_0^{1,p(\cdot)}(\Omega),$$

$$J(u) = \int_\Omega \left(\frac{1}{p(x)}|\nabla u(x)|^{p(x)} - \frac{1}{p_1^\sharp(x)}u^+(x)^{p_1^\sharp(x)}\right) dx \quad \textit{for } u \in W_0^{1,p(\cdot)}(\Omega),$$

$$\mathcal{N}_I = \left\{u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\} : \int_\Omega |\nabla u(x)|^{p(x)}\, dx = \int_\Omega u^+(x)^{q(x)}\, dx\right\},$$

$$\mathcal{N}_J = \left\{u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\} : \int_\Omega |\nabla u(x)|^{p(x)}\, dx = \int_\Omega u^+(x)^{p_1^\sharp(x)}\, dx\right\}.$$

*Then problem* (1.3) *has a nontrivial nonnegative weak solution.*

*Proof.* We set $c = \inf_{u\in\mathcal{N}_I} I(u)$, and define $\Gamma$ by (4.4) with $X = W_0^{1,p(\cdot)}(\Omega)$. Along the similar lines as those in the proof of Theorem 4.1, we can easily see that $\Gamma \neq \emptyset$, $\mathcal{N}_J \neq \emptyset$, $\mathcal{N}_I \neq \emptyset$ and (4.5) holds for small $r > 0$.

First we show

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I(\gamma(t)). \tag{5.1}$$

Let $u \in \mathcal{N}_I$. For $\alpha_u > 1$ large enough, consider the path $\gamma_u \in \Gamma$ defined by $\gamma_u(t) = t\alpha_u u$ for $t \in [0,1]$. Since $I(u) = \max_{0\le t\le 1} I(\gamma_u(t))$, we have

$$c \ge \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I(\gamma(t)).$$

On the other hand, let $\gamma \in \Gamma$. Then

$$\int_\Omega (|\nabla\gamma(1)|^{p(x)} - (\gamma(1)^+)^{q(x)})\,dx < 0.$$

As in the proof of Theorem 4.1, we find a small $t > 0$ satisfying

$$\int_\Omega (|\nabla\gamma(t)|^{p(x)} - (\gamma(t)^+)^{q(x)})\,dx > 0.$$

By the intermediate value theorem, there exists $t \in (0,1)$ such that $\gamma(t) \in \mathcal{N}_I$, which implies $c \le \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I(\gamma(t))$. Thus (5.1) holds.

Now, in view of Theorem 4.4, $c > 0$. Moreover there exists $\{u_n\} \subset W_0^{1,p(\cdot)}(\Omega)$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in $(W_0^{1,p(\cdot)}(\Omega))'$. By Proposition 4.2 and $c > 0$, we find a constant $C > 0$ such that

$$\frac{1}{C} \le \int_\Omega |\nabla u_n(x)|^{p(x)}\,dx \le C \quad \text{for large } n \in \mathbb{N}. \tag{5.2}$$

Here we may assume that $\{u_n\}$ converges weakly to some $u \in W_0^{1,p(\cdot)}(\Omega)$; further $\{u_n(x)\}$ and $\{\nabla u_n(x)\}$ converge to $u(x)$ and $\nabla u(x)$ for almost every $x \in \Omega$, respectively. Then it follows that $I'(u) = 0$. If we show that $u \neq 0$, then $u$ is a nontrivial nonnegative weak solution of (1.3).

On the contrary, suppose $u = 0$. Since $I(u_n) \to c > 0$ and $\langle I'(u_n), u_n\rangle \to 0$, taking a subsequence if necessary, we may assume $u_n^+ \neq 0$ for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, there exists a unique $t_n \in (0,\infty)$ such that

$$\int_\Omega |\nabla(t_n u_n(x))|^{p(x)}\,dx = \int_\Omega (t_n u_n^+(x))^{p_1^\sharp(x)}\,dx,$$

i.e., $t_n u_n \in \mathcal{N}_J$. We will show $t_n \le 1 + o(1)$. On the contrary, if there exists $\varepsilon > 0$ such that $t_n \ge 1 + \varepsilon$ for all $n \in \mathbb{N}$, then

$$\begin{aligned} t_n^{p^*} \int_\Omega |\nabla u_n(x)|^{p(x)}\,dx &\ge \int_\Omega |\nabla(t_n u_n(x))|^{p(x)}\,dx \\ &= \int_\Omega (t_n u_n^+(x))^{p_1^\sharp(x)}\,dx \ge t_n^{q_*} \int_\Omega u_n^+(x)^{p_1^\sharp(x)}\,dx \end{aligned}$$

for all $n \in \mathbb{N}$. Using Lebesgue's convergence theorem, we have

$$\begin{aligned}\int_\Omega |\nabla u_n(x)|^{p(x)}\,dx &= \int_\Omega u_n^+(x)^{q(x)}\,dx + o(1)\\ &= \int_{\{x\in\Omega: u_n(x)\le 1\}} u_n^+(x)^{q(x)}\,dx + \int_{\{x\in\Omega: u_n(x)>1\}} u_n^+(x)^{q(x)}\,dx + o(1)\\ &\le \int_\Omega \min\{u_n^+(x),1\}\,dx + \int_\Omega u_n^+(x)^{p_1^\sharp(x)}\,dx + o(1)\\ &\le \int_\Omega u_n^+(x)^{p_1^\sharp(x)}\,dx + o(1).\end{aligned}$$

Hence it follows that

$$\begin{aligned}\int_\Omega |\nabla u_n(x)|^{p(x)}\,dx &\ge t_n^{q_*-p^*}\int_\Omega u_n^+(x)^{p_1^\sharp(x)}\,dx \ge (1+\varepsilon)^{q_*-p^*}\int_\Omega u_n^+(x)^{p_1^\sharp(x)}\,dx\\ &\ge (1+\varepsilon)^{q_*-p^*}\left(\int_\Omega |\nabla u_n(x)|^{p(x)}\,dx + o(1)\right),\end{aligned}$$

which together with (5.2) yields a contradiction. Thus we have shown $t_n \le 1+o(1)$. On the other hand, for each $n \in \mathbb{N}$, take a unique number $s_n > 0$ such that

$$\int_\Omega |\nabla(s_n u_n(x))|^{p(x)}\,dx = \int_\Omega (s_n u_n^+(x))^{q(x)}\,dx, \tag{5.3}$$

i.e., $s_n u_n \in \mathcal{N}_I$. We see easily that $I(s_n u_n) = \max_{s\ge 0} I(su_n)$ for each $n \in \mathbb{N}$. By (5.2), (5.3) and $\langle I'(u_n), u_n\rangle = o(1)$, we infer that $s_n = 1+o(1)$, so that

$$I(u_n) = I(s_n u_n) + o(1) = \max_{s\ge 0} I(su_n) + o(1) \ge I(t_n u_n) + o(1).$$

Let $\varepsilon \in (0,1)$. Then, noting

$$\begin{aligned}&\int_{\{x\in\Omega: q(x)\le p_1^\sharp(x)-\varepsilon\}} (t_n u_n^+(x))^{q(x)}\,dx\\ &\qquad\le \int_\Omega \min\{t_n u_n^+(x),1\}\,dx + \int_\Omega (t_n u_n^+(x))^{p_1^\sharp(x)-\varepsilon}\,dx = o(1),\end{aligned}$$

we obtain

$$\begin{aligned}c &= I(u_n) + o(1) \ge I(t_n u_n) + o(1)\\ &\ge \int_\Omega \left(\frac{1}{p(x)}|\nabla(t_n u_n(x))|^{p(x)} - \frac{1}{p_1^\sharp(x)-\varepsilon}(t_n u_n^+(x))^{p_1^\sharp(x)}\right)dx + o(1)\\ &= J(t_n u_n) + \int_\Omega \left(\frac{1}{p_1^\sharp(x)} - \frac{1}{p_1^\sharp(x)-\varepsilon}\right)(t_n u_n^+(x))^{p_1^\sharp(x)}\,dx + o(1)\\ &\ge \inf_{v\in\mathcal{N}_J} J(v) - C\varepsilon,\end{aligned}$$

where $C$ is a constant which is independent of $\varepsilon \in (0,1)$. Since $\varepsilon \in (0,1)$ is arbitrary, we conclude that $c \ge \inf_{v\in\mathcal{N}_J} J(v)$, which contradicts our assumption. Hence it follows that $u \ne 0$, as required. □

We denote by $\mathcal{D}^{1,p(\cdot)}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ by the norm $\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^N)}$ in $C_0^\infty(\mathbb{R}^N)$.

**Theorem 5.2.** *Let $p(\cdot) : \mathbb{R}^N \to \mathbb{R}$ be a log-Hölder continuous function with $1 < p_* \le p^* < N$, and let $q(\cdot) : \mathbb{R}^N \to \mathbb{R}$ be a measurable function such that $p^* < q_* \le q(x) \le p_1^\sharp(x)$ for almost every $x \in \mathbb{R}^N$. Assume that $\mathcal{D}^{1,p(\cdot)}(\mathbb{R}^N)$ is continuously embedded into $L^{p_1^\sharp(\cdot)}(\mathbb{R}^N)$, i.e., there exists a constant $C > 0$ such that*

$$\|u\|_{L^{p_1^\sharp(\cdot)}(\mathbb{R}^N)} \le C\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^N)} \quad \text{for all } u \in \mathcal{D}^{1,p(\cdot)}(\mathbb{R}^N). \tag{5.4}$$

*Assume also that there exist a measurable subset $D$ of $\mathbb{R}^N$ and a number $q_0$ such that*

$$\varlimsup_{R\to\infty} |\{x \in B_1(0) : Rx \in D\}| < |B_1(0)|, \tag{5.5}$$

*$N\underline{p}/(N + p_* - \underline{p}) < q_0 < N\underline{p}/(N - \underline{p})$, and $\operatorname{ess\,sup}_{x\in\mathbb{R}^N\setminus D} q(x) \le q_0$, where $\underline{p} = \varliminf_{|x|\to\infty} p(x)$. Then there exists $R > 0$ such that for each bounded open set $\Omega$ in $\mathbb{R}^N$ which contains $B_R(0)$, problem (1.3) has a nontrivial nonnegative weak solution.*

*Proof.* We set

$$J_{\mathbb{R}^N}(u) = \int_{\mathbb{R}^N} \left( \frac{1}{p(x)}|\nabla u(x)|^{p(x)} - \frac{1}{p_1^\sharp(x)} u^+(x)^{p_1^\sharp(x)} \right) dx \quad \text{for } u \in \mathcal{D}^{1,p(\cdot)}(\mathbb{R}^N),$$

$$\mathcal{N}_{J_{\mathbb{R}^N}} = \left\{ u \in \mathcal{D}^{1,p(\cdot)}(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |\nabla u(x)|^{p(x)}\, dx = \int_{\mathbb{R}^N} u^+(x)^{p_1^\sharp(x)}\, dx \right\}.$$

By Lemma 2.1 we have for $u \in \mathcal{N}_{J_{\mathbb{R}^N}}$

$$\begin{aligned}\min\left\{ \|\nabla u\|^{p_*}_{L^{p(\cdot)}(\mathbb{R}^N)}, \|\nabla u\|^{p^*}_{L^{p(\cdot)}(\mathbb{R}^N)} \right\} &\le \int_{\mathbb{R}^N} |\nabla u(x)|^{p(x)}\, dx \\ &= \int_{\mathbb{R}^N} u^+(x)^{p_1^\sharp(x)}\, dx \le \max\left\{ \|u^+\|^{(p_1^\sharp)_*}_{L^{p_1^\sharp(\cdot)}(\mathbb{R}^N)}, \|u^+\|^{(p_1^\sharp)^*}_{L^{p_1^\sharp(\cdot)}(\mathbb{R}^N)} \right\},\end{aligned}$$

which together with (5.4) implies that

$$\inf_{u\in\mathcal{N}_{J_{\mathbb{R}^N}}} \|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^N)} > 0.$$

Hence we infer that

$$\inf_{u\in\mathcal{N}_{J_{\mathbb{R}^N}}} J_{\mathbb{R}^N}(u) > 0.$$

Choose any $p_0$ such that

$$1 < p_0 < \underline{p} \quad \text{and} \quad \frac{Np_0}{N + p_* - p_0} < q_0 < \frac{Np_0}{N - p_0}. \tag{5.6}$$

Let $\bar{u}_1 \in W_0^{1,p_0}(B_1(0))$ be a weak solution of the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u(x)|^{p_0-2}\, \nabla u(x)\right) = u(x)^{q_0-1} & \text{in } B_1(0), \\ u(x) > 0 & \text{in } B_1(0), \\ u(x) = 0 & \text{on } \partial B_1(0). \end{cases} \tag{5.7}$$

According to [24, Theorem 1] or [33, Proposition 2.1], we see that $\bar{u}_1 \in C^{1,\beta}(\overline{B_1(0)})$ for some $\beta \in (0, 1)$. Hence, for each $R > 0$, $\bar{u}_R(x) \equiv R^{-p_0/(q_0-p_0)}\bar{u}_1(x/R)$ is a weak solution of (5.7). Take $R_1 > 0$ such that $\max_{|x|\le R} \bar{u}_R(x) \le 1$ for $R \ge R_1$. For each $R > 0$, there exists a unique $t_R \in (0, \infty)$ such that

$$\int_{B_R(0)} |\nabla(t_R\bar{u}_R(x))|^{p(x)}\, dx = \int_{B_R(0)} |t_R\bar{u}_R(x)|^{q(x)}\, dx.$$

From (5.5), we find $\delta > 0$ and $R_2 \geq R_1$ such that

$$|\{x \in B_1(0) : Rx \in D\}| \leq |B_1(0)| - \delta \quad \text{for each } R \geq R_2.$$

We will show $\{t_R : R \geq R_2\}$ is bounded. If $t_R > 1$ with $R \geq R_2$, then we have

$$t_R^{p^*} \int_{B_R(0)} |\nabla \bar{u}_R(x)|^{p(x)}\, dx \geq \int_{B_R(0)} |t_R \bar{u}_R(x)|^{q(x)}\, dx \geq t_R^{q_*} \int_{B_R(0)\setminus D} |\bar{u}_R(x)|^{q_0}\, dx$$
$$= t_R^{q_*} \left( \int_{B_R(0)} |\bar{u}_R(x)|^{q_0}\, dx - \int_{B_R(0)\cap D} |\bar{u}_R(x)|^{q_0}\, dx \right),$$

which implies

$$t_R^{q_*-p^*} \leq \frac{\int_{B_1(0)} R^{\frac{q_0(p_0-p(Rx))}{q_0-p_0}} |\nabla \bar{u}_1(x)|^{p(Rx)}\, dx}{\int_{B_1(0)} |\bar{u}_1(x)|^{q_0}\, dx - \sup\{\int_A |\bar{u}_1(x)|^{q_0}\, dx : A \subset B_1(0), |A| \leq |B_1(0)| - \delta\}}.$$

Let $r_0 > 0$ such that $p(x) > p_0$ for all $x \in \mathbb{R}^N$ with $|x| \geq r_0$. By (5.6) and the boundedness of $|\nabla \bar{u}_1|$, we have for $R \geq r_0$,

$$\int_{B_1(0)} R^{\frac{q_0(p_0-p(Rx))}{q_0-p_0}} |\nabla \bar{u}_1(x)|^{p(Rx)}\, dx$$
$$\leq C \left( \int_{|x|<r_0/R} R^{\frac{q_0(p_0-p(Rx))}{q_0-p_0}}\, dx + \int_{r_0/R \leq |x| \leq 1} R^{\frac{q_0(p_0-p(Rx))}{q_0-p_0}}\, dx \right)$$
$$\leq C \left( R^{\frac{q_0(p_0-p_*)}{q_0-p_0}} \left(\frac{r_0}{R}\right)^N + 1 \right) \leq C,$$

where each $C$ is a positive constant which is independent of $R$. Hence we insist that $\{t_R : R \geq R_2\}$ is bounded. Then we have

$$\int_{B_R(0)} \left( \frac{1}{p(x)} |\nabla (t_R \bar{u}_R(x))|^{p(x)} - \frac{1}{q(x)} |t_R \bar{u}_R(x)|^{q(x)} \right) dx \leq C \int_{B_R(0)} |\nabla \bar{u}_R(x)|^{p(x)}\, dx$$
$$= C \int_{B_1(0)} R^{-\frac{q_0 p(Rx)}{q_0-p_0}+N} |\nabla \bar{u}_1(x)|^{p(Rx)}\, dx \leq C \left( R^{-\frac{q_0 p_*}{q_0-p_0}} r_0^N + R^{-\frac{q_0 p_0}{q_0-p_0}+N} \right) \to 0$$

as $R \to \infty$. Hence we can find $R \geq R_2$ satisfying

$$\int_{B_R(0)} \left( \frac{1}{p(x)} |\nabla (t_R \bar{u}_R(x))|^{p(x)} - \frac{1}{q(x)} |t_R \bar{u}_R(x)|^{q(x)} \right) dx < \inf_{v \in \mathcal{N}_{J_{\mathbb{R}^N}}} J_{\mathbb{R}^N}(v).$$

Now, let $\Omega$ be any bounded open set which contains $B_R(0)$. Extending $\bar{u}_R$ on $\Omega$ with $\bar{u}_R(x) = 0$ for $x \in \Omega \setminus B_R(0)$, we have $\bar{u}_R \in W_0^{1,p(\cdot)}(\Omega)$. Letting $I$, $J$, $\mathcal{N}_I$ and $\mathcal{N}_J$ be as in the previous proposition, we have

$$\inf_{v \in \mathcal{N}_I} I(v) \leq I(t_R \bar{u}_R) < \inf_{v \in \mathcal{N}_{J_{\mathbb{R}^N}}} J_{\mathbb{R}^N}(v) \leq \inf_{v \in \mathcal{N}_J} J(v).$$

Hence problem (1.3) has a nontrivial nonnegative weak solution on $\Omega$ by the proposition. □

Finally, we give a sufficient condition for (5.4). We recall the following result, which is a special case of [6, Theorem 1.8].

**Lemma 5.3.** *Let $p(\cdot) : \mathbb{R}^N \to \mathbb{R}$ be a log-Hölder continuous function which satisfies $1 < p_* \le p^* < N$ and*

$$|p(x) - p(y)| \le \frac{C}{\log(e + |x|)} \quad \text{for each } x, y \in \mathbb{R}^N \text{ with } |y| \ge |x|.$$

*Then the fractional integral operator*

$$u \mapsto \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-1}}\, dy$$

*is bounded from $L^{p(\cdot)}(\mathbb{R}^N)$ to $L^{p_1^\sharp(\cdot)}(\mathbb{R}^N)$.*

**Corollary 5.4.** *Let $p(\cdot) : \mathbb{R}^N \to \mathbb{R}$ be as in the previous lemma, and let $D$, $q_0$ and $q(\cdot)$ be as in Theorem 5.2. Then there exists $R > 0$ such that for each bounded open set $\Omega$ in $\mathbb{R}^N$ which contains $B_R(0)$, problem (1.3) has a nontrivial nonnegative weak solution.*

*Proof.* Using the previous lemma, we can show that $\mathcal{D}^{1,p(\cdot)}(\mathbb{R}^N)$ is continuously embedded into $L^{p_1^\sharp(\cdot)}(\mathbb{R}^N)$ by similar lines as those in [35, p. 88]. Hence we obtain the conclusion by Theorem 5.2. □

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(Y. Mizuta) Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima 739-8521, Japan
*E-mail address*: mizuta@mis.hiroshima-u.ac.jp

(T. Ohno) General Arts, Hiroshima National College of Maritime Technology, Higashino Oosakikamijima Toyotagun 725-0231, Japan
*E-mail address*: ohno@hiroshima-cmt.ac.jp

(T. Shimomura) Department of Mathematics, Graduate School of Education, Hiroshima University, Higashi-Hiroshima 739-8524, Japan
*E-mail address*: tshimo@hiroshima-u.ac.jp

(N. Shioji) Department of Mathematics, Graduate School of Environment and Information Sciences, Yokohama National University, Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan
*E-mail address*: shioji@math.sci.ynu.ac.jp